Problem 21)

$$\frac{d}{dx} e^{-\pi x^2} = -2\pi x\, e^{-\pi x^2}$$

$$\frac{d^2}{dx^2} e^{-\pi x^2} = -2\pi e^{-\pi x^2} + 4\pi^2 x^2 e^{-\pi x^2} = (4\pi^2 x^2 - 2\pi) e^{-\pi x^2}$$

$$\frac{d^3}{dx^3} e^{-\pi x^2} = 8\pi^2 x e^{-\pi x^2} - 2\pi x (4\pi^2 x^2 - 2\pi) e^{-\pi x^2} = (12\pi^2 x - 8\pi^3 x^3) e^{-\pi x^2}$$

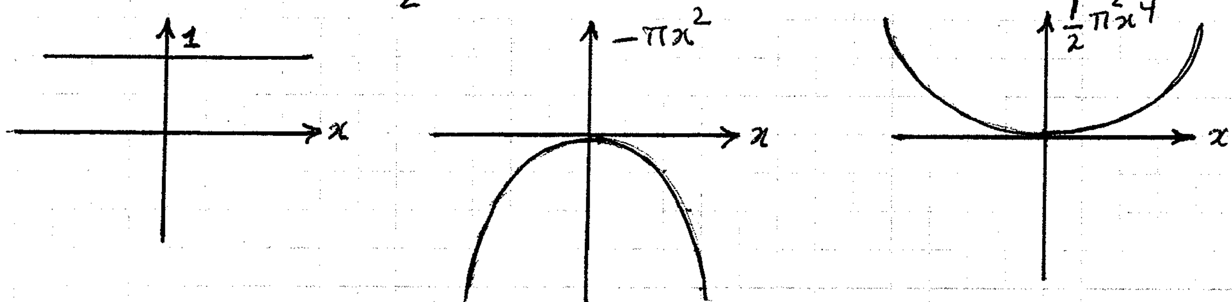
$$\frac{d^4}{dx^4} e^{-\pi x^2} = (12\pi^2 - 24\pi^3 x^2) e^{-\pi x^2} - 2\pi x (12\pi^2 x - 8\pi^3 x^3) e^{-\pi x^2}$$

$$= (12\pi^2 - 48\pi^3 x^2 + 16\pi^4 x^4) e^{-\pi x^2}$$

$$f(x) = f(x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f''''(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$ ← because $f'(x_0)=0$ and $f'''(x_0)=0$.

Therefore,

$$e^{-\pi x^2} = 1 - \pi x^2 + \tfrac{1}{2}\pi^2 x^4 + \cdots$$



When these functions are added together, at first they approximate $\exp(-\pi x^2)$ in the vicinity of $x=0$. Successive terms will have less and less influence in the neighborhood of the origin, but they make corrections at larger and larger values of $|x|$.