Average and RMS Values of a Periodic Waveform: (Nofziger, 2008)

Begin by defining the average value of any time-varying function over a time interval $\Delta t \equiv t_2 - t_1$ as the integral of the function over this time interval, divided by Δt:

$$f_{avg} \equiv \frac{\int_{t_1}^{t_2} f(t)\,dt}{\Delta t} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)\,dt \tag{1.1}$$

Numerically, this is an extension of the basic definition of the average for a discrete variable, $\bar{x} \equiv \frac{\sum_{i=1}^{N} x_i}{N}$, applied to a continuously-varying function.

Graphically, this means that the area under the function (between times t_1 and t_2) is equivalent to the area of a rectangle of height f_{avg} multiplied by width $(t_2 - t_1)$:

$$f_{avg} \cdot (t_2 - t_1) = \int_{t_1}^{t_2} f(t)\,dt \tag{1.2}$$

- The average of a sine wave over one half-cycle:

Consider a sine wave of peak amplitude I_p and frequency f:

$$I(t) = I_p \sin(2\pi f \cdot t) = I_p \sin(\omega t) \tag{1.3}$$

The period of the wave, τ, is given as:

$$\tau \equiv \frac{1}{f} = \frac{2\pi}{\omega} \tag{1.4}$$

The average value of one half-cycle in time, is then calculated as:

$$I_{avg} = \frac{1}{(\tau/2)}\int_0^{\tau/2} I_p \sin(\omega t)dt \quad = \quad \frac{2I_p}{\tau}\int_0^{\tau/2} \sin(\omega t)dt \tag{1.5}$$

$$I_{avg} = \left(\frac{2I_p}{\tau}\right)\left[\frac{-\cos(\omega t)}{\omega}\right]\Bigg|_0^{\tau/2} = \left(\frac{2I_p}{\omega\tau}\right)\left[-\cos\left(\frac{\omega\tau}{2}\right) - (-)\cos(0)\right]$$

$$= \left(\frac{2I_p}{\omega\tau}\right)\left[-\cos(\pi) + 1\right] = \left(\frac{4I_p}{\omega\tau}\right) = \left(\frac{4I_p}{2\pi}\right) = \frac{2I_p}{\pi} \tag{1.6}$$

$$\boxed{I_{avg} = \frac{2I_p}{\pi} = (.637)\,I_p} \text{ calculated over one half-cycle of the sine wave} \tag{1.7}$$

- <u>The average of a sine wave over one full-cycle</u>:

$$I_{avg} = \frac{1}{(\tau)} \int_0^{\tau} I_p \sin(\omega t) dt \qquad (1.8)$$

$$I_{avg} = \left(\frac{I_p}{\tau}\right)\left[\frac{-\cos(\omega t)}{\omega}\right]\Big|_0^{\tau} = \left(\frac{I_p}{\omega\tau}\right)\left[-\cos(\omega\tau) - (-)\cos(0)\right]$$

$$= \left(\frac{I_p}{\omega\tau}\right)\left[-\cos(2\pi) + 1\right] = \left(\frac{I_p}{\omega\tau}\right)\left[-1+1\right] = 0$$

$$\boxed{I_{avg} = 0} \quad \text{calculated over one full-cycle of the sine wave} \qquad (1.9)$$

While this result is mathematically correct, it doesn't provide any physical insight into what this sine wave can "accomplish" over one complete cycle. Consider the AC sinusoidal voltage delivered by Tucson Electric Power at a wall outlet. Its average value may be 0, yet we know from experience that this sine wave will light up fluorescent bulbs, heat up tungsten filaments in light bulbs, heat wires in toasters, etc. This is because these devices absorb energy (power) from the sine wave, whether the voltage is positive-going or negative-going. <u>Useful work is done during both half-cycles of the sinusoidal waveform</u>.

The problem, mathematically, is that we have added up negative as well as positive values. To account for this and make sure that all parts of the waveform contribute to its calculated effective value, first square the function $f(t)$ so that all values are positive. Then add up (integrate) all values of $f(t)$ over one complete cycle, and finally take the square root. This is known as the "Root Mean Square" or RMS value of any time-varying (or spatially-varying) waveform, and is defined as:

$$Y_{RMS} \equiv \sqrt{\left\{f^2(t)\right\}_{AVG}} = \sqrt{\frac{1}{\tau}\int_0^{\tau} f^2(t)dt} \qquad (1.10)$$

The physical meaning of the RMS value is this—it is the constant, or "DC" value that would cause the same physical effect as the actual time-varying waveform does, during one complete period. This might be to deliver the same power in a circuit, to cause the same heating effect in a toaster, to light up a bulb with the same brightness, etc.

Note that, in general, a periodic waveform may not have symmetry like sinusoidal or triangular waveforms have. In this case, you still calculate the RMS value according to equation (1.10), by integrating over one complete cycle. Modern-day instrumentation (the DMMs and oscilloscopes in our lab) digitally sample a waveform, and numerically integrate the values to calculate the RMS value, according to equation (1.10).

● <u>RMS Value of a Sinusoidal Waveform</u>:

Consider a sinusoidal voltage of peak amplitude V_p and frequency f:

$$V(t) = V_p \sin(2\pi f \cdot t) = V_p \sin(\omega t) \quad (1.11)$$

The RMS value is calculated as:

$$V_{RMS} = \sqrt{\frac{1}{\tau}\int_0^\tau V^2(t)dt} = \sqrt{\frac{1}{\tau}\int_0^\tau V_p^{\ 2}\sin^2(\omega t)dt} \quad (1.12)$$

$$= \sqrt{\frac{V_p^{\ 2}}{\tau}\int_0^\tau \frac{1}{2}(1-\cos 2\omega t)dt} = \sqrt{\frac{V_p^{\ 2}}{2\tau}\left[\int_0^\tau dt - \int_0^\tau \cos(2\omega t)dt\right]} \quad (1.13)$$

$$= \sqrt{\frac{V_p^{\ 2}}{2\tau}\left[(t)\Big|_0^\tau - \left(\frac{\sin(2\omega t)}{2\omega}\right)\Bigg|_0^\tau\right]} \quad (1.14)$$

$$= \sqrt{\frac{V_p^{\ 2}}{2\tau}\left[\tau - \left(\frac{\sin(2\omega\tau)}{2\omega} - 0\right)\right]} = \sqrt{\frac{V_p^{\ 2}}{2\tau}\left[\tau - \left(\frac{\sin(2\cdot 2\pi)}{2\omega} - 0\right)\right]} \quad (1.15)$$

$$= \sqrt{\frac{V_p^{\ 2}}{2\tau}[\tau - 0 + 0]} = \sqrt{\frac{V_p^{\ 2}}{2}} \quad (1.16)$$

$$\boxed{V_{RMS} = \frac{V_p}{\sqrt{2}} = (.707)V_p} \quad \text{for a sinusoidal waveform} \quad (1.17)$$

(Note that you get the same result whether you integrate from $(0 - \tau)$, $(0 - \tau/2)$, or even from $(0 - \tau/4)$, because of the symmetry of this waveform. For non-symmetrical waveforms, you have to integrate over the complete cycle.)

The quoted value of 120VAC for the voltage at the wall socket, as delivered by Tucson Electric Power is, in fact, the RMS value of the (60Hz) sinusoidal voltage. The peak voltage is therefore $V_p = \sqrt{2}\cdot 120 = 169$ V, and the peak-to-peak value is 339V!!!

- <u>RMS Value of a Triangular Waveform</u>:

Consider a triangular voltage waveform that is bi-polar, has a 50% duty-cycle (symmetrical about V=0), and a frequency f. Let the maximum voltage be V_p, and the minimum voltage be $-V_p$.

Because of the symmetry, integrate from $(0 - \tau/4)$ to calculate the RMS value:

$$V(t) = \frac{4V_p}{\tau} t \qquad \text{(valid between } t = 0 \text{ and } t = \tau/4\text{)} \tag{1.18}$$

$$V_{RMS} = \sqrt{V^2{}_{AVG}} = \sqrt{\frac{1}{\tau/4 - 0} \int_0^{\tau/4} V^2(t)dt} \tag{1.19}$$

$$= \sqrt{\frac{4}{\tau} \int_0^{\tau/4} \left(\frac{4V_p}{\tau}\right)^2 t^2 dt} = \sqrt{\frac{4}{\tau} \left(\frac{4V_p}{\tau}\right)^2 \cdot \int_0^{\tau/4} t^2 dt} \tag{1.20}$$

$$= \sqrt{\frac{64V_p{}^2}{\tau^3} \left[\frac{t^3}{3} \bigg|_0^{\tau/4} \right]} = \sqrt{\frac{64V_p{}^2}{\tau^3} \left(\frac{\tau^3}{3 \cdot 4^3} \right)} = \sqrt{\frac{V_p{}^2}{3}} \tag{1.21}$$

$$\boxed{V_{RMS} = \frac{V_p}{\sqrt{3}} = (.577)V_p} \qquad \text{for a triangular waveform} \tag{1.22}$$

- <u>RMS Value of a Rectangular Waveform</u>:

Consider a rectangular voltage waveform that is bi-polar, has a 50% duty-cycle (symmetrical about V=0), and a frequency f. Let the maximum voltage be V_p, and the minimum voltage be $-V_p$.

Because of the symmetry, integrate from $(0 - \tau/2)$ to calculate the RMS value:

$$V(t) = V_p \qquad \text{(valid between } t=0 \text{ and } t=\tau/2\text{)} \qquad (1.23)$$

$$V_{RMS} = \sqrt{V^2{}_{AVG}} = \sqrt{\frac{1}{\tau/2}\int_0^{\tau/2} V^2(t)dt} = \sqrt{\frac{2}{\tau}\int_0^{\tau/2}\left(V_p\right)^2 dt} \qquad (1.24)$$

$$= \sqrt{\frac{2}{\tau}\left(V_p\right)^2\int_0^{\tau/2} dt} = \sqrt{\frac{2}{\tau}\left(V_p\right)^2 t\Big|_0^{\tau/2}} = \sqrt{\frac{2}{\tau}\left(V_p\right)^2\left(\tau/2-0\right)} \qquad (1.25)$$

$$\boxed{V_{RMS} = V_p} \qquad \text{for a rectangular waveform} \qquad (1.26)$$