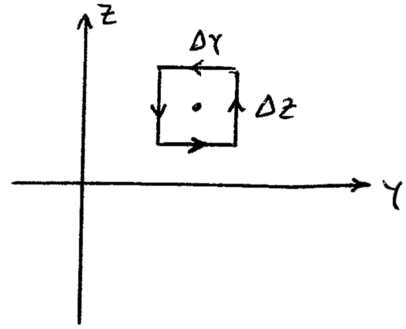
Problem 24)

a) Cartesian Coordinates:



$$\vec{\nabla}\times\vec{f}(x,y,z) = \frac{\oint \vec{f}\cdot d\vec{\ell}}{\Delta s}\Big|_{\Delta s\to 0} \quad \text{(3 Components)}$$

x-component of Curl $\vec{f} = \frac{1}{\Delta y \Delta z}\Big\{ f_z(x, y+\tfrac{1}{2}\Delta y, z)\,\Delta z$

$- f_y(x,y,z+\tfrac{1}{2}\Delta z)\,\Delta y - f_z(x, y-\tfrac{1}{2}\Delta y, z)\,\Delta z + f_y(x,y,z-\tfrac{1}{2}\Delta z)\,\Delta y \Big\} = \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}.$

The y- and z-components of $\vec{\nabla}\times\vec{f}$ may be obtained in a similar way.

Consequently: $$\vec{\nabla}\times\vec{f}(\vec{r}) = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{x} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{y} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{z}.$$

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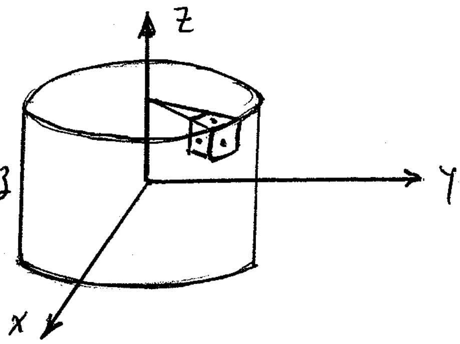
b) Cylindrical Coordinates:



ρ-Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\Delta\phi\Delta z}\Big\{ f_z(\rho, \phi+\tfrac{\Delta\phi}{2}, z)\,\Delta z$

$- f_\phi(\rho,\phi,z+\tfrac{\Delta z}{2})\,\rho\Delta\phi - f_z(\rho,\phi-\tfrac{\Delta\phi}{2},z)\,\Delta z$

$+ f_\phi(\rho,\phi,z-\tfrac{\Delta z}{2})\,\rho\Delta\phi \Big\} = \frac{1}{\rho}\frac{\partial}{\partial\phi}f_z - \frac{\partial f_\phi}{\partial z}.$

ϕ-Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\Delta\rho\Delta z}\Big\{ -f_z(\rho+\tfrac{\Delta\rho}{2},\phi,z)\,\Delta z + f_\rho(\rho,\phi,z+\tfrac{\Delta z}{2})\,\Delta\rho$

$+ f_z(\rho-\tfrac{\Delta\rho}{2},\phi,z)\,\Delta z - f_\rho(\rho,\phi,z-\tfrac{\Delta z}{2})\Big\} = \frac{-\partial}{\partial\rho}f_z + \frac{\partial}{\partial z}f_\rho.$

z-Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\Delta\rho\Delta\phi}\Big\{ f_\phi(\rho+\tfrac{\Delta\rho}{2},\phi,z)(\rho+\tfrac{\Delta\rho}{2})\Delta\phi - f_\rho(\rho,\phi+\tfrac{\Delta\phi}{2},z)\,\Delta\rho$

$- f_\phi(\rho-\tfrac{\Delta\rho}{2},\phi,z)(\rho-\tfrac{\Delta\rho}{2})\Delta\phi + f_\rho(\rho,\phi+\tfrac{\Delta\phi}{2},z)\,\Delta\rho\Big\} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\phi) - \frac{1}{\rho}\frac{\partial}{\partial\phi}f_\rho.$

Therefore, $$\vec{\nabla}\times\vec{f}(\vec{r}) = \left(\frac{1}{\rho}\frac{\partial}{\partial\phi}f_z - \frac{\partial}{\partial z}f_\phi\right)\hat{\rho} + \left(\frac{\partial}{\partial z}f_\rho - \frac{\partial}{\partial\rho}f_z\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho f_\phi) - \frac{\partial}{\partial\phi}f_\rho\right)\hat{z}.$$

c) Spherical Coordinates:

ρ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \dfrac{1}{\rho^2 \sin\theta\, \Delta\theta\, \Delta\phi} \Big\{ -f_\theta(\rho,\theta,\phi+\tfrac{\Delta\phi}{2})\,\rho\Delta\theta -$

$f_\phi(\rho,\theta-\tfrac{\Delta\theta}{2},\phi)\,\rho\sin(\theta-\tfrac{\Delta\theta}{2})\Delta\phi + f_\theta(\rho,\theta,\phi-\tfrac{\Delta\phi}{2})\,\rho\Delta\theta + f_\phi(\rho,\theta+\tfrac{\Delta\theta}{2},\phi)\,\rho\sin(\theta+\tfrac{\Delta\theta}{2})\Delta\phi \Big\}$

$$= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\phi) - \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi} f_\theta .$$

θ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \dfrac{1}{\rho\sin\theta\,\Delta\phi\,\Delta\rho} \Big\{ -f_\phi(\rho+\tfrac{\Delta\rho}{2},\theta,\phi)(\rho+\tfrac{\Delta\rho}{2})\sin\theta\,\Delta\phi$

$- f_\rho(\rho,\theta,\phi-\tfrac{\Delta\phi}{2})\Delta\rho + f_\phi(\rho-\tfrac{\Delta\rho}{2},\theta,\phi)(\rho-\tfrac{\Delta\rho}{2})\sin\theta\,\Delta\phi + f_\rho(\rho,\theta,\phi+\tfrac{\Delta\phi}{2})\Delta\rho \Big\}$

$$= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi} f_\rho - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\phi) .$$

ϕ- Component of $\vec{\nabla}\times\vec{f}(\vec{r}) = \dfrac{1}{\rho\,\Delta\theta\,\Delta\rho} \Big\{ f_\theta(\rho+\tfrac{\Delta\rho}{2},\theta,\phi)(\rho+\tfrac{\Delta\rho}{2})\Delta\theta$

$- f_\rho(\rho,\theta+\tfrac{\Delta\theta}{2},\phi)\Delta\rho - f_\theta(\rho-\tfrac{\Delta\rho}{2},\theta,\phi)(\rho-\tfrac{\Delta\rho}{2})\Delta\theta + f_\rho(\rho,\theta-\tfrac{\Delta\theta}{2},\phi)\Delta\rho \Big\}$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\theta) - \frac{1}{\rho}\frac{\partial}{\partial\theta} f_\rho .$$

Therefore,
$$\vec{\nabla}\times\vec{f}(\vec{r}) = \frac{1}{\rho\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta f_\phi) - \frac{\partial}{\partial\phi} f_\theta\Big]\hat{\rho} + \frac{1}{\rho}\Big[\frac{1}{\sin\theta}\frac{\partial f_\rho}{\partial\phi} - \frac{\partial}{\partial\rho}(\rho f_\phi)\Big]\hat{\theta} + \frac{1}{\rho}\Big[\frac{\partial}{\partial\rho}(\rho f_\theta) - \frac{\partial}{\partial\theta} f_\rho\Big]\hat{\phi} .$$