Introduction to aberrations
OPTI 518
Lecture 12

\[ \bar{y}_O \cdot \vec{H} \]

\[ y_E \cdot (\vec{\rho} + \Delta \vec{\rho}) \]

\[ y'_E \cdot \vec{\rho} \]

\[ \bar{y}_I \cdot (\vec{H} + \Delta \vec{H}) \]
Topics

• Aspheric surfaces
• Aspheric contributions to the Seidel sums
• Stop shifting formulas
• Sine condition from flux conservation
Aspheric Surfaces

- Meaning not spherical
- Conic surfaces: Sphere, prolate ellipsoid, hyperboloid, paraboloid, oblate ellipsoid or spheroid
- Cartesian Ovals
- Polynomial surfaces
- Infinite possibilities for an aspheric surface
- Ray tracing for quadric surfaces uses closed formulas; for other surfaces iterative algorithms are used
Aspheric surfaces

The concept of the sag (or depth) of a surface at a given position $S$

$$Z(S) = \frac{cS^2}{1 + \sqrt{1 - (K+1)c^2S^2}} + A_2S^2 + A_4S^4 + A_6S^6 + A_8S^8 + A_{10}S^{10} + \ldots$$

$$K = -\varepsilon^2$$

$K$ is the conic constant
- $K=0$, sphere
- $K=-1$, parabola
- $K<-1$, hyperbola
- $-1<K<0$, prolate ellipsoid
- $K>0$, oblate ellipsoid

$C$ is $1/r$ where $r$ is the radius of curvature; $K$ is the conic constant (the eccentricity squared); $A$'s are aspheric coefficients
Conic surfaces focal properties

- Focal points for mirrors
- Focal points of lenses

Ellipsoid case
Hyperboloid case
Oblate ellipsoid

\[ K = -n^2 \]
Aspheric surface description

\[
Z(S) = \frac{cS^2}{1 + \sqrt[3]{1 - (K + 1)c^2S^2}} + A_2S^2 + A_4S^4 + A_6S^6 + A_8S^8 + A_{10}S^{10} + \ldots
\]

\[
Z_{aspheric} = \frac{1}{2r}(x^2 + y^2) + \frac{1}{8r}(1 + K)(x^2 + y^2)^2 + A_4(x^2 + y^2)^2 + \ldots
\]

\[S = \sqrt{x^2 + y^2}\]
Cartesian Ovals

\[ l \cdot n + l' \cdot n' = Cte. \]
The aspheric surface can be thought of as comprising a base sphere and an aspheric cap. The equation is:

\[ Z(S) = \text{Base sphere} + \text{Aspheric cap} \]
Freeform surfaces

- Freeform surfaces are aspheric surfaces often without rotational symmetry
- Freeform surfaces have degrees of freedom for adjustment
- Bezier curves and splines are some examples
Polynomials as freeform surfaces

\[ Z(S) = \frac{cS^2}{1 + \sqrt{1 - (K + 1)c^2 S^2}} + Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 \\
+ Hx^4 + Ix^3y + Jx^2y^2 + Kxy^3 + Ly^4 + ... \]

Superposition of conic and a polynomial on x and y

\[ Z(S) = \frac{cS^2}{1 + \sqrt{1 - (K + 1)c^2 S^2}} + Cy^2 + Ex^2y + Gy^3 + Hx^4 + Jx^2y^2 + Ly^4 + ... \]

Plane symmetric about Y axis
Basic plane-symmetric freeform surface

\[ Z(S) = \frac{cS^2}{1 + \sqrt{1 - (K + 1)c^2 S^2}} + \alpha (\tilde{i} \cdot \tilde{\rho})^2 + \beta (\tilde{\rho} \cdot \tilde{\rho})(\tilde{i} \cdot \tilde{\rho}) + \gamma (\tilde{\rho} \cdot \tilde{\rho})^2 \]

\[ Z(S) = \text{conic} + \text{astigmatic} + \text{comatic} + \text{spherical} \]
Aspheric surface contributions

\[ Z(S) = \frac{cS^2}{1 + \sqrt{1 - (K + 1)c^2 S^2}} + A_4S^4 \]

• When the stop is at the aspheric surface only spherical aberration is contributed given that all the beams see the same portion of the surface
• When the stop is away from the surface, different field beams pass through different parts of the aspheric surface and other aberrations are contributed
Aspheric contributions

\[
\delta W_{040} = \frac{1}{8} a \\
\delta W_{131} = \frac{1}{2} \frac{\bar{y}}{y} a \\
\delta W_{222} = \frac{1}{2} \left( \frac{\bar{y}}{y} \right)^2 a \\
\delta W_{220} = \frac{1}{4} \left( \frac{\bar{y}}{y} \right)^2 a \\
\delta W_{311} = \frac{1}{2} \left( \frac{\bar{y}}{y} \right)^3 a
\]

\[
\delta C_L = 0 \\
\delta C_T = 0 \\
a = -\varepsilon^2 c^3 y^4 \Delta n \\
a = 8 A_4 y^4 \Delta n
\]
Stop at aspheric surface

$$W_{cap} \left( \vec{H}, \vec{\rho} \right) = A_4 y^4 \Delta (n)(\vec{\rho} \cdot \vec{\rho})^2$$

$$OPL_1 = n \cdot (t + \Delta t)$$
$$OPL_2 = n \cdot t + \Delta t$$
$$OPD = (n - 1) \Delta t$$
Upon stop shifting

\[ W_{\text{cap}} \left( \tilde{H}, \tilde{\rho} \right) = A_4 y^4 \Delta(n) \left( \tilde{\rho}_{\text{shift}} \cdot \tilde{\rho}_{\text{shift}} \right)^2 \]
Stop shifting

\[ W_{cap}(\vec{H}, \vec{\rho}) = A_4 y^4 \Delta(n) \left( \vec{\rho}_{\text{shift}} \cdot \vec{\rho}_{\text{shift}} \right)^2 \]

\[ \vec{\rho}_{\text{shift}} = \vec{\rho} + \overline{S} \vec{H} \]

\[ \overline{S} = \frac{\overline{y}_{\text{new}} - \overline{y}_{\text{old}}}{\overline{y}} = \frac{\overline{y}_{\text{new}} - 0}{\overline{y}} = \frac{\overline{y}}{\overline{y}} \]
Expansion of \( \left( \vec{\rho}_{\text{shift}} \cdot \vec{\rho}_{\text{shift}} \right)^2 \)

\[
\left( \vec{\rho}_{\text{shift}} \cdot \vec{\rho}_{\text{shift}} \right)^2 = \left[ \vec{\rho} \cdot \vec{\rho} + 2\vec{S}\vec{H} \cdot \vec{\rho} + \left( \vec{S} \right)^2 \vec{H} \cdot \vec{H} \right] \times 
\[
\left[ \vec{\rho} \cdot \vec{\rho} + 2\vec{S}\vec{H} \cdot \vec{\rho} + \left( \vec{S} \right)^2 \vec{H} \cdot \vec{H} \right] =
\]
\[
= \left( \vec{\rho} \cdot \vec{\rho} \right)^2 + 4\vec{S} \left( \vec{H} \cdot \vec{\rho} \right) \left( \vec{\rho} \cdot \vec{\rho} \right) + 4\vec{S}^2 \left( \vec{H} \cdot \vec{\rho} \right)^2 
+ 2\vec{S}^2 \left( \vec{H} \cdot \vec{H} \right) \left( \vec{\rho} \cdot \vec{\rho} \right) + 4\vec{S}^3 \left( \vec{H} \cdot \vec{H} \right) \left( \vec{H} \cdot \vec{\rho} \right) + \vec{S}^4 \left( \vec{H} \cdot \vec{H} \right)^2
\]
Aberration function upon stop shifting for an aspheric cap

\[ W_{cap}(\bar{H}, \bar{p}) = A_4 y^4 \Delta(n) (\bar{p}_{shift} \cdot \bar{p}_{shift})^2 \]

\[ = A_4 y^4 \Delta(n) (\bar{p} \cdot \bar{p})^2 + 4 A_4 y^4 \Delta(n) \bar{S} (\bar{H} \cdot \bar{p}) (\bar{p} \cdot \bar{p}) \]

\[ + 4 A_4 y^4 \Delta(n) \bar{S}^2 (\bar{H} \cdot \bar{p})^2 + 2 A_4 y^4 \Delta(n) \bar{S}^2 (\bar{H} \cdot \bar{H}) (\bar{p} \cdot \bar{p}) \]

\[ + 4 A_4 y^4 \Delta(n) \bar{S}^3 (\bar{H} \cdot \bar{H}) (\bar{H} \cdot \bar{p}) + A_4 y^4 \Delta(n) \bar{S}^4 (\bar{H} \cdot \bar{H})^2 \]

\[ = \frac{1}{8} a (\bar{p} \cdot \bar{p})^2 + \frac{1}{2} a \bar{S} (\bar{H} \cdot \bar{p}) (\bar{p} \cdot \bar{p}) + \frac{1}{2} a \bar{S}^2 (\bar{H} \cdot \bar{p})^2 \]

\[ + \frac{1}{4} a \bar{S}^2 (\bar{H} \cdot \bar{H}) (\bar{p} \cdot \bar{p}) + \frac{1}{2} a \bar{S}^3 (\bar{H} \cdot \bar{H}) (\bar{H} \cdot \bar{p}) + \frac{1}{8} a \bar{S}^4 (\bar{H} \cdot \bar{H})^2 \]

\[ a = 8 A_4 y^4 \Delta(n) \]
Aspheric contributions

\[ \delta S_1 = a \]

\[ \delta S_\Pi = \left( \frac{\bar{y}}{y} \right) a \]

\[ \delta S_\III = \left( \frac{\bar{y}}{y} \right)^2 a \]

\[ \delta S_\IV = 0 \]

\[ \delta S_\V = \left( \frac{\bar{y}}{y} \right)^3 a \]

\[ \delta S_\VI = \left( \frac{\bar{y}}{y} \right)^4 a \]

\[ \delta C_L = 0 \]

\[ \delta C_T = 0 \]

\[ a = -\varepsilon^2 c^3 y^4 \Delta(n) \]

For a conic surface of eccentricity \( \varepsilon \)

\[ a = 8A_4 y^4 \Delta(n) \]

For an aspheric surface with fourth order coefficient \( A_4 \)
Aspheric cap contributions
Aberration function of base sphere and aspheric cap

\[ W(\vec{H}, \vec{\rho}) = W_{spherical}(\vec{H}, \vec{\rho}) + W_{cap}(\vec{H}, \vec{\rho}) \]

Case of an aspherical surface with the stop at the surface:

\[ W_{040}(\vec{\rho} \cdot \vec{\rho})^2 = -\frac{1}{8} A^2 y \Delta \left( \frac{u}{n} \right)(\vec{\rho} \cdot \vec{\rho})^2 + A_4 y^4 \Delta(n) (\vec{\rho} \cdot \vec{\rho})^2 \]

\[ Sag = Sphere + A_4 (x^2 + y^2)^2 \]
Example

Aspheric coefficient to correct the spherical aberration of a spherical mirror, f/4, r=1200 mm, y=75 mm, object at infinity:

\[
W_{040} (\tilde{\rho} \cdot \tilde{\rho})^2 = -\frac{1}{8} A^2 y \Delta \left( \frac{u}{n} \right) (\tilde{\rho} \cdot \tilde{\rho})^2 + A_4 y^4 \Delta (n)(\tilde{\rho} \cdot \tilde{\rho})^2 \\
= -\frac{1}{4} \frac{y^4}{r^3} - 2 y^4 A_4 = 0
\]

\[
A_4 = \frac{1}{8} \frac{1}{r^3}
\]

\[
r = 1200 \text{mm}
\]

\[
y = 75 \text{mm}
\]

\[
A_4 = 2.289 \times 10^{-3}
\]
Stop shifting

As the stop shifts different off-axis rays are selected and the on-axis rays remain the same. The stop diameter changes to maintain the F/#.
Stop shifting

Stop-shift formulas

The process of moving the stop to a new location to obtain a different aberration content is known as stop-shift. The aberration description of an optical system depends on the stop position along the optical axis. As the stop changes position the chief ray is redefined and the Seidel coefficients change. There is no need to recompute the Seidel coefficients; the thus called stop-shift formulas provide the new Seidel coefficients from the old Seidel coefficients and the ratio \( \frac{\delta y}{y} \). These changes upon stop-shift are given in the following Table where the ratio \( \frac{\delta y}{y} \) is the change in chief ray height divided by the marginal ray height. The ratio \( \frac{\delta y}{y} \) is an invariant and can be computed at any plane of the optical system, and in particular it is easier to compute it at the plane of the old stop surface where \( \delta y = y_{new} - y_{old} = y_{new} - 0 = y_{new} \).
Stop shifting

• How do the aberration coefficients change as we shift the stop?
• Used to ease calculations
• Most importantly, the formulas give insights
• The concept of object shifting
Stop shifting parameter

\[ S = \frac{u_{new} - u_{old}}{u} = \frac{y_{new} - y_{old}}{y} = \frac{A_{new} - A_{old}}{A} \]

\[ A_{new} = A_{old} + SA \]
Derivation for \( \bar{A}^* = \bar{A}_{new} \)

\( \mathcal{K} = \bar{A}_1 y - A\bar{y}_1 \)

\( \mathcal{K} = \bar{A}_2 y - A\bar{y}_2 \)

\[ 0 = \left( \bar{A}_1 - \bar{A}_2 \right) y - A \left( \bar{y}_1 - \bar{y}_2 \right) \]

\[ \Delta \bar{A} = A \frac{\bar{y}_2 - \bar{y}_1}{y} = A \frac{\Delta \bar{y}}{y} = A \bar{S} \]

\( \bar{A}^* = \bar{A} + A\bar{S} \)
Stop shifting formulas

\[
\begin{align*}
S^*_I &= S_I \\
S^*_II &= S_{II} + \bar{S}S_I \\
S^*_III &= S_{III} + 2 \cdot \bar{S}S_{II} + \bar{S}^2 S_I \\
S^*_IV &= S_{IV} \\
S^*_V &= S_V + \bar{S} \left( S_{IV} + 3 \cdot S_{III} \right) + 3 \cdot \bar{S}^2 S_{II} + \bar{S}^3 S_I \\
C^*_L &= C_L \\
C^*_T &= C_T + \bar{S}C_L
\end{align*}
\]
Derivation of new Seidel sums upon stop shifting

\[ \overline{A}^* = \overline{A} + \overline{S}A \]

\[ S_I = - \sum A^2 y \Delta \left( \frac{u}{n} \right) \]

\[ S_I^* = S_I \]

\[ S_{II}^* = - \sum A\overline{A}^* y \Delta \left( \frac{u}{n} \right) = - \sum A (\overline{A} + \overline{S}A) y \Delta \left( \frac{u}{n} \right) \]

\[ = - \sum A\overline{A} y \Delta \left( \frac{u}{n} \right) - \overline{S} \sum A^2 y \Delta \left( \frac{u}{n} \right) \]

\[ S_{II}^* = S_{II} + \overline{S} S_I \]

\[ S_{III}^* = - \sum (\overline{A}^*)^2 y \Delta \left( \frac{u}{n} \right) = - \sum (\overline{A} + \overline{S}A)^2 y \Delta \left( \frac{u}{n} \right) \]

\[ = - \sum (\overline{A}^2 + 2 \overline{A} \overline{S} + \overline{S}^2 A^2) y \Delta \left( \frac{u}{n} \right) \]

\[ S_{III}^* = S_{III} + 2 \overline{S} S_{II} + \overline{S}^2 S_I \]

And similarly for:

\[ S_{IV}^* = S_{IV} \]

\[ S_{v}^* = S_v + \overline{S} (S_{IV} + 3 S_{III}) + 3 \overline{S}^2 S_{II} + \overline{S}^3 S_I \]
Summary of stop shifting

\[
\bar{y}_{\text{new}} = \bar{y}_{\text{old}} + \bar{S} y \\
\bar{u}_{\text{new}} = \bar{u}_{\text{old}} + \bar{S} u \\
\bar{A}_{\text{new}} = \bar{A}_{\text{old}} + \bar{S} A
\]

\[
W_{\text{new}} \left( \bar{H}, \bar{\rho} \right) = W_{\text{old}} \left( \bar{H}, \bar{\rho}_{\text{shift}} \right)
\]

\[
\bar{\rho}_{\text{shift}} = \bar{\rho} + \bar{S} \bar{H}
\]

| \( S^*_I \) | \( S_I \) |
| \( S^*_II \) | \( S_{II} + \bar{S} S_I \) |
| \( S^*_III \) | \( S_{III} + 2 \cdot \bar{S} S_{II} + \bar{S}^2 S_I \) |
| \( S^*_IV \) | \( S_{IV} \) |
| \( S^*_V \) | \( S_V + \bar{S} \left( S_{IV} + 3 \cdot S_{III} \right) + 3 \cdot \bar{S}^2 S_{II} + \bar{S}^3 S_I \) |
| \( C^*_L \) | \( C_L \) |
| \( C^*_T \) | \( C_T + \bar{S} C_L \) |
Sine condition from optical flux conservation and radiance theorem

Etendue considerations
Throughput or Etendue

\[ T = \varepsilon = n^2 A\Omega = n'^2 A'\Omega' \cong \pi \mathcal{K}^2 \]
Sine condition

\[ \frac{u'}{u} = \frac{\sin(U')}{\sin(U)} \]

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Optical flux from a Lambertian source

\[ \Phi(\theta) = 2\pi AL_0 \int_0^\theta \cos(\zeta) \sin(\zeta) d\zeta = \pi AL_0 \sin^2(\theta) \]

\[ L_0 \text{ is assumed uniform} \]

Compare with isotropic source

\[ \Phi(\theta) = 2\pi AL_0 \int_0^\theta \sin(\zeta) d\zeta = 2\pi AL_0 \left(1 - \cos(\theta)\right) = AL_0 \Omega \]
Optical flux = radiance x throughput

\[ \Phi(\theta) = \frac{L_0}{n^2} T \]

\[ \Phi(\theta') = \frac{L_{0'}}{n'^2} T \]

\[ \frac{n^2}{L_0} \Phi(\theta) = \frac{n'^2}{L'_{0'}} \Phi(\theta') = T \]

\[ \frac{n^2}{L_0} = \frac{n'^2}{L'_{0'}} \]

Radiance theorem

\[ U = \theta \]

\[ \frac{n^2}{L_0} \Phi(U) = \frac{n'^2}{L'_{0'}} \Phi(U') \]
Sine condition from optical flux conservation

\[
\frac{n^2}{L_0} \Phi(U) = \frac{n'^2}{L'_0} \Phi(U')
\]

\[
n^2 A \sin^2(U) = n'^2 A' \sin^2(U')
\]

\[
\pi n^2 h^2 \sin^2(U) = \pi h'^2 n'^2 \sin^2(U')
\]

\[
n^2 h^2 \sin^2(U) = h'^2 n'^2 \sin^2(U')
\]

\[
nh \sin(U) = n' h' \sin(U')
\]

\[
\frac{u'}{u} = \frac{\sin(U')}{\sin(U)} \quad \text{Sine condition}
\]
Throughput
Etendue
(area-omega product)

\[ \mathcal{E} = n^2 A \Omega = n'^2 A' \Omega' \]

\[ T = \pi \mathcal{K}^2 \]

"Capacity to transfer optical flux"

\[ \Omega = 2\pi \left(1 - \cos(\theta)\right) \]
Even better

\[ \varepsilon = n^2 A \Omega = n'^2 A' \Omega' \]

For isotropic source

\[ \varepsilon = A (NA)^2 = n^2 A \sin^2 (\theta) \]

For Lambertian source
Etendu considerations are key to design an optical system.

\[ \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4 \leq \epsilon_5 \]

Start with the sensor at the end. \[ \epsilon_5 = A_5 (NA)^2 \]

Every optics component has associated an etendue. The component with the smallest etendue limits the amount of optical flux transferred by the system.

Assumption: lossless and not active components.
Summary

• Variety of aspheric surfaces
• Aspheric surface contributions to fourth-order aberrations
• Stop shifting and how aberrations change
• Etendue considerations