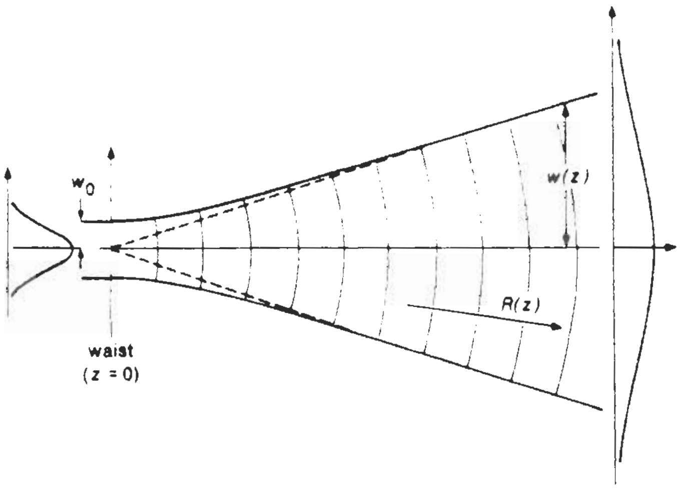
Gaussian Beams

<u>Reference</u>: A. Siegman, <u>LASERS</u>, Ch 17.



Lowest-order gaussian beam field amplitude (normalized):

$$\tilde{u}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{\tilde{q}_0}{w_0\,\tilde{q}(z)}\exp\left\{-jkz - jk\frac{x^2+y^2}{2\tilde{q}(z)}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\underbrace{\frac{\exp\{-jkz + j\psi(z)\}}{w(z)}}\,\exp\left\{\underbrace{-\frac{x^2+y^2}{w^2(z)}}\;\underbrace{-jk\frac{x^2+y^2}{2R(z)}}\right\}$$

tilde (~) => complex quantity

$w(0) = w_0$ = beam waist

z_R = Rayleigh range

leading phase and amplitude terms

decaying transverse exponential

parabolic approximation to spherical wave

$$\frac{1}{\tilde{q}(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

free space:

$$\tilde{q}(z) = \tilde{q}_0 + z = z + jz_R$$

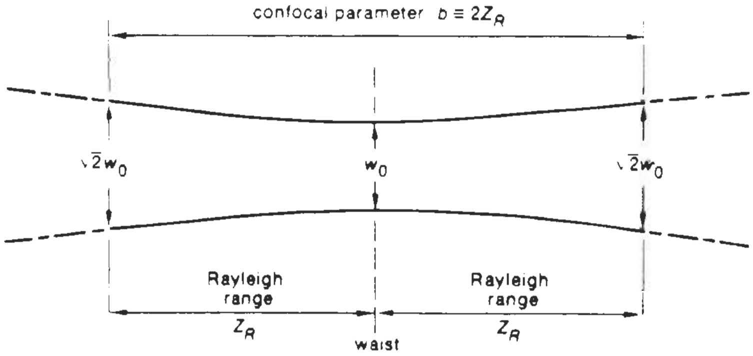
$$\tilde{q}_0 = j\frac{\pi w_0^2}{\lambda} = jz_R$$

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

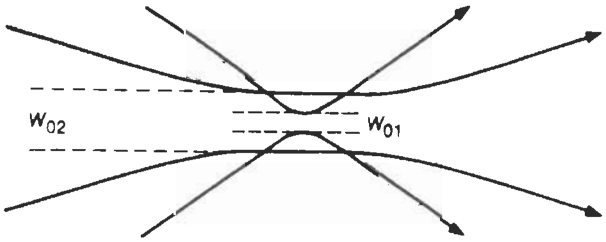
$$R(z) = z + z_R^2/z$$

$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$

Collimated region of beam around waist given by Rayleigh range, where beam radius increases by $\sqrt{2}$, thus area increases by 2.



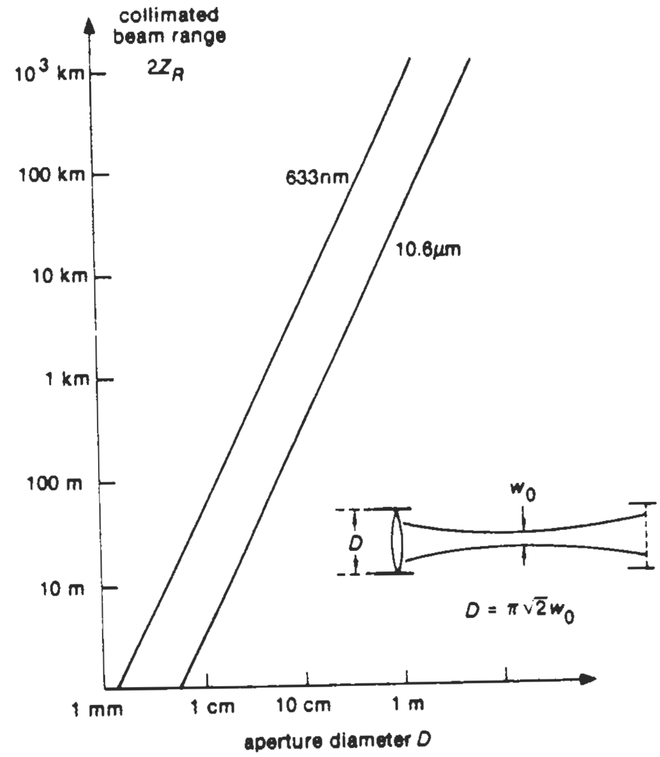
Beams with smaller waist sizes spread more rapidly.



The collimated range is given by $2z_R = \dfrac{2\pi w_0^2}{\lambda}$

Gaussian Beams - Beam Collimation

Assume we have an aperture given by $D = \pi \times \sqrt{2}\, w_0$ at one end of the Rayleigh range. The reason that we choose D this size is to limit effects of diffraction from the aperture. The collimated range, $2z_R$, is given versus the aperture diameter, D, in the following figure and table for two laser wavelengths.

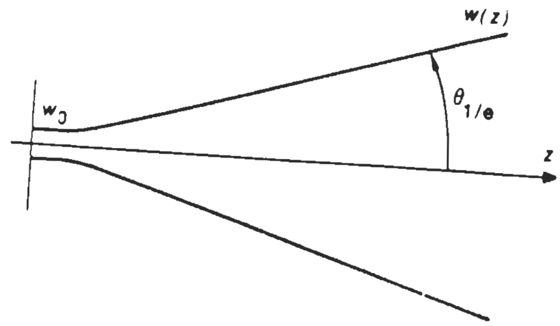


Collimated Laser Beam Ranges

Aperture diameter D	Waist spot size w_0	Collimated range, $2z_r$ (10.6 μm)	Collimated range, $2r_R$ (633 nm)
1 cm	2.25 mm	3 m	45 m
10 cm	2.25 cm	300 m	5 km
1 m	22.5 cm	30 km	500 km

Gaussian Beams - Far Field Angle

We use the $1/e^2$ irradiance beam diameter to define the far-field angle. Note that this is the $1/e$ point in field amplitude.



$$\theta_{1/e^2} = \lim_{z\to\infty} \frac{w(z)}{z} = \frac{\lambda}{\pi w_0}$$

The solid angle containing this angular spread will contain 86% of the total power in the far field.

$$\Omega_{1/e^2} = \pi\,\theta^2_{1/e^2} = \frac{\lambda^2}{\pi w_0^2}$$

If we define the effective area at the beam waist as

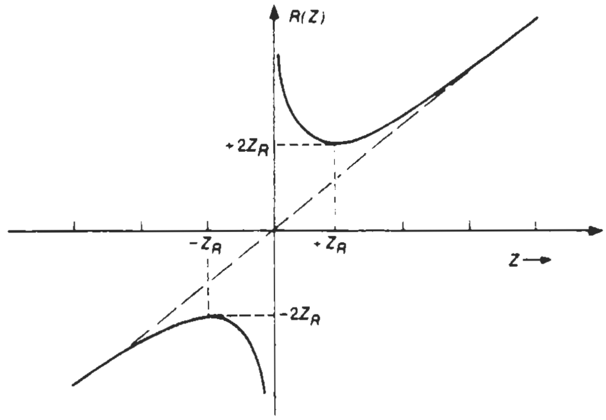
$$A_{1/e^2} = \pi w_0^2 \quad ,$$

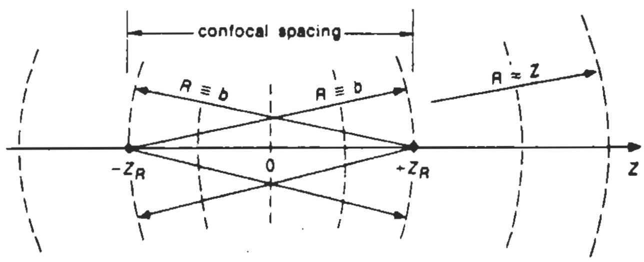
the product of A_{1/e^2} and Ω_{1/e^2} is

$$A_{1/e^2}\,\Omega_{1/e^2} = (\pi w_0^2)\left(\frac{\lambda^2}{\pi w_0^2}\right) = \lambda^2 \quad .$$

Gaussian Beams - Wavefront Radius of Curvature

$$R(z) = z + \frac{z_R^2}{z} \approx \begin{cases} \infty & \text{for } z \ll z_R \\ 2z_R & \text{for } z = z_R \\ z & \text{for } z \gg z_R \end{cases}$$





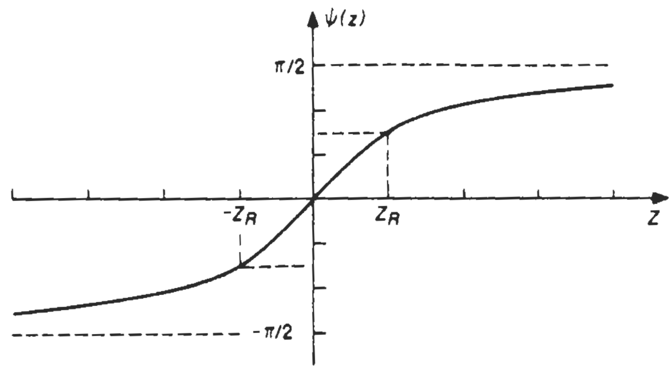
Minimum curvature at $z = z_R$, where center of curvature is at opposite Rayleigh range.

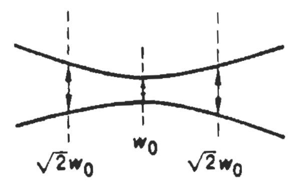
$$R_{z_R} = 2z_R \equiv b$$

Gaussian Beams - Axial Phase Shift

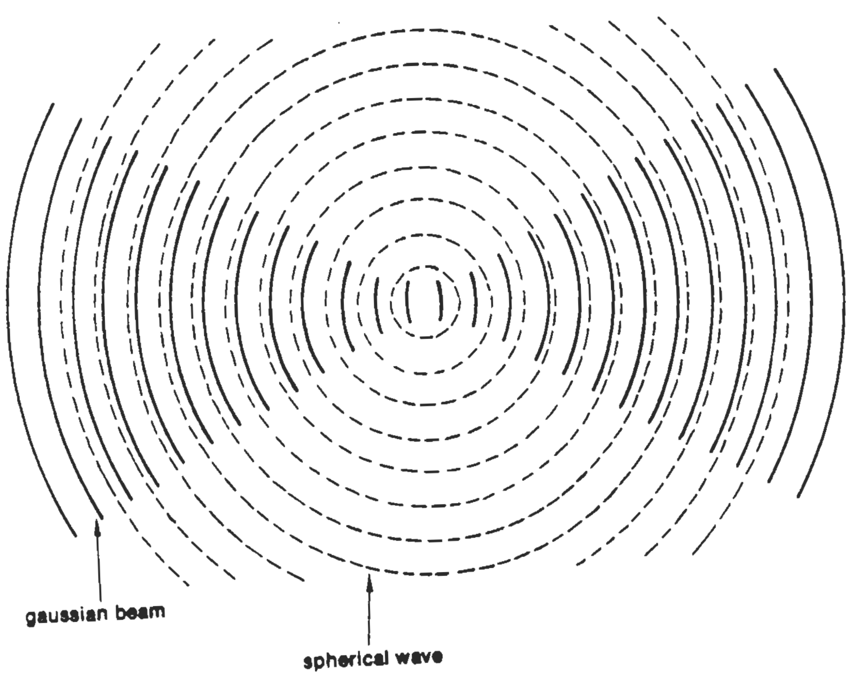
For the lowest-order gaussian mode,

$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right).$$





Note the ±90° phase on either side of focus and the sign change at focus. The sign change is called the Guoy Effect, and it is valid for any kind of optical beam passing through focus.

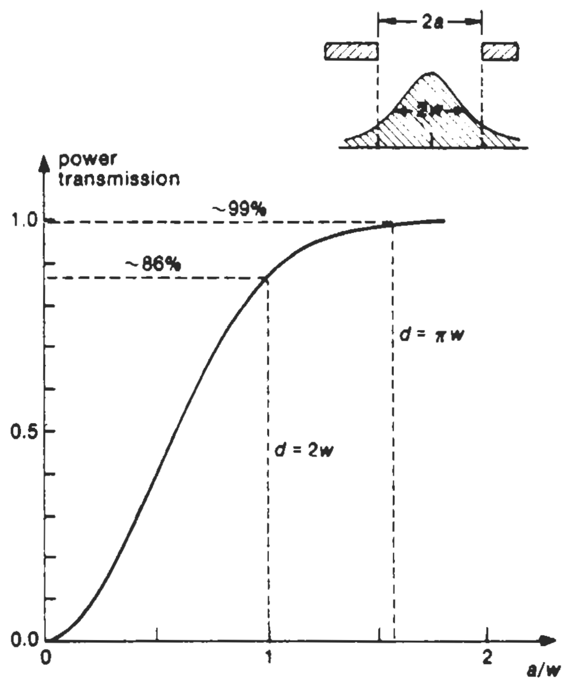


Gaussian Beams - Aperture Transmission

Power transmission through an aperture of radius a is given by

$$\text{power transmission} = \frac{2}{\pi w^2} \int_0^a 2\pi r e^{-2r^2/w^2} dr$$

$$= 1 - e^{-2a^2/w^2}$$

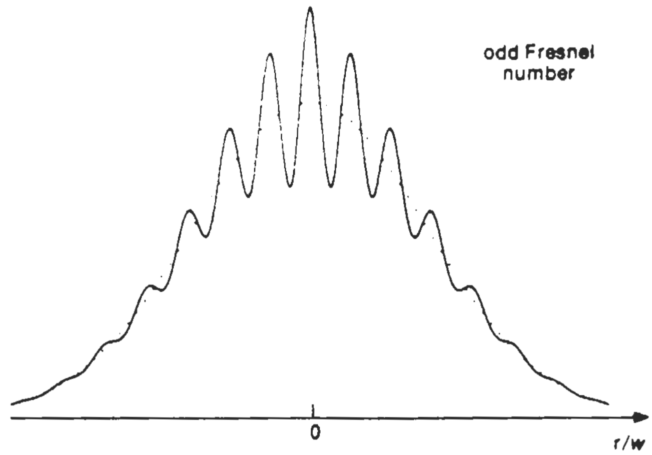


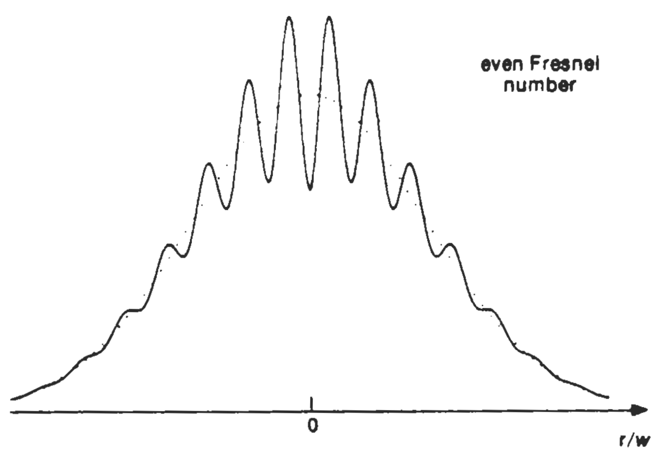
Note than an aperture of width $a = w$ transmits $\approx 86\%$ of the total power in the gaussian beam.

If $a = \frac{\pi w}{2}$, the aperture passes just over 99% of the total power in the gaussian beam.

Gaussian Beams - Diffraction Effects

When a gaussian beam is passed through an aperture, propagation in the Fresnel region will cause diffraction ringing. The amount of ringing will depend on the overfill of the aperture.





We refer to the ringing as "ripple."

$$\frac{\Delta I}{I} \times 100\% = \text{ripple}.$$

We must use an aperture diameter of $d \approx 4.6w$ to get down to $\pm 1\%$ diffraction ripple.

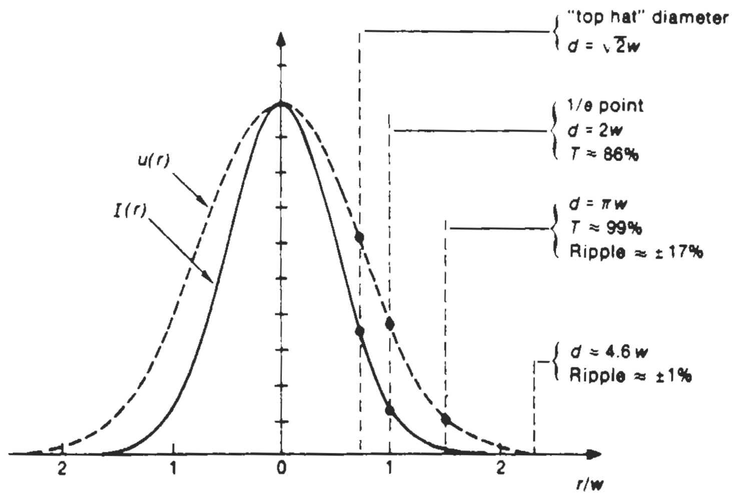
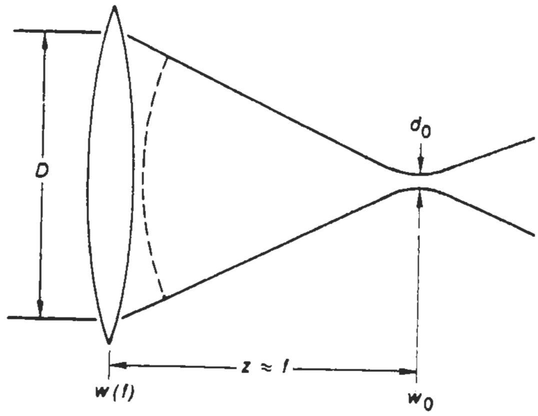


FIGURE 17.4
Significant diameters for hard-edged truncation of a cylindrical gaussian beam. Note that the $d = \pi w$ criterion gives 99% power transmission, but also $\pm 17\%$ intensity ripples and intensity reduction in the near and far fields.

Gaussian Beams - Focusing



If $D = \pi\, \omega(f)$ (99% power through)

and we define $d_o = 2\omega_o$,

$$d_o \approx \frac{2 f \lambda}{D} = \boxed{2\lambda f\#} \quad \text{at the focused spot.}$$

The <u>depth of focus</u> is given by $2z_R$, or

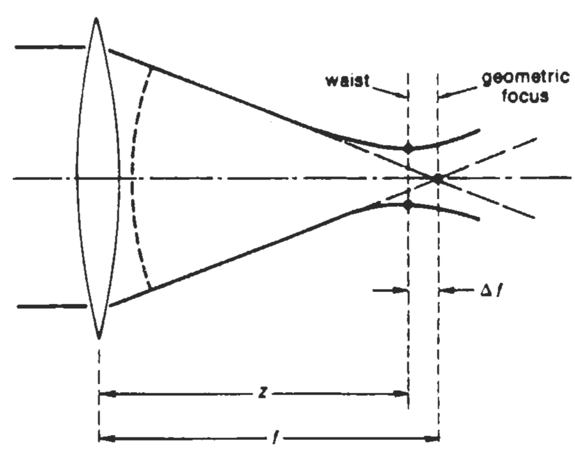
$$\text{depth of focus} = 2z_R \approx \boxed{2\pi \lambda (f^{\#})^2}$$

There is a small error when using thin lenses to focus gaussian beams. If, after the lens, the beam aquires a wave-front radius of curvature given by f, the difference is

$$\Delta f \equiv f - z = z_R^2 / z$$

$$\approx \boxed{\frac{z_R^2}{f}}$$

Since f is usually >> z_R, Δf is usually insignificant.



Hermite - Gaussians :

$$n = 0, 1, \cdots$$

$$\tilde{u}_n(x,z) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{e^{j(2n+1)\psi(z)}}{2^n n!\, w(z)}\right)^{1/2}$$

$$\times\; H_n\left(\frac{\sqrt{2}\, x}{w(z)}\right) \exp\left\{-jkz - \frac{jkx^2}{2R(z)} - \frac{x^2}{w^2(z)}\right\}$$

$H_n(x)$ are the Hermite polynomials given by

$$H_0(x) = 1 \qquad H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x$$

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x) \quad .$$

The net Guoy phase shift is $(n+\frac{1}{2})\,\psi(z)$ in traveling from the waist to any other plane z.

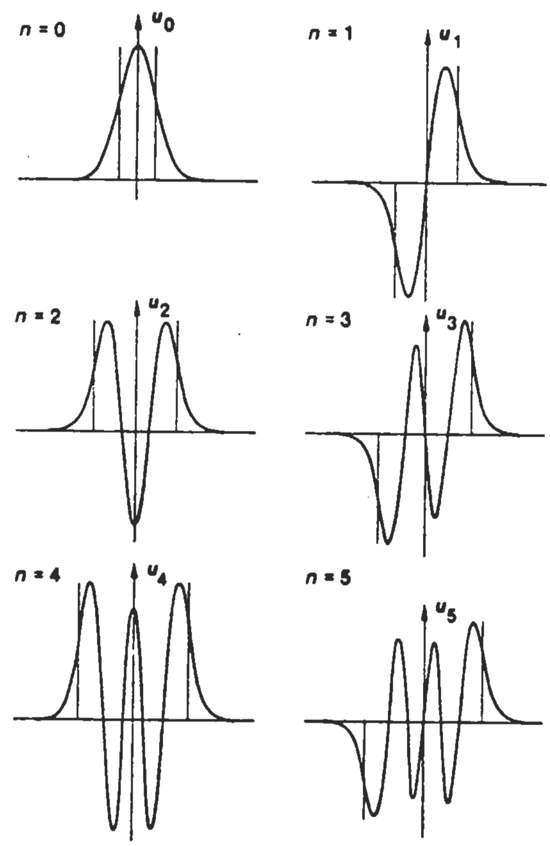


FIGURE 17.18
Amplitude profiles for low-order Hermite-gaussian modes.

Gaussian Beams - Hermite Functions

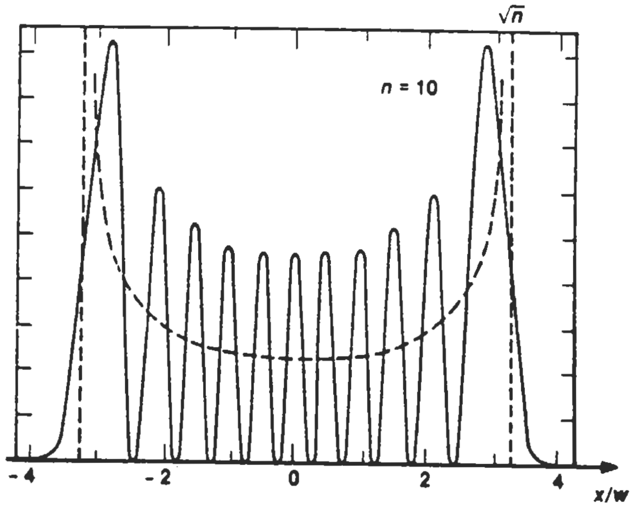


FIGURE 17.19
Intensity profile for the Hermite-gaussian mode pattern with $n = 10$.

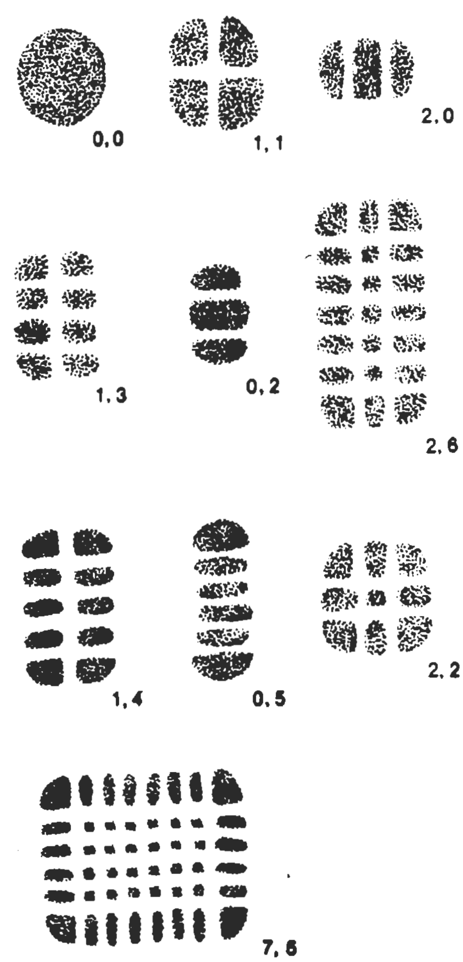


FIGURE 17.20
Transverse mode patterns for Hermite-gaussian modes of various orders.

Laguerre- Gaussian Modes

Laguerre- Gaussian Modes are written in cylindrical rather than rectangular coordinates.

$$\tilde{u}_{pm}(r,\theta,z) = \sqrt{\frac{2p!}{(1+\delta_{0m})\pi(m+p)!}}\;\frac{\exp\{j(2p+m+1)(\psi(z)-\psi_0)\}}{w(z)}$$

$$\times \left(\frac{\sqrt{2}\,r}{w(z)}\right)^m L_p^m\left(\frac{2r^2}{w^2(z)}\right)\exp\left\{-jk\frac{r^2}{2\tilde{q}(z)} + im\theta\right\}$$

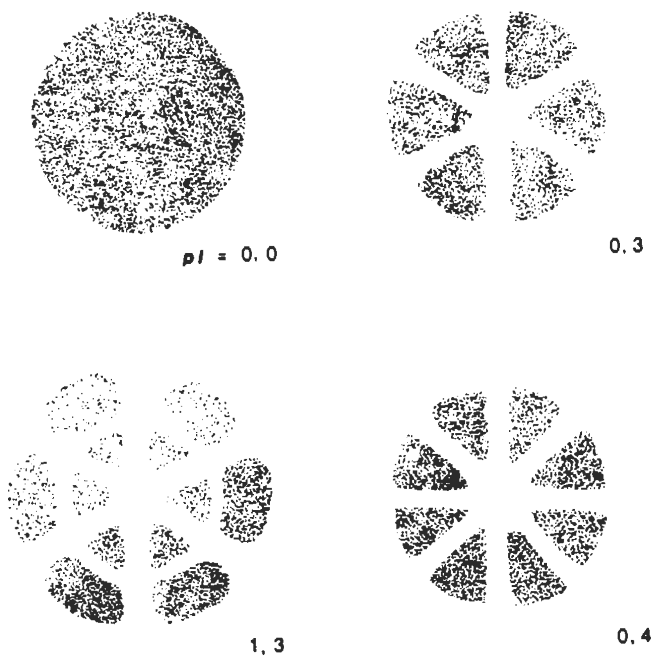


FIGURE 17.21
Transverse mode patterns for Laguerre-gaussian modes of various orders.

Comparison of Bessel and Gaussian beams

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A comparison of beam divergence and power-transport efficiency is made between Gaussian and Bessel beams when both beams have the same initial total power and the same initial full width at half-maximum.

Recently we reported some experimental observations[1] that verified the theoretical predictions by Durnin[2] on the depth of field of Bessel modes. Some of the features of these modes are strikingly different from corresponding features of the well-known Gaussian modes of the paraxial wave equation. For example, in their ideal form these modes represent perfectly diffraction-free waves with a sharply defined central maximum. Here diffraction-free means that the wave amplitude $\psi(x, y, z, t)$ has the property

$$\psi(x, y, z, t) = \exp[-i(\omega t - \beta z)]\psi(x, y, z = 0, t = 0), \tag{1}$$

so that the time-averaged wave intensity satisfies

$$I(x, y, z) = |\psi(x, y, z, t)|^2 = |\psi(x, y, z = 0, t = 0)|^2 \tag{2}$$

That is, according to Eq. (1) the wave travels in the z direction, and according to Eq. (2) *the transverse intensity distribution is independent of the propagation distance z.*

The simplest diffraction-free beam solution is[2]

$$\psi(x, y, z = 0, t = 0) = J_0(\alpha\rho), \tag{3}$$

where J_0 is the zero-order Bessel function of the first kind, $\rho = \sqrt{(x^2 + y^2)}$, α and β are real constants that satisfy $\alpha^2 + \beta^2 = (\omega/c)^2$, and c is the speed of light. The effective width of the beam is proportional to $1/\alpha$ and can be extremely small, of the order of one wavelength (FWHM). We have shown[1] that finite aperture approximations to a Bessel beam are easily realizable in the laboratory by using only simple optical elements.

Although Bessel beams are diffraction-free in the sense defined above, they are not so well localized as Gaussian beams. Each lobe (i.e., the area between successive zeros) of a Bessel beam carries approximately the same energy as the central spot, and thus a Bessel beam with 20 lobes, for example, contains only about 5% of the total energy within the central spot. This property of Bessel beams appears strongly disadvantageous when questions of power transport are considered. In comparison with Gaussian beams, which contain 50% of their total energy within their FWHM, it appears that a Bessel beam wastes most of its energy off axis.

The purpose of this Letter is to explain that this conclusion, although apparently obvious, is not at all correct. We present order-of-magnitude calculations that compare the properties of Bessel and Gaussian beams with respect to both beam divergence and power transport efficiency. The comparisons are based on observations made at a plane $z = d$ given the following two initial constraints on field distribution in the plane $z = 0$: same initial total power and same initial FWHM.

Consider an aperture of radius R at $z = 0$ and an observation plane at $z = d$ having a circle of radius a upon it, as shown in Fig. 1. We shall assume that the field distributions for the Gaussian and Bessel beams in the plane $z = 0$ are of the form

$$E(x, y, z = 0) = \xi_G \exp(-\rho^2/2\sigma^2) \quad \text{Gaussian}, \tag{4}$$

$$E(x, y, z = 0) = \xi_B J_0(\alpha\rho) \quad \text{Bessel} \tag{5}$$

for all $\rho < R$, and zero for all $\rho > R$. The total initial power contained in each beam is given by

$$P_T = \left(\frac{c}{2\pi}\right)\frac{\pi}{2}\xi_G{}^2\sigma^2 \quad \text{Gaussian}, \tag{6}$$

$$P_T = \left(\frac{c}{2\pi}\right)\xi_B{}^2\frac{R}{\alpha} \quad \text{Bessel}, \tag{7}$$

assuming that $R \gg \sigma, 1/\alpha$ (i.e., the aperture radius R is large compared with the central spot sizes of the two beams).

We now pose the following question: Given the wavelength λ, the total initial power P_T, and a propagation distance d, what is the maximum possible fraction F of total initial power P_T that can be transported to the disk of radius a? The case of greatest practical interest, and that which we shall consider, is when $d \gg a > \lambda$. The parameters to be optimized are ξ_G and σ for the Gaussian beam and ξ_B and R for the Bessel beam.

Because of the finite aperture radius R, the Bessel beam will have a propagation range given by[2]

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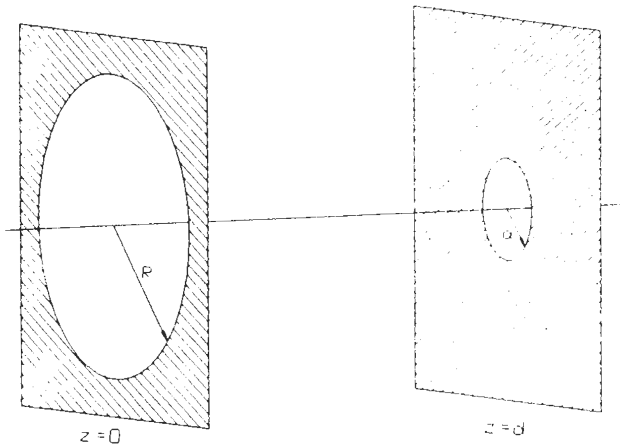

Fig. 1. Aperture geometry. Both beams are assumed to be confined to a finite aperture of radius R in the plane $z = 0$. A disk of radius a located in the plane $z = d$ defines the area over which the beam intensity is to be integrated.

$$Z_{\max} = R[(\kappa/\alpha)^2 - 1]^{+1/2}, \tag{8}$$

provided that $\kappa > \alpha > 2\pi/R$, where $\kappa = 2\pi/\lambda$ is the wave number. Over that distance $Z_{\max}$ the central spot radius of the Bessel beam will remain fixed, while the peak intensity oscillates slightly about its initial value. We shall now choose the central spot radius of the Bessel beam to equal the disk radius a so that the fraction of total power contained within that disk at the plane $z = d$ is approximately the same as in the initial plane, namely,

$$F \approx aR \approx 1/(1 + 4N/3), \tag{9}$$

where N is the number of rings in the Bessel function within the aperture radius R. In terms of a, the maximum propagation range [Eq. (8)] of the Bessel function is $Z_{\max} \approx \pi Ra/\lambda$, and we therefore find that if we choose the aperture radius R to be

$$R = \lambda d/\pi a, \tag{10}$$

the fraction F of total initial power P_T that is contained within the disk of radius a becomes

$$F \approx \pi a^2/\lambda d. \tag{11}$$

This also determines the Bessel beam amplitude to be given by

$$\xi_B{}^2 = (4\pi/c)(P_T/Ra). \tag{12}$$

In the case of the Gaussian beam, one can easily show that both the peak intensity at $z = d$ and the fractional power contained within a disk of radius a located at $z = d$ will be maximized by choosing

$$\sigma^2 = d/\kappa, \tag{13}$$

$$\xi_G{}^2 = (8\pi/c)(P_T/\lambda d),$$

regardless of the value of a. The maximum fraction total initial power contained within the disk of radi a at $z = d$ is then found to be

$$F = 1 - \exp(-\pi a^2/\lambda d) \approx \pi(a^2/\lambda d).$$

Thus, just as in the case of the Bessel beam, we fi that fractional power incident upon the disk is appro imately π times the Fresnel number of the disk, pr vided that the Fresnel number is much less than (When the Fresnel number is comparable with greater than 1 the effects of diffraction are sufficient negligible that a plane wave of radius a suffices f power transport.) With regard to beam divergenc the Gaussian beam spreads by a factor of $\sqrt{2}$, while t central maximum of the Bessel beam does not spre at all.

Let us now consider the case in which the Bessel a Gaussian beams are assumed to have the same FWH in the plane $z = 0$, i.e., $\sigma^2 \approx a^2$. If we define the ran of the Gaussian beam as the distance z at which peak intensity falls by a factor of 2, we find that

$$\text{Bessel range} \approx N \times \text{Gaussian range},$$

where N is again the total number of rings in t Bessel function in the plane $z = 0$. However, since t total integrated power contained in each ring is a proximately the same as that contained in the centr maximum of the Bessel beam, we also find that

$$\text{total Bessel power} \approx N \times \text{total Gaussian power}.$$

Increased depth of field is therefore obtained at t expense of power.

In conclusion, we have shown that (1) comparab efficiencies in power transport are obtained with op mized Gaussian and Bessel beams in practical situ tions (i.e., when the Fresnel number of the target much smaller than unity) and (2) the depth of field o Bessel beam can be made arbitrarily larger than th of a Gaussian beam having the same spot si (FWHM), but at the expense of power. The appar advantage of greater localization in Gaussian bea relative to that found in Bessel beams, is theref nonexistent.

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1. J. Durnin, J. J. Miceli, Jr., and J. H. Eberly, Phys. R Lett. 58, 1499 (1987).
2. J. Durnin, J. Opt. Soc. Am. A 4, 651 (1987).