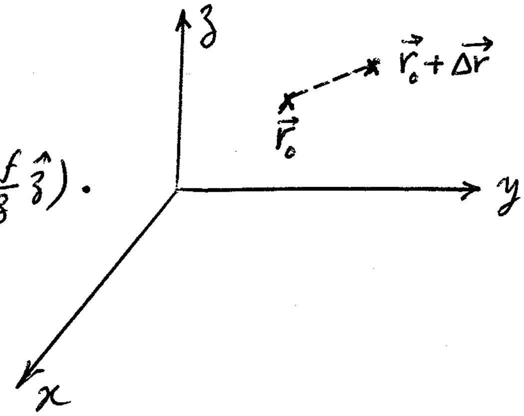
Problem 22)

a) Cartesian Coordinates; $f(\vec{r}) = f(x,y,z)$



$$f(\vec{r}_0+\Delta\vec{r}) = f(\vec{r}_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

$$\Rightarrow \Delta f = f(\vec{r}_0+\Delta\vec{r}) - f(\vec{r}_0) = \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)\cdot (\Delta x\,\hat{x} + \Delta y\,\hat{y} + \Delta z\,\hat{z})$$

Defining $\vec{\nabla} f(\vec{r}) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$,
We can write the above expression as follows:

$$\Delta f = \vec{\nabla} f \cdot \Delta\vec{r}$$

Now, the above dot-product is equal to $|\vec{\nabla} f|\,|\Delta\vec{r}|\cos\theta$, where $|\vec{\nabla} f|$ is the magnitude (or length) of $\vec{\nabla} f$, $|\Delta\vec{r}|$ is the magnitude of $\Delta\vec{r}$, and θ is the angle between $\vec{\nabla} f$ and $\Delta\vec{r}$. If we fix the length of $\Delta\vec{r}$, we'll find that Δf is maximized when $\theta = 0°$. Therefore, $|\vec{\nabla} f| = \max\left(\frac{\Delta f}{\Delta r}\right)$ for a fixed $|\Delta\vec{r}|$, and the direction of $\vec{\nabla} f$ is the same as the direction of $\Delta\vec{r}$ for which Δf is a maximum.

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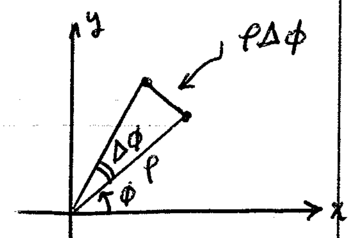
b) Cylindrical Coordinates: $f(\vec{r}) = f(\rho,\phi,z) \Rightarrow$



$$f(\vec{r}_0+\Delta\vec{r}) = f(\vec{r}_0) + \frac{\partial f}{\partial \rho}\Delta\rho + \frac{\partial f}{\partial \phi}\Delta\phi + \frac{\partial f}{\partial z}\Delta z$$

$$= f(\vec{r}_0) + \underbrace{\left(\frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\partial f}{\rho\,\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}\right)}_{\vec{\nabla} f}\cdot\underbrace{(\Delta\rho\,\hat{\rho} + \rho\Delta\phi\,\hat{\phi} + \Delta z\,\hat{z})}_{\Delta\vec{r}}$$

The rest of the argument is the same as in part (a).

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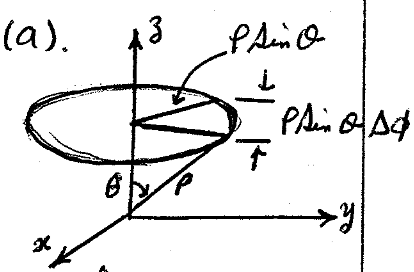
c) Spherical Coordinates: $f(\vec{r}) = f(\rho,\theta,\phi) \Rightarrow$



$$f(\vec{r}_0+\Delta\vec{r}) = f(\vec{r}_0) + \frac{\partial f}{\partial \rho}\Delta\rho + \frac{\partial f}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial \phi}\Delta\phi$$

$$= f(\vec{r}_0) + \underbrace{\left(\frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\partial f}{\rho\,\partial\theta}\hat{\theta} + \frac{\partial f}{\rho\sin\theta\,\partial\phi}\hat{\phi}\right)}_{\vec{\nabla} f}\cdot\underbrace{(\Delta\rho\,\hat{\rho} + \rho\Delta\theta\,\hat{\theta} + \rho\sin\theta\,\Delta\phi\,\hat{\phi})}_{\Delta\vec{r}}$$