Problem 26)

$$\int_a^b [f(x)g(x)]'\mathrm{d}x = \int_a^b f'(x)g(x)\mathrm{d}x + \int_a^b f(x)g'(x)\mathrm{d}x$$

$$\rightarrow \quad \int_a^b f'(x)g(x)\mathrm{d}x = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\mathrm{d}x$$

$$= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\mathrm{d}x.$$

a) $$\int_0^\infty xe^{-\kappa x}\mathrm{d}x = \cancelto{0}{-(x/\kappa)e^{-\kappa x}|_0^\infty} + (1/\kappa)\int_0^\infty e^{-\kappa x}\mathrm{d}x = -(1/\kappa^2)e^{-\kappa x}|_0^\infty = 1/\kappa^2.$$

b) $$\int_0^\infty x^2 e^{-\kappa x}\mathrm{d}x = \cancelto{0}{-(x^2/\kappa)e^{-\kappa x}|_0^\infty} + (2/\kappa)\int_0^\infty xe^{-\kappa x}\mathrm{d}x = 2/\kappa^3.$$ ← see part (a)

c) $$\int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x = \cancelto{0}{-\tfrac{1}{\kappa}\sin(\omega x)\, e^{-\kappa x}\Big|_0^\infty} + \tfrac{\omega}{\kappa}\int_0^\infty \cos(\omega x)\, e^{-\kappa x}\mathrm{d}x$$

$$= \tfrac{\omega}{\kappa}\left[-\tfrac{1}{\kappa}\cos(\omega x)\, e^{-\kappa x}\Big|_0^\infty - \tfrac{\omega}{\kappa}\int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x\right]$$

$$= \tfrac{\omega}{\kappa}\left[\tfrac{1}{\kappa} - \tfrac{\omega}{\kappa}\int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x\right].$$

$$\rightarrow \left(1 + \tfrac{\omega^2}{\kappa^2}\right)\int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x = \tfrac{\omega}{\kappa^2} \qquad \rightarrow \qquad \int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x = \tfrac{\omega}{\omega^2+\kappa^2}.$$

d) Using the result in part (c), we also find

$$\int_0^\infty \cos(\omega x)\, e^{-\kappa x}\mathrm{d}x = (\kappa/\omega)\int_0^\infty \sin(\omega x)\, e^{-\kappa x}\mathrm{d}x \quad \rightarrow \quad \int_0^\infty \cos(\omega x)\, e^{-\kappa x}\mathrm{d}x = \tfrac{\kappa}{\omega^2+\kappa^2}.$$

e) $$\int_0^\infty x^2 e^{-\pi x^2}\mathrm{d}x = \cancelto{0}{-(x/2\pi)e^{-\pi x^2}\Big|_0^\infty} + (1/2\pi)\int_0^\infty e^{-\pi x^2}\mathrm{d}x = 1/4\pi.$$ ← see Problem 27

f) $$\int_0^\infty x^3 e^{-\pi x^2}\mathrm{d}x = \cancelto{0}{-(x^2/2\pi)e^{-\pi x^2}\Big|_0^\infty} + \int_0^\infty (2x/2\pi)e^{-\pi x^2}\mathrm{d}x = -\tfrac{1}{2\pi^2}e^{-\pi x^2}\Big|_0^\infty = \tfrac{1}{2\pi^2}.$$

g) $$\int_0^a x\ln x\,\mathrm{d}x = (\tfrac12 x^2\ln x)|_0^a - \int_0^a (\tfrac12 x^2)(1/x)\mathrm{d}x = \tfrac12 a^2\ln a - (\tfrac14 x^2)|_0^a$$

$$= a^2(2\ln a - 1)/4.$$

h) $$\int_0^a x^\kappa \ln^2 x\,\mathrm{d}x = \tfrac{x^{\kappa+1}}{\kappa+1}\ln^2 x\Big|_0^a - \int_0^a \left(\tfrac{x^{\kappa+1}}{\kappa+1}\right)\left(\tfrac{2}{x}\ln x\right)\mathrm{d}x$$

$$= \tfrac{a^{\kappa+1}}{\kappa+1}\ln^2 a - \tfrac{2}{\kappa+1}\int_0^a x^\kappa \ln x\,\mathrm{d}x$$

$$= \tfrac{a^{\kappa+1}}{\kappa+1}\ln^2 a - \tfrac{2}{\kappa+1}\left[\tfrac{x^{\kappa+1}}{\kappa+1}\ln x\Big|_0^a - \int_0^a \left(\tfrac{x^{\kappa+1}}{\kappa+1}\right)\left(\tfrac{1}{x}\right)\mathrm{d}x\right]$$

$$= \tfrac{a^{\kappa+1}}{\kappa+1}\ln^2 a - \tfrac{2a^{\kappa+1}}{(\kappa+1)^2}\ln a + \tfrac{2a^{\kappa+1}}{(\kappa+1)^3}.$$