Fast Phase Retrieval: Unique and Stable Complex Object Recovery in O(NLogN) Time

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Abstract: We present a novel Phase Retrieval algorithm that is able to uniquely recover a complex object from its noise-corrupted Fourier magnitudes in an arbitrary number of dimensions while running in O(N Log N) time.

1. Introduction

Phase retrieval - recovering a complex object from the magnitude of its Fourier transform - is ubiquitous in computational imaging. [1]. Despite the importance of this problem, and myriad attempts at solving it, most algorithms are unable to recover arbitrary complex objects from only noise-corrupted Fourier magnitudes. Algorithms that can recover phase require either extensive prior information, can only reconstruct positive real objects, or require high oversampling ratios [2]. We close this major gap in the literature by presenting an efficient, noise-resilient, and stable Phase Retrieval algorithm requiring an oversampling ratio of only 2.

2. Algorithm Description

Fast Phase Retrieval requires very little prior information. The only input is the Fourier intensities, $\mathbf{y} = |\mathscr{F}\{\mathbf{x}\}|^2$. To achieve unique recovery, the length of $\mathbf{y}$ needs to be at least twice the length of the support of $\mathbf{x}$ in each dimension. Optionally, the values for $\mathbf{x}$ outside this support can be specified; by default they are set to 0.
Fast Phase Retrieval is based on computing the outer component [3] of the trigonometric polynomial representing $\mathbf{y}$. The outer component is always guaranteed to exist and is unique. We accomplish outer component calculation by first solving the following Second Order Cone Program (SOCP)

$$\begin{aligned} \min_{\mathbf{z},\mathbf{t}} \quad & \langle \mathbf{z}, \mathbf{y} \rangle \\ \text{s.t} \quad & \langle \mathbf{1}, \mathbf{t} \rangle \leq N \\ & (t_i, z_i, \sqrt{2}) \in Q_r^3, \quad i \in [1, N] \end{aligned} \tag{1}$$

Where, $\mathbf{y}$ is flattened to 1-dimension, $\mathbf{z}$ and $\mathbf{t}$ are decision variables, N is the total number of intensity samples, and Q_r^3 is the third order rotated Lorentzian Cone. This SOCP can be solved in linear time, as its Hessian consists of a diagonal matrix plus a rank 1 update. Commercial solvers such as MOSEK are able to take advantage of this structure to solve problem instances with sizes well in excess of a million (Fig. 1f).

Given the optimal value of the SOCP, d^*, the intial estimate for $\mathbf{x}$ can be extracted by unflattening $\mathbf{z}$ and applying:

$$\hat{\mathbf{x}}_0 = \mathscr{F}\{\mathbf{z}^{-1} \odot \mathscr{F}\{d^* \mathbf{e}\}\} \tag{2}$$

Where $\mathbf{e}$ is the n-dimensional vector with value 1 in the first entry, and zeros otherwise. $\odot$ represents the Hadamard product. Finally, we extract the maximum likelihood estimator $\hat{\mathbf{x}}_{ML}$ by running Newton's method (complexity O(N Log N)) initialized with $\hat{\mathbf{x}}_0$ on the following cost function

$$\hat{\mathbf{x}}_{ML} = \min_{\mathbf{x}} \left|\left| |\mathscr{F}\{\mathbf{x}\}|^2 - \mathbf{y} \right|\right|^2 \tag{3}$$

While this is a non-convex optimization, it can be shown that when initialized with $\hat{\mathbf{x}}_0$ from (2) the Hessian of (3) is positive definite.

Fast Phase Retrieval achieves unique recovery when the true $\mathbf{x}$ contains several reference pixels of significantly higher magnitude than the average pixel magnitude - this ensures an irreducible Z-Transform [1]. While the exact requirement is a complicated function of oversampling ratio and noise, for an oversampling ratio of 2 a few pixels with magnitudes 10-100 times the average pixel magnitude is sufficient to achieve recovery. This is in stark contrast to other methods relying on polynomial factorization that require unphysically large impulses and work only in one dimension [4].

3. Results

We present the results of a simulated diffraction pattern representing the pillars of creation from JWST(Credits: NASA, ESA, CSA, STScI; Joseph DePasquale, Anton M. Koekemoer, Alyssa Pagan) . The image is represented in HSL format, with Hue encoding phase, Light encoding magnitude, and Saturation being fixed. The Fourier magnitudes were corrupted with Gaussian noise resulting in a signal-to-noise ratio of 10dB.
With the memory requirements of PhaseLift [5] being too large, and alternating projections being unsuited to complex recovery with loose support [1] we tested Fast Phase Retrieval only against Wirtinger flow. We implemented Wirtinger flow using spectral initialization, backtracking line search for the learning parameter and included support constraints to optimize recovery chances [2]. However, even after over 100 restarts, and extensive parameter tuning, Wirtinger flow was able to achieve a relative error of only 10^{-4} yielding an image that appears as only static. The reason for the failure of Wirtinger flow was the presence of noise and the oversampling ratio being only 2. Fast Phase Retrieval meanwhile was able to achieve a relative error of 10^{-17}, and the recovered image remains high quality over a range of reference pixel magnitudes. The results are summarized in Figure 1.

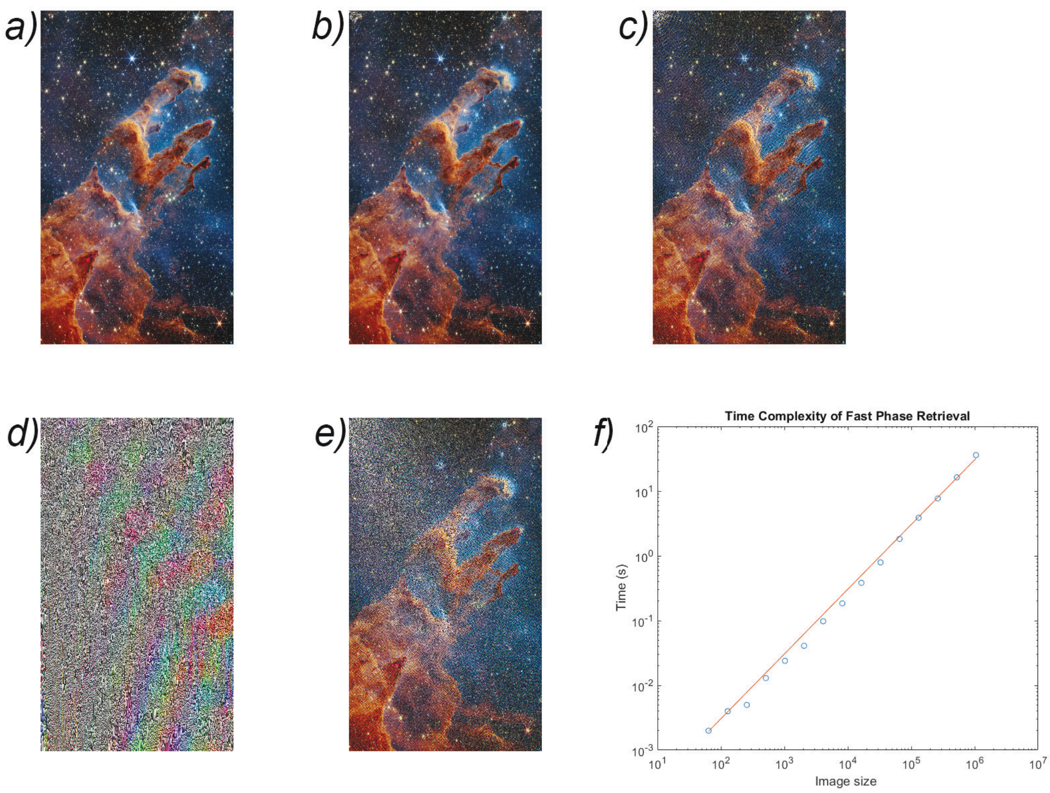

Fig. 1. (a-c): Fast Phase Retrieval results with reference pixels of magnitude 2^{20} (a), 2^{10} (b), and 2^{7} (c). (d): Best Wirtinger Flow Reconstruction. (e) Result of initializing eq. 3 with ground truth pixels. (f) The required running time for Fast Phase Retrieval as a function of the total number of pixels in an image.

References

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