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# Geometrical irradiance changes in a symmetric optical system

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**Abstract.** The concept of the aberration function is extended to define two functions that describe the light irradiance distribution at the exit pupil plane and at the image plane of an axially symmetric optical system. Similar to the wavefront aberration function, the irradiance function is expanded as a polynomial, where individual terms represent basic irradiance distribution patterns. Conservation of flux in optical imaging systems is used to derive the specific relation between the irradiance coefficients and wavefront aberration coefficients. It is shown that the coefficients of the irradiance functions can be expressed in terms of wavefront aberration coefficients and first-order system quantities. The theoretical results—these are irradiance coefficient formulas—are in agreement with real ray tracing.  [DOI: 10.1117/1.OE.56.1.015104]

Keywords: aberration theory; wavefront aberrations; irradiance.

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## 1 Introduction

In the development of the theory of aberrations in optical imaging systems, emphasis has been given to the study of image aberrations, which are described as wave, angular, transverse, or longitudinal quantities.[1] The light irradiance variation, specifically at the exit pupil plane and at the image plane of an optical system, is a radiometric aspect of a system that is also of interest.

The geometrical irradiance calculation is useful for determining the apodization of the wavefront at the exit pupil. The point spread function (PSF) describes the response of an imaging system to a point object. An accurate diffraction calculation of a system's PSF requires knowledge not only of the wavefront phase but also of its amplitude. These, the phase and amplitude, are usually calculated geometrically at the exit pupil of the system. While the wavefront phase is evaluated by calculating the optical path length for rays, the wavefront amplitude is estimated by the square root of the geometrical irradiance.

The image plane illumination or relative illumination is another characteristic that may have a strong impact on the performance of a lens system.

The light irradiance distribution at the exit pupil plane and/or at the image plane of an optical system can be derived from basic radiometric principles, such as conservation of flux.[2] In this paper, we provide a study of the relationship between irradiance at these two planes and the system's wavefront aberration coefficients. Our study is based on geometrical optics. Edge diffraction effects are not considered in this study. An extended object that has a uniform or Lambertian radiance is assumed. The geometrical irradiance that we discuss describes the illumination distribution on the image plane or on the exit pupil plane of an optical system.

The concept of the wavefront aberration function is well established. The wavefront aberration function $W(\vec{H},\vec{\rho})$ of an axially symmetric system gives the geometrical wavefront deformation (from a sphere) at the exit pupil as a function of the normalized field $\vec{H}$ and aperture $\vec{\rho}$ vectors. The field vector and the aperture vector may be defined in either the object or the image spaces. Two vectors uniquely specify a ray propagating in the lens system. The wavefront aberration function is expanded into a polynomial of the rotational invariants as dot products of the field and aperture vectors, specifically $(\vec{H}\cdot\vec{H})$, $(\vec{H}\cdot\vec{\rho})$, and $(\vec{\rho}\cdot\vec{\rho})$, and to the fourth order of approximation on $\vec{H}$ and $\vec{\rho}$ is written as follows:

$$\begin{aligned} W(\vec{H},\vec{\rho}) &= \sum_{j,m,l} W_{k,l,m}(\vec{H}\cdot\vec{H})^j(\vec{H}\cdot\vec{\rho})^m(\vec{\rho}\cdot\vec{\rho})^n \\ &= W_{000} + W_{200}(\vec{H}\cdot\vec{H}) + W_{111}(\vec{H}\cdot\vec{\rho}) \\ &\quad + W_{020}(\vec{\rho}\cdot\vec{\rho}) + W_{040}(\vec{\rho}\cdot\vec{\rho})^2 \\ &\quad + W_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) + W_{222}(\vec{H}\cdot\vec{\rho})^2 \\ &\quad + W_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + W_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) + W_{400}(\vec{H}\cdot\vec{H})^2, \end{aligned} \tag{1}$$

where each aberration coefficient $W_{k,l,m}$ represents the amplitude of basic wavefront deformation forms and subindices $j$, $m$, and $n$ represent integers $k=2j+m$ and $l=2n+m$.[3] Similarly, a pupil aberration function $\overline{W}(\vec{H},\vec{\rho})$ can be defined to describe the aberration between the pupils. This function is constructed by interchanging the role of the field and aperture vectors, and to fourth order, it is as follows:

$$\begin{aligned} \overline{W}(\vec{H},\vec{\rho}) &= \sum_{j,m,l} \overline{W}_{k,l,m}(\vec{H}\cdot\vec{H})^j(\vec{H}\cdot\vec{\rho})^m(\vec{\rho}\cdot\vec{\rho})^n \\ &= \overline{W}_{000} + \overline{W}_{200}(\vec{\rho}\cdot\vec{\rho}) + \overline{W}_{111}(\vec{H}\cdot\vec{\rho}) \\ &\quad + \overline{W}_{020}(\vec{H}\cdot\vec{H}) + \overline{W}_{040}(\vec{H}\cdot\vec{H})^2 \\ &\quad + \overline{W}_{131}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) + \overline{W}_{222}(\vec{H}\cdot\vec{\rho})^2 \\ &\quad + \overline{W}_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + \overline{W}_{311}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) + \overline{W}_{400}(\vec{\rho}\cdot\vec{\rho})^2, \end{aligned} \tag{2}$$

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where the pupil aberration coefficients are barred to distinguish them from the image aberration coefficients and subindices $j$, $m$, and $n$ represent integers $k = 2j + m$ and $l = 2n + m$.

In analogy to the wavefront aberration function, we define the irradiance function $I(\vec{H}, \vec{\rho})$ that gives the irradiance at the image plane of an optical system. To the fourth order of approximation, this irradiance function is expressed as follows:

$$\begin{aligned} I(\vec{H},\vec{\rho}) &= \sum_{j,m,n} I_{l,k,m} (\vec{H}\cdot\vec{H})^j \cdot (\vec{H}\cdot\vec{\rho})^m \cdot (\vec{\rho}\cdot\vec{\rho})^n \\ &= I_{000} + I_{200}(\vec{H}\cdot\vec{H}) + I_{111}(\vec{H}\cdot\vec{\rho}) + I_{020}(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + I_{040}(\vec{\rho}\cdot\vec{\rho})^2 + I_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) + I_{222}(\vec{H}\cdot\vec{\rho})^2 \\ &\quad + I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) + I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) \\ &\quad + I_{400}(\vec{H}\cdot\vec{H})^2, \end{aligned} \tag{3}$$

where the irradiance coefficients $I_{k,l,m}$ represent basic illumination distribution patterns at the image plane, as shown in Fig. 1, and subindices $j$, $m$, and $n$ represent integers $k = 2j + m$ and $l = 2n + m$. We also define another irradiance function $\bar{I}(\vec{H}, \vec{\rho})$ that gives the irradiance of the beam at the exit pupil plane of an optical system:

$$\begin{aligned} \bar{I}(\vec{H},\vec{\rho}) &= \sum_{j,m,l} \bar{I}_{l,k,m} (\vec{H}\cdot\vec{H})^j (\vec{H}\cdot\vec{\rho})^m (\vec{\rho}\cdot\vec{\rho})^n \\ &= \bar{I}_{000} + \bar{I}_{200}(\vec{\rho}\cdot\vec{\rho}) + \bar{I}_{111}(\vec{H}\cdot\vec{\rho}) + \bar{I}_{020}(\vec{H}\cdot\vec{H}) \\ &\quad + \bar{I}_{040}(\vec{H}\cdot\vec{H})^2 + \bar{I}_{131}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) \\ &\quad + \bar{I}_{222}(\vec{H}\cdot\vec{\rho})^2 + \bar{I}_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + \bar{I}_{311}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) + \bar{I}_{400}(\vec{\rho}\cdot\vec{\rho})^2, \end{aligned} \tag{4}$$

where each irradiance coefficient $\bar{I}_{l,k,m}$ represents basic apodization distributions at the exit pupil, as shown in Fig. 1, and subindices $j$, $m$, and $n$ represent integers $k = 2j + m$ and $l = 2n + m$. In these functions, it is assumed that the irradiance is greater than zero for any point defined by the normalized field $\vec{H}$ and aperture $\vec{\rho}$ vectors.

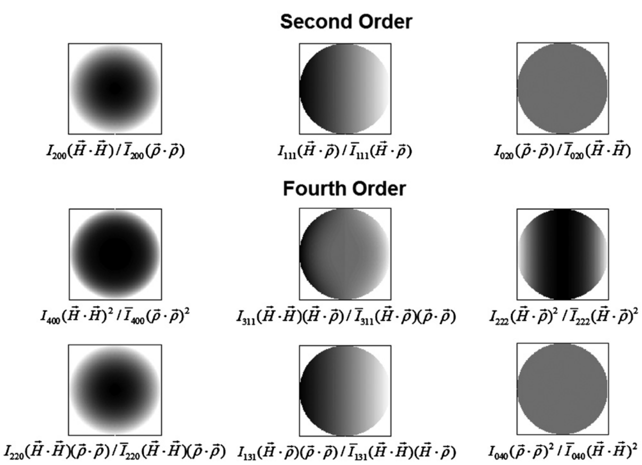


**Fig. 1** Second- and fourth-order apodization terms at the exit pupil given by the coefficients of the pupil irradiance function presented as a function of the aperture vector $\vec{\rho}$ or irradiance patterns at the image given by the coefficients of the irradiance function presented as a function of the field vector $\vec{H}$.

The question we pose and answer is: what is the relationship between the wavefront aberration functions $W(\vec{H}, \vec{\rho})$ and $\overline{W}(\vec{H}, \vec{\rho})$ and the irradiance functions $I(\vec{H}, \vec{\rho})$ and $\bar{I}(\vec{H}, \vec{\rho})$? We use conservation of flux in an optical system to determine the relationship between wavefront and irradiance coefficients. We also discuss in some detail the particular case of relative illumination at the image plane. Our theoretical results are in agreement with real ray tracing. Overall, the paper provides new insights into the irradiance changes in an optical system and furthers the theory of aberrations.

## 2 Radiative Transfer in an Optical System

In this section, we review radiometry and show how aberrations relate to conservation of optical flux $\Phi$ in an optical system. Figure 2 illustrates the basic elements of an axially symmetric optical system and the geometry defining the transfer of radiant energy. Rays from a differential area $dA$ in the object plane pass through the optical system and converge at the conjugate area $dA'$ in the image plane. The differential cross sections of the beam $dS$ in the entrance pupil plane and $dS'$ in the exit pupil plane are optically conjugated to the first order.

In a lossless and passive optical system, the element of radiant flux $d\Phi$ is conserved through all cross sections of the beam. The equation for the conservation of flux is written as follows:

$$d\Phi = L_o \frac{dA\,dS \cos^4(\theta)}{t^2} = L_o \frac{dA' dS' \cos^4(\theta')}{t'^2} = d\Phi', \tag{5}$$

where $L_o$ is the source radiance. We assume that the source is Lambertian, and, consequently, $L_o$ is constant. The object space angle $\theta$ is between the ray connecting $dA$ and $dS$ and the optical axis of the lens system. Similarly, $\theta'$ is the image space angle between the ray connecting $dA'$ and $dS'$ and the optical axis. $t$ ($t'$) is the axial distance between the object (image) plane and the entrance (exit) pupil plane, respectively.

Equation (5) provides the radiant flux along a particular ray in an optical system. Since the normalized field $\vec{H}$ and aperture $\vec{\rho}$ vectors uniquely specify any ray propagating in the optical system, Eq. (5) gives the radiant flux as a function of the normalized field $\vec{H}$ and pupil coordinates $\vec{\rho}$.

The irradiance on the exit pupil plane is obtained by dividing the radiant flux that is incident on a surface by the unit area. It follows that

$$\begin{aligned} d\bar{I}(\vec{H},\vec{\rho}) &= \frac{d\Phi'}{dS'} = L_o \frac{dA'}{t'^2} \cos^4(\theta') \\ &= L_o \frac{dA}{t^2} \frac{dS}{dS'} \cos^4(\theta) = \frac{d\Phi}{dS'} \end{aligned} \tag{6}$$

where $d\bar{I}(\vec{H}, \vec{\rho})$ is the differential of irradiance on the exit pupil plane. Similarly, the irradiance on the image plane is obtained by dividing the radiant flux by the image surface unit area as in

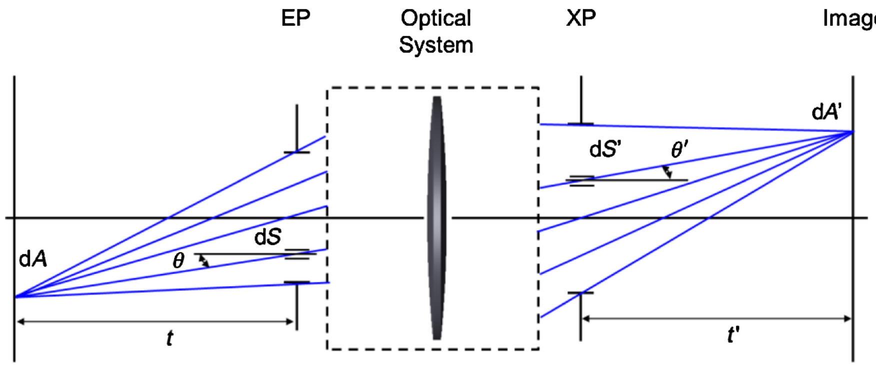


**Fig. 2** Geometry defining the transfer of radiant energy from differential area $dA$ in the object space to the conjugate area $dA'$ in the image space. The radiant flux through all cross sections of the beam is the same.

$$dI(\vec{H},\vec{\rho}) = \frac{d\Phi'}{dA'} = L_o \frac{dS'}{t'^2}\cos^4(\theta') = L_o \frac{dS'}{t^2}\frac{dA}{dA'}\cos^4(\theta) = \frac{d\Phi}{dA'} \tag{7}$$

and $dI(\vec{H},\vec{\rho})$ is now the differential of irradiance on the image plane. The differentials of irradiance $d\bar{I}(\vec{H},\vec{\rho})$ and $dI(\vec{H},\vec{\rho})$ are functions of the field and aperture vectors.

Equations (6) and (7) are general and do not involve any approximations. However, care must be taken in evaluating these expressions since the angles $\theta$ and $\theta'$ may vary due to aberrations. In addition, the differential areas $dA$, $dA'$, $dS$, and $dS'$ may also vary for different points in the aperture and in the field of the lens.

## 3 Irradiance Function of a Pinhole Camera

In this section, we calculate coefficients of the irradiance function defined in Eqs. (6) and (7) for a pinhole camera. The pinhole camera produces images of illuminated objects as light passes through an aperture, as shown in Fig. 3.

In this model, we define the field vector $\vec{H}$ on the image plane and the aperture vector $\vec{\rho}$ on the plane of the aperture. The irradiance functions $\bar{I}(\vec{H},\vec{\rho})$ and $I(\vec{H},\vec{\rho})$ give the relative irradiance along the ray specified by the field and aperture vectors. It follows from Eqs. (6) and (7) that irradiance distribution at a point specified by $\vec{H}$ on the focal plane or at a point specified by $\vec{\rho}$ on the aperture plane are proportional to $\cos^4(\theta')$ of the particular ray. To calculate the coefficients of the irradiance functions to fourth order, we express $\cos^4(\theta')$ in terms of first-order system quantities as (see Appendix A):

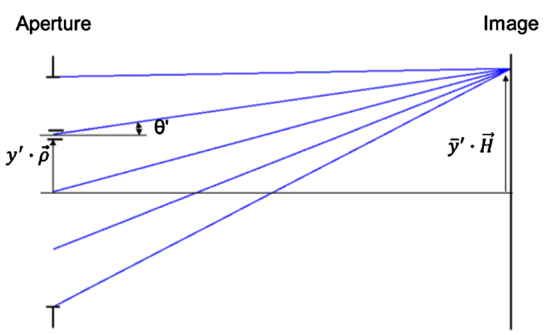


**Fig. 3** Geometry defining a pinhole camera. Images are formed as light passes through the aperture.

$$\begin{aligned} I_{\text{pinhole}}(\vec{H},\vec{\rho}) &= \bar{I}_{\text{pinhole}}(\vec{H},\vec{\rho}) = I_{0/\text{pinhole}} \cdot \cos^4(\theta') \\ &= I_{0/\text{pinhole}} \cdot [1 - 2\cdot \bar{u}'^2(\vec{H}\cdot\vec{H}) - 2\cdot u'^2(\vec{\rho}\cdot\vec{\rho}) \\ &\quad - 4\cdot u'\bar{u}'(\vec{H}\cdot\vec{\rho})\ldots \\ &\quad + 3\cdot\bar{u}'^4(\vec{H}\cdot\vec{H})^2 + 12\cdot u'^3\bar{u}'(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + 12\cdot u'^2\bar{u}'^2(\vec{H}\cdot\vec{\rho})^2\ldots \\ &\quad + 6\cdot u'^2\bar{u}'^2(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) \\ &\quad + 12\cdot u'\bar{u}'^3(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) + 3\cdot u'^4(\vec{\rho}\cdot\vec{\rho})^2], \end{aligned} \tag{8}$$

where $u'$ and $\bar{u}'$ are the first-order marginal and chief ray slopes in the image space and $I_{0/\text{pinhole}}$ is a constant corresponding to the irradiance value for the on-axis field point or zero-order term of the irradiance function. Without loss of generality, we set this zero-order term of the irradiance function equal to 1.

In aberration theory, the reference sphere is used as the reference to measure the wavefront deformation. In analogy, the geometry in Fig. 3 can be used to define a model of irradiance $I_{\text{pinhole}}(\vec{H},\vec{\rho})$, which is helpful as a reference and for further calculations. In an actual system, image and pupil aberrations affect the irradiance distribution.

## 4 Irradiance on the Image Plane

The aperture vector $\vec{\rho}$ is selected at the exit pupil plane, and, thus, it defines the ray intersection with this plane. Rays at the exit pupil pass through a uniform grid by construction, and differential areas $dS'$ are constant given that we choose to define rays at the exit pupil. To calculate the image plane irradiance, we evaluate Eq. (7) in the image space as follows:

$$dI = I_0 \cdot \cos^4(\theta'), \tag{9}$$

where

$$I_0 = L_o \frac{dS'}{t'^2}, \tag{10}$$

where $I_0 = 1$ is a constant corresponding to the irradiance value for the on-axis field point. In contrast to a pinhole camera, in an actual system, the ray angle in Eq. (9) is modified by image aberrations. To calculate the irradiance distribution

**Table 1** Second-order irradiance coefficients $I^{(2)}_{k,l,m}$ at the image plane of an optical system.

| |
|---|
| $I_{020}(\vec{\rho}\cdot\vec{\rho}) = -2u'^2(\vec{\rho}\cdot\vec{\rho})$ |
| $I_{111}(\vec{H}\cdot\vec{\rho}) = -4u'\bar{u}'(\vec{H}\cdot\vec{\rho})$ |
| $I_{200}(\vec{H}\cdot\vec{H}) = -2\bar{u}'^2(\vec{H}\cdot\vec{H})$ |

at the image plane of an optical system with the aperture stop at the exit pupil, we express $\cos^4(\theta')$ in terms of first-order system quantities and wavefront aberration coefficients (see Appendix B). Table 1 provides a summary of the second-order image plane irradiance coefficients. We conclude that the second-order irradiance terms are not affected by image aberrations and that the image plane irradiance to the second order is equivalent to the ideal irradiance of a pinhole camera:

$$\begin{aligned} I(\vec{H},\vec{\rho})^{(2)} &= I_{\text{pinhole}}(\vec{H},\vec{\rho})^{(2)} \\ &= 1 - 2u'^2(\vec{\rho}\cdot\vec{\rho}) - 4u'\bar{u}'(\vec{H}\cdot\vec{\rho}) - 2\bar{u}'^2(\vec{H}\cdot\vec{H}). \end{aligned} \tag{11}$$

Table 2 summarizes the fourth-order image plane irradiance coefficients. The fourth-order irradiance coefficients are the sum of two components. The first component is represented by products of the first-order ray slopes $u'$ and $\bar{u}'$ in the image space. The second component includes additional terms that are functions of the fourth-order image aberration coefficients and the first-order ray slopes. The constant Ж in Table 2 represents the Lagrange invariant of the system.

## 5 Irradiance on the Exit Pupil Plane

To calculate the irradiance distribution on the exit pupil plane of an optical system, we evaluate Eq. (6) in the image space as follows:

$$d\bar{I} = \bar{I}_0 \cdot \frac{dA'}{dA} \cdot \cos^4(\theta') \tag{12}$$

**Table 2** Fourth-order irradiance coefficients $I^{(4)}_{k,l,m}$ at the image plane of an optical system.

| |
|---|
| $I_{040}(\vec{\rho}\cdot\vec{\rho})^2 = \left[3u'^4 + \frac{16}{\text{Ж}}W_{040}u'\bar{u}'\right](\vec{\rho}\cdot\vec{\rho})^2$ |
| $I_{131}(\vec{\rho}\cdot\vec{\rho})(\vec{H}\cdot\vec{\rho}) = \left[12u'^3\bar{u}' + \frac{16}{\text{Ж}}W_{040}\bar{u}'^2 + \frac{12}{\text{Ж}}W_{131}u'\bar{u}'\right](\vec{\rho}\cdot\vec{\rho})(\vec{H}\cdot\vec{\rho})$ |
| $I_{222}(\vec{H}\cdot\vec{\rho})^2 = \left[12u'^2\bar{u}'^2 + \frac{8}{\text{Ж}}W_{131}\bar{u}'^2 + \frac{8}{\text{Ж}}W_{222}u'\bar{u}'\right](\vec{H}\cdot\vec{\rho})^2$ |
| $I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) = \left[6u'^2\bar{u}'^2 + \frac{4}{\text{Ж}}W_{131}\bar{u}'^2 + \frac{8}{\text{Ж}}W_{220}u'\bar{u}'\right](\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ |
| $I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) = \left[12u'\bar{u}'^3 + \frac{8}{\text{Ж}}W_{222}\bar{u}'^2 + \frac{8}{\text{Ж}}W_{220}\bar{u}'^2 + \frac{4}{\text{Ж}}W_{311}u'\bar{u}'\right] \times (\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ |
| $I_{400}(\vec{H}\cdot\vec{H})^2 = \left[3\bar{u}'^4 + \frac{4}{\text{Ж}}W_{311}\bar{u}'^2\right](\vec{H}\cdot\vec{H})^2$ |

and

$$\bar{I}_0 \equiv \left(L_o \frac{dA}{t^2}\right), \tag{13}$$

where $\bar{I}_0 = 1$ is a constant corresponding to the irradiance value for the on-axis field point. A comparison of Eqs. (12) and (9) reveals an additional term $\frac{dA'}{dA}$ that is given by the ratio between the elements of the area at the object and image planes. In an optical system with aberrations, the ratio $\frac{dA'}{dA}$ may vary over the pupil and over the field. Two differential areas are related by the determinant of the Jacobian $J(\vec{H},\vec{\rho})$ of the transformation:

$$dA' = J(\vec{H},\vec{\rho})dA. \tag{14}$$

**Table 3** Second-order irradiance coefficients $\bar{I}^{(2)}_{k,l,m}$ at the exit pupil plane of an optical system.

| |
|---|
| $\bar{I}_{020}(\vec{H}\cdot\vec{H}) = \left[-2\bar{u}'^2 - \frac{4}{\text{Ж}}W_{311}\right](\vec{H}\cdot\vec{H})$ |
| $\bar{I}_{111}(\vec{H}\cdot\vec{\rho}) = \left[-4u'\bar{u}' - \frac{4}{\text{Ж}}W_{220} - \frac{6}{\text{Ж}}W_{222}\right](\vec{H}\cdot\vec{\rho})$ |
| $\bar{I}_{200}(\vec{\rho}\cdot\vec{\rho}) = \left[-2u'^2 - \frac{4}{\text{Ж}}W_{131}\right](\vec{\rho}\cdot\vec{\rho})$ |

**Table 4** Fourth-order irradiance coefficients $\bar{I}^{(4)}_{k,l,m}$ at the exit pupil plane of an optical system.

| |
|---|
| $\bar{I}_{040}(\vec{H}\cdot\vec{H})^2 = \left[3\bar{u}'^4 - \frac{6}{\text{Ж}}W_{511} + \frac{3}{\text{Ж}}W_{311}\bar{u}'^2 + \frac{3}{\text{Ж}^2}W_{311}W_{311}\right](\vec{H}\cdot\vec{H})^2$ |
| $\bar{I}_{131}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) = \left[12u'\bar{u}'^3 - \frac{10}{\text{Ж}}W_{422} - \frac{8}{\text{Ж}}W_{420} + \frac{5}{\text{Ж}}W_{222}\bar{u}'^2 + \frac{2}{\text{Ж}}W_{220}\bar{u}'^2 + \frac{6}{\text{Ж}}W_{311}u'\bar{u}' + \frac{4}{\text{Ж}^2}W_{220}W_{311} + \frac{10}{\text{Ж}^2}W_{222}W_{311}\right] \times (\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ |
| $\bar{I}_{222}(\vec{H}\cdot\vec{\rho})^2 = \left[12u'^2\bar{u}'^2 - \frac{12}{\text{Ж}}W_{333} - \frac{4}{\text{Ж}}W_{331} - \frac{2}{\text{Ж}}W_{131}\bar{u}'^2 + \frac{12}{\text{Ж}}W_{222}u'\bar{u}' + \frac{8}{\text{Ж}}W_{220}u'\bar{u}' - \frac{2}{\text{Ж}}W_{311}u'^2 + \frac{8}{\text{Ж}^2}W_{222}W_{222} + \frac{16}{\text{Ж}^2}W_{222}W_{220} - \frac{4}{\text{Ж}^2}W_{311}W_{131}\right](\vec{H}\cdot\vec{\rho})^2$ |
| $\bar{I}_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) = \left[6u'^2\bar{u}'^2 - \frac{6}{\text{Ж}}W_{331} + \frac{5}{\text{Ж}}W_{131}\bar{u}'^2 - \frac{2}{\text{Ж}}W_{222}u'\bar{u}' - \frac{4}{\text{Ж}}W_{220}u'\bar{u}' + \frac{5}{\text{Ж}}W_{311}u'^2 + \frac{10}{\text{Ж}^2}W_{311}W_{131} - \frac{8}{\text{Ж}^2}W_{222}W_{220}\right] \times (\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ |
| $\bar{I}_{311}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) = \left[12u'^3\bar{u}' - \frac{10}{\text{Ж}}W_{242} - \frac{8}{\text{Ж}}W_{240} + \frac{5}{\text{Ж}}W_{222}u'^2 + \frac{2}{\text{Ж}}W_{220}u'^2 + \frac{6}{\text{Ж}}W_{131}u'\bar{u}' + \frac{4}{\text{Ж}^2}W_{220}W_{131} + \frac{10}{\text{Ж}^2}W_{222}W_{131}\right] \times (\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ |
| $\bar{I}_{400}(\vec{\rho}\cdot\vec{\rho})^2 = \left[3u'^4 - \frac{6}{\text{Ж}}W_{151} + \frac{3}{\text{Ж}}W_{131}u'^2 + \frac{3}{\text{Ж}^2}W_{131}W_{131}\right](\vec{\rho}\cdot\vec{\rho})^2$ |

If the Jacobian determinant is expressed in terms of wavefront aberration coefficients, the differential of irradiance on the exit pupil plane is calculated by evaluating Eq. (12) and keeping terms to the fourth order (see Appendix C). Tables 3 and 4 provide a summary of the exit pupil plane irradiance coefficients to the fourth order of approximation.

The coefficients $\bar{I}_{k,l,m}$ are the sum of several components. The first component is represented by products of the first-order ray slopes $u'$ and $\bar{u}'$ in the image space and is equivalent to the first component of $I_{k,l,m}$ coefficients. The second component includes additional terms that are functions of the fourth-order image aberration coefficients and first-order ray slopes. Note that the second-order terms of the exit pupil irradiance function are modified by image aberrations, in contrast to the second-order terms of the image plane irradiance. The third component of the coefficients $\bar{I}^{(4)}_{k,l,m}$ is proportional to the six-order image aberration coefficients. Finally, we have additional terms that involve products of the fourth-order aberrations.

**Table 5** Relationships between second-order irradiance coefficients $I^{(2)}_{k,l,m}$ and $\bar{I}^{(2)}_{k,l,m}$.

| |
|---|
| $I_{020} - \bar{I}_{200} = \frac{4}{\text{Ж}} W_{131}$ |
| $I_{111} - \bar{I}_{111} = \frac{4}{\text{Ж}} W_{220} + \frac{6}{\text{Ж}} W_{222}$ |
| $I_{200} - \bar{I}_{020} = \frac{4}{\text{Ж}} W_{311}$ |

We have set the field vector $\vec{H}$ at the object plane and the aperture vector $\vec{\rho}$ at the exit pupil plane. We assume that the aperture stop coincides with the location of the aperture vector, which is the exit pupil plane. Our formulas would change depending on whether the field vector $\vec{H}$ is at the object or image plane and on whether the aperture vector $\vec{\rho}$ and the stop are at the entrance pupil, the exit pupil, or an intermediate pupil.

## 6 Coefficient Relationship

For the case in consideration of having the field vector at the object plane and the aperture vector at the exit pupil plane, we can write the second order relationships in Table 5.

Thus, it is possible to determine aberration coefficients from measurements of the irradiance at the exit pupil plane and at the image plane of an optical system.[4] However, since the field vector is defined at the object plane, the image plane irradiance refers to image points $\vec{H} + \Delta\vec{H}$, and, as a consequence, measurements should be made at conjugate object-image points.

## 7 Relative Illumination

A practical and important case is the relative illumination $RI(\vec{H})$ at the image plane of an optical system, which can be written to the fourth order as follows:

$$RI(\vec{H}) = 1 + I_{200}(\vec{H}\cdot\vec{H}) + I_{400}(\vec{H}\cdot\vec{H})^2. \tag{15}$$

This is given in the limit of a small aperture by the terms $I_{200}(\vec{H}\cdot\vec{H})$ and $I_{400}(\vec{H}\cdot\vec{H})^2$ of the irradiance function

**Table 6** Second and fourth-order irradiance coefficients describing relative illumination.

| Case | $I_{200}$ | $I_{400}$ |
|---|---|---|
| Pinhole camera. | $-2\cdot\bar{u}'^2$ | $3\cdot\bar{u}'^4$ |
| Field vector at the image plane. Stop and aperture vector at the exit pupil plane. | $-2\cdot\bar{u}'^2$ | $3\cdot\bar{u}'^4$ |
| Field vector at the object plane. Stop and aperture vector at the exit pupil plane. | $-2\cdot\bar{u}'^2$ | $3\cdot\bar{u}'^4 + \frac{4}{\text{Ж}}W_{311}\cdot\bar{u}'^2$ |
| Field vector at the image plane. Stop and aperture vector at the entrance pupil plane. | $-2\bar{u}'^2 + \frac{4}{\text{Ж}}\overline{W}_{131}$ | $3\bar{u}'^4 + \frac{6}{\text{Ж}}\overline{W}_{151} - \frac{3}{\text{Ж}}\overline{W}_{131}\bar{u}'^2 + \frac{3}{\text{Ж}^2}\overline{W}_{131}\overline{W}_{131} - \frac{1}{\text{Ж}^2}[32\overline{W}_{040}W_{220} + 24\overline{W}_{040}W_{222}]$ |
| Field vector at the object plane. Stop and aperture vector at the entrance pupil plane. | $-2\bar{u}'^2 + \frac{4}{\text{Ж}}\overline{W}_{131}$ | $3\bar{u}'^4 + \frac{4}{\text{Ж}}W_{311}\bar{u}'^2 + \frac{6}{\text{Ж}}\overline{W}_{151} - \frac{3}{\text{Ж}}\overline{W}_{131}\bar{u}'^2 + \frac{3}{\text{Ж}^2}\overline{W}_{131}\overline{W}_{131} - \frac{1}{\text{Ж}^2}[32\overline{W}_{040}W_{220} + 24\overline{W}_{040}W_{222} + 8\overline{W}_{131}W_{311}]$ |
| Stop and aperture vector at plane between components of the lens. Field vector at the image plane. | $-2\bar{u}'^2 + \frac{4}{\text{Ж}}\overline{W}_{131B}$ | $3\bar{u}'^4 + \frac{6}{\text{Ж}}\overline{W}_{151B} - \frac{3}{\text{Ж}}\overline{W}_{131B}\bar{u}'^2 + \frac{3}{\text{Ж}^2}\overline{W}_{131B}\overline{W}_{131B} - \frac{1}{\text{Ж}^2}[32\overline{W}_{040B}W_{220B} + 24\overline{W}_{040B}W_{222B}]$ |
| Stop and aperture vector at plane between components of the lens. Field vector at the object plane. | $-2\bar{u}'^2 + \frac{4}{\text{Ж}}\overline{W}_{131B}$ | $3\bar{u}'^4 + \frac{4}{\text{Ж}}W_{311A}\bar{u}'^2 + \frac{4}{\text{Ж}}W_{311B}\bar{u}'^2 + \frac{6}{\text{Ж}}\overline{W}_{151B} - \frac{3}{\text{Ж}}\overline{W}_{131B}\bar{u}'^2 - \frac{1}{\text{Ж}^2}[32\overline{W}_{040B}W_{220B} + 24\overline{W}_{040B}W_{222B} + 8\overline{W}_{131B}W_{311B}] + \frac{3}{\text{Ж}^2}\overline{W}_{131B}\overline{W}_{131B} - \frac{8}{\text{Ж}^2}\overline{W}_{131B}W_{311A}$ |

$I(\vec{H},\vec{\rho})$. Although the calculation of the relative illumination is typically done over the entire pupil, we have previously shown that the approximation of a small diaphragm is sufficiently accurate for practical purposes.[5] The relative illumination on the image plane depends on the position of the aperture stop, and the irradiance coefficients also depend on the location of the field and aperture vectors.

Table 6 presents the second $I_{200}$ and fourth order $I_{400}$ coefficients of the relative illumination function $RI(\vec{H})$ for several cases in the location of the aperture and field vectors. When the stop is between system components, the system is then divided into part A preceding the stop and part B following the stop, and aberrations of each part are then used to define the irradiance coefficients. Note that both image and pupil aberration coefficients are used.[5]

Examination of the functional form of the coefficients allows one to make several points of interest. First, to the second–order, pupil coma influences the relative illumination and can nullify the coefficient $I_{200}$ in some cases. This is known as the Slyusarev effect.[6] Alternatively, the relationship between image and pupil aberrations is as follows:

$$\overline{W}_{131} = W_{311} - \frac{\text{Ж}}{2}\Delta\{\bar{u}^2\}, \tag{16}$$

which can be used to determine how image distortion $W_{311}$ influences relative illumination.[7] In this case, similar to the standard $\cos^4$-law of illumination fall off, slopes in the object space are used in the expression of the relative illumination.[5]

Second, if the system is telecentric in the image space $\bar{u}' = 0$ and if the system is also aplanatic $\overline{W}_{131} = 0$, $\overline{W}_{151} = 0$, and $\overline{W}_{040} = 0$, then the relative illumination is uniform to the fourth order. Such a system would have a $f\ \sin(\theta)$ image height mapping.[8] Other cases for uniform illumination are also possible, requiring $\frac{4}{\text{Ж}}\overline{W}_{131} - 2\bar{u}'^2 = 0$.

Third, if the system is telecentric in the object space $\bar{u} = 0$ and if the Herschel condition of the pupils is satisfied as

$$\overline{W}_{131} = \frac{\text{Ж}}{8}\Delta\{\bar{u}^2\}, \tag{17}$$

then the relative illumination would follow to the second order $\cos^3(\theta')$ rule in a system, where the aperture vector is defined at the entrance pupil plane or at the intermediate pupil.[1] This follows since

$$\begin{aligned} RI(\vec{H}) &= 1 + \left[-2\bar{u}'^2 + \frac{4}{\text{Ж}}\overline{W}_{131}\right](\vec{H}\cdot\vec{H}) \\ &= 1 + \left[-2\bar{u}'^2 + \frac{1}{2}\bar{u}'^2\right](\vec{H}\cdot\vec{H}) \end{aligned} \tag{18}$$

and

$$\cos^3(\theta') = 1 - \frac{3}{2}\bar{u}'^2 + \cdots . \tag{19}$$

Fourth, contrary to intuition, pupil spherical aberration $\bar{W}_{040}$ may not influence the relative illumination when $W_{220} = W_{222} = 0$. However, pupil spherical aberration can change the angle of incidence of light on the image plane.

If all fourth-order irradiance terms are considered, it is possible to analytically calculate relative illumination more accurately for a lens system with a finite aperture. The relative illumination is obtained by the integration of the irradiance function over the exit pupil area as follows:

$$RI(\vec{H}) = \frac{1}{\pi}\iint_{\rho,\alpha} I(\vec{H},\vec{\rho})\rho \mathrm{d}\rho\, \mathrm{d}\alpha, \tag{20}$$

where $\alpha$ is the angle between the field vector $\vec{H}$ and aperture vector $\vec{\rho}$. The irradiance terms that are proportional to $(\vec{H}\cdot\vec{\rho})$ do not contribute to the relative illumination of a lens since the integral of $\cos(\alpha)$ over the circle is zero. The field-independent irradiance terms give a constant integral value, while other fourth-order irradiance terms modify second- and fourth-order relative illumination coefficients. Table 7 summarizes the integrated values of the irradiance terms over the exit pupil. It follows that the relative illumination $RI(\vec{H})$ for an optical system with a finite aperture can be written as follows:

$$\begin{aligned} RI(\vec{H}) = \frac{1}{1+\frac{1}{2}I_{020}+\frac{1}{3}I_{040}}\Bigg[1 + \left(I_{200} + \frac{1}{4}I_{222} + \frac{1}{2}I_{220}\right) \\ \times (\vec{H}\cdot\vec{H}) + I_{400}(\vec{H}\cdot\vec{H})^2\Bigg]. \end{aligned} \tag{21}$$

As an example, we analyze the image plane illumination of a mirror system in Fig. 4. This optical system is telecentric in the image space and is designed to satisfy the aplanatic condition between the pupils to the sixth order. As a consequence of aplanatic imaging between the pupils, the system suffers from residual image aberrations. The mirror system operates at $f/10$ and covers a field-of-view (FoV) of 20 deg.

**Table 7** Second and fourth-order irradiance coefficients integrated over the exit pupil area.

| Irradiance term | Integral value |
|---|---|
| $I_{020}(\vec{\rho}\cdot\vec{\rho})$ | $\frac{1}{2}I_{020}$ |
| $I_{111}(\vec{H}\cdot\vec{\rho})$ | 0 |
| $I_{200}(\vec{H}\cdot\vec{H})$ | $I_{200}H^2$ |
| $I_{040}(\vec{\rho}\cdot\vec{\rho})^2$ | $\frac{1}{3}I_{040}$ |
| $I_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ | 0 |
| $I_{222}(\vec{H}\cdot\vec{\rho})^2$ | $\frac{1}{4}I_{222}H^2$ |
| $I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ | $\frac{1}{2}I_{220}H^2$ |
| $I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | 0 |
| $I_{400}(\vec{H}\cdot\vec{H})^2$ | $I_{400}H^4$ |

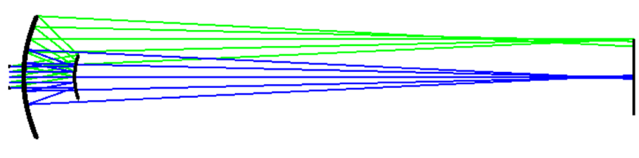

**Fig. 4** An image space telecentric mirror system. The mirror system is designed to satisfy the aplanatic condition between the pupils to six order.

**Table 8** Second and fourth-order irradiance coefficients of the mirror system in Fig. 4.

| Irradiance term | Value |
|---|---|
| $I_{020}(\vec{\rho}\cdot\vec{\rho})$ | 0.141052 |
| $I_{111}(\vec{H}\cdot\vec{\rho})$ | −0.132590 |
| $I_{200}(\vec{H}\cdot\vec{H})$ | 0.000000 |
| $I_{040}(\vec{\rho}\cdot\vec{\rho})^2$ | −0.008925 |
| $I_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ | 0.026928 |
| $I_{222}(\vec{H}\cdot\vec{\rho})^2$ | −0.011326 |
| $I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ | −0.016355 |
| $I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | 0.009353 |
| $I_{400}(\vec{H}\cdot\vec{H})^2$ | 0.000000 |

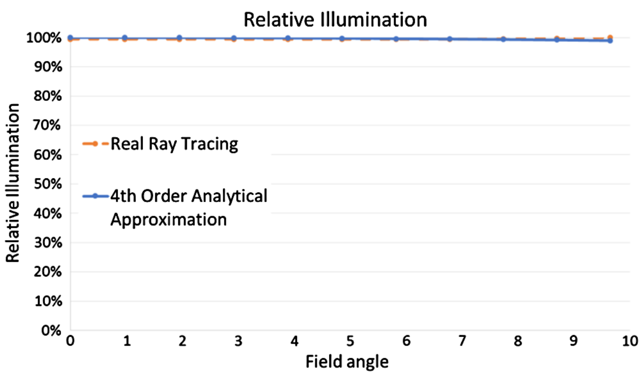


**Fig. 5** Relative illumination curves calculated with real ray tracing and with the fourth-order analytical approximation for the mirror system in Fig. 4 show excellent agreement over the entire FoV. The image is equally illuminated over the entire area.

The coefficients of the irradiance function for this mirror system are calculated and summarized in Table 8. The irradiance terms $I_{200}$ and $I_{400}$ are zero since $\bar{u}=\overline{W}_{131}=\overline{W}_{151}=\overline{W}_{040}=0$ by design. The relative illumination curves calculated with real ray tracing in Zemax OpticsStudio lens design software and with the fourth-order analytical approximation in Eq. (21) show excellent agreement over the entire FoV, as presented in Fig. 5. The image is equally illuminated over the entire area. More analysis examples of lenses that show improved image plane illumination can be found in our previous publications.[5]

## 8 Combination of Irradiance Coefficients

In the derivation of the irradiance coefficients, we assume that the object has a constant or Lambertian radiance. However, it may be desirable to determine the irradiance coefficients when the object has a different emission profile. In this case, the source radiance is expanded as a polynomial series of dot products of the field and aperture vectors and to the second order, for example, can be written as follows:

$$L(\vec{H},\vec{\rho})=L_0\cdot[1+A(\vec{\rho}\cdot\vec{\rho})+B(\vec{H}\cdot\vec{\rho})+C(\vec{H}\cdot\vec{H})], \tag{22}$$

where $A$, $B$, and $C$ are coefficients describing the emission profile. We substitute Eq. (22) into Eqs. (9) and (10) and write the irradiance coefficients to the second order as follows:

$$I(\vec{H},\vec{\rho})^{(2)}=1+[-2u'^2+A](\vec{\rho}\cdot\vec{\rho})+[-4u'\bar{u}'+B](\vec{H}\cdot\vec{\rho})+[-2\bar{u}'^2+C](\vec{H}\cdot\vec{H}). \tag{23}$$

Thus, second-order variations in the source radiance produce second-order variations in irradiance at the image plane. In addition, other higher-order irradiance terms, not presented here, result from different combinations of the source radiance terms with irradiance terms, as shown in Tables 1 and 2. These higher order terms can be considered as extrinsic irradiance aberrations that result from the interaction of the incoming irradiance variations and aberration in the system.

However, Eq. (23) shows that, with respect to the irradiance of a pinhole camera $I_{\mathrm{pinhole}}(\vec{H},\vec{\rho})$, second-order variations in the object space simply add to obtaining the irradiance in the image space.

## 9 Irradiance Coefficients and Choice of Coordinates

We have pointed out that the irradiance coefficients depend on the location of the field and aperture vectors. Tables 1–4 give the coefficients for the case of having the field vector at the object plane and the aperture vector at the exit pupil plane. This case is an important one, for a diffraction calculation knowledge of the amplitude of the field at the exit pupil is necessary; this amplitude is taken to be equal to the square root of the irradiance function $I(\vec{H},\vec{\rho})$. For completeness, we present in Appendices D and E the corresponding formulas for other cases on the location of the field and aperture vectors.

The derivation of irradiance coefficients requires several steps:

1. Choose the location of the field and aperture vectors and determine the relationships according to the flow chart in Fig. 6.
2. With those relationships, determine $\cos^4[\theta'(\vec{H}',\vec{\rho}')]$. This calculation is similar to Appendix B.
3. If not a constant, determine $dA'$ or $dS'$ by evaluating the Jacobian determinant. This calculation is similar to Appendix C.
4. Calculate the irradiance coefficients with Eqs. (6) or (7).

## 10 Coefficients Verification

To support the analytical derivation, the magnitude of the irradiance coefficients was determined both through the formulas derived and numerically. A macro program was written to calculate the irradiance coefficients by making an iterative fit to a selected set of irradiance values across the aperture and field of an optical system. The iterative algorithm is similar to one used by Sasian to fit aberration coefficients.[9]

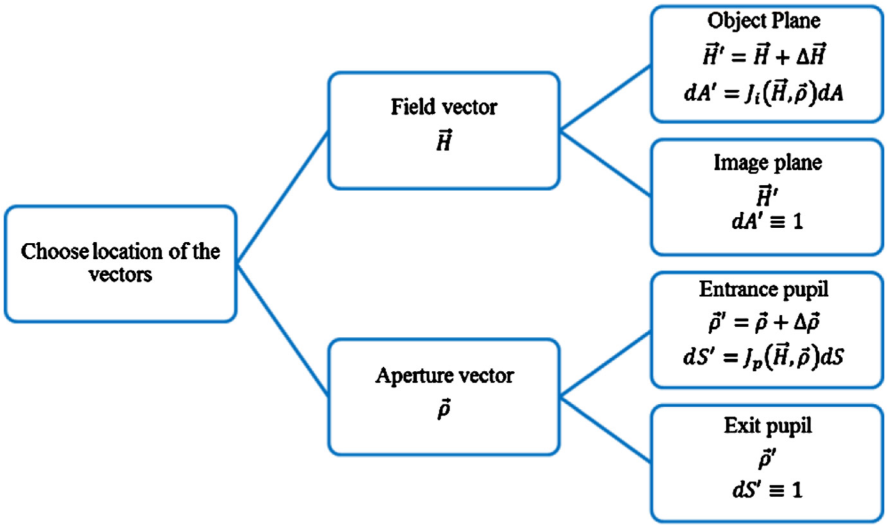


**Fig. 6** Flow chart that defines the relationships for different selections of the coordinates.

For example, for an optical system with the stop aperture at the entrance pupil, the routine in Table 9 was executed to find the magnitude of the irradiance coefficients $I_{020}(\vec{\rho}\cdot\vec{\rho})$ and $I_{040}(\vec{\rho}\cdot\vec{\rho})^2$. According to Eq. (7), the normalized irradiance at the specified field and aperture points is given by $\frac{dS'}{dS}\cdot\cos^4[\theta'(\vec{H},\vec{\rho})]$. The quantity $\frac{dS'}{dS}\cdot\cos^4[\theta'(\vec{H},\vec{\rho})]$ was computed in a lens design program by defining a small circle at the entrance pupil and tracing real rays to calculate the area of the corresponding ellipse at the exit pupil.

**Table 9** Iterative algorithm that was used to fit irradiance coefficients.

FOR $i = 1$ to 100

$\rho = 0.2$

$I_{\mathrm{real}} = \frac{dS}{dS'}\cos^4(\theta)\{0,\rho\}$

$I_{200} = (I_{\mathrm{real}} - 1 - I_{400}\rho^4 - I_{600}\rho^6 - I_{800}\rho^8 - I_{1000}\rho^{10})\cdot\rho^{-2}$

$\rho = 0.4$

$Ri = \frac{dS}{dS'}\cos^4(\theta)\{0,\rho\}$

$I_{400} = (I_{\mathrm{real}} - 1 - I_{200}\rho^2 - I_{600}\rho^6 - I_{800}\rho^8 - I_{1000}\rho^{10})\cdot\rho^{-4}$

$\rho = 0.6$

$Ri = \frac{dS}{dS'}\cos^4(\theta)\{0,\rho\}$

$I_{600} = (I_{\mathrm{real}} - 1 - I_{200}\rho^2 - I_{400}\rho^4 - I_{800}\rho^8 - I_{1000}\rho^{10})\cdot\rho^{-6}$

$\rho = 0.8$

$Ri = \frac{dS}{dS'}\cos^4(\theta)\{0,\rho\}$

$I_{800} = (I_{\mathrm{real}} - 1 - I_{200}\rho^2 - I_{400}\rho^4 - I_{600}\rho^6 - I_{1000}\rho^{10})\cdot\rho^{-8}$

$\rho = 1$

$Ri = \frac{dS}{dS'}\cos^4(\theta)\{0,\rho\}$

$I_{1000} = (I_{\mathrm{real}} - 1 - I_{200}\rho^2 - I_{400}\rho^4 - I_{600}\rho^6 - I_{800}\rho^8)\cdot\rho^{-10}$

NEXT

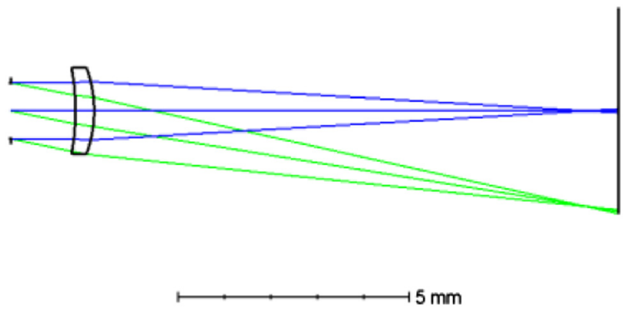


**Fig. 7** Layout of the Landscape lens used to compare irradiance coefficients computed analytically and numerically. To minimize fitting errors, the FoV of the lens is limited to 30 deg.

**Table 10** Comparison of irradiance coefficients computed analytically and numerically. The agreement to eighth digits supports the correctness of the formulas.

| Irradiance coefficient $I_{k,l,m}$ | Analytical formula | Numerical calculation |
|---|---|---|
| $I_{020}(\vec{\rho}\cdot\vec{\rho})$ | −0.0051310**721** | −0.0051310**695** |
| $I_{111}(\vec{H}\cdot\vec{\rho})$ | 0.00514131**31** | 0.00514131**22** |
| $I_{200}(\vec{H}\cdot\vec{H})$ | −0.0563954**333** | −0.0563954**267** |
| $I_{040}(\vec{\rho}\cdot\vec{\rho})^2$ | −0.0002767**939** | −0.0002768**392** |
| $I_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ | 0.0003447**530** | 0.0003447**656** |
| $I_{222}(\vec{H}\cdot\vec{\rho})^2$ | 0.0003958**170** | 0.0003958**224** |
| $I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ | 0.0004**320213** | 0.0004**318147** |
| $I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | −0.0001299**094** | −0.0001299**152** |
| $I_{400}(\vec{H}\cdot\vec{H})^2$ | 0.002860**3412** | 0.002860**2385** |

After a few iterations of the loop in Table 9, the coefficients $I_{020}(\vec{\rho}\cdot\vec{\rho})$ and $I_{040}(\vec{\rho}\cdot\vec{\rho})^2$ converged to the theoretical values with insignificant error. A similar approach was applied to validate the remaining irradiance coefficients. The iterative fit methodology was used to test the coefficients' values at several conjugate distances and aperture stop positions for both single surface and a system of several surfaces. The obvious agreement of the formulas with the coefficients found with the iterative fit supports the validity of the theory.

As an example, the irradiance coefficients for a Landscape lens[10] shown in Fig. 7 were calculated both with the analytical formulas and with real ray tracing. The lens operates at $f/8$ and the FoV is limited to 30 deg. Table 10 presents a comparison of coefficients where the differences in the eighth decimal place are due to numerical computation errors.

## 11 Conclusion

In this paper, we present a second- and a fourth-order theory of irradiance changes in axially symmetric optical systems. The concept of the irradiance function is presented and an interpretation of the irradiance aberrations is discussed. The irradiance function terms represent basic distribution patterns in the irradiance of a beam at the exit pupil plane or at the image plane of an imaging system.

The irradiance coefficients are found via basic radiometric principles, such as conservation of flux, and are derived from purely geometric considerations. This approach gives us the specific relationship between the irradiance distribution and wavefront aberration coefficients. The edge diffraction effects are not considered in this study, and unclipped and unfolded beams are assumed.

Tables 1–4 give the irradiance coefficients in terms of wavefront aberration coefficients and first-order system quantities to the fourth order of approximation. Specific formulas are provided for irradiance at the image and at the exit pupil of an optical system. The irradiance coefficients depend on the selection of coordinates. In this manuscript, the field vector is defined at the object plane, while the aperture vector is defined at the exit pupil plane. The formulas for the irradiance coefficients in Tables 1–4 show excellent agreement with the results from real ray tracing. We also have discussed relative illumination and several cases of interest where the coordinate position changes.

The theory of irradiance aberrations enhances our knowledge about the behavior of light as it propagates in optical systems and provides insights into how individual wavefront aberration terms affect the light irradiance produced by a lens system at its image plane or at the exit pupil plane.

## Appendix A: Series Expansion of the Cosine-to-the-Fourth-Power of the Ray Angle

By definition, the cosine-to-the-fourth-power of the angle between the ray connecting the point $(x'_s, y'_s)$ on the aperture plane and point $(x'_i, y'_i)$ on the image plane and the optical axis is as follows:

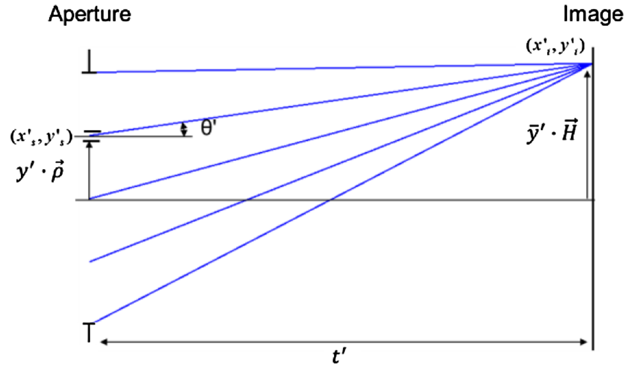


**Fig. 8** Geometrical variables involved in computing irradiance of a pinhole camera.

$$\cos^4(\theta') = \left\{\frac{t'}{[(x'_s - x'_i)^2 + (y'_s - y'_i)^2 + e'^2]^{1/2}}\right\}^4 = \left\{\frac{1}{\left[1 + \frac{(x'_s - x'_i)^2}{t'^2} + \frac{(y'_s - y'_i)^2}{t'^2}\right]^{1/2}}\right\}^4 = \frac{1}{\left[1 + \frac{(x'^2_s + y'^2_s)}{t'^2} + \frac{(x'^2_i + y'^2_i)}{t'^2} - \frac{2\cdot(x'_s x'_i + y'_s y'_i)}{t'^2}\right]^2}, \tag{24}$$

where $t'$ is the axial distance between the image plane and the aperture plane.

Points $(x'_s, y'_s)$ and $(x'_i, y'_i)$ are specified by the field $\vec{H}$ aperture $\vec{\rho}$ vectors, as in Fig. 8, and are expressed in terms of the first-order system quantities as follows:

$$(x_i^2 + y_i^2)^{1/2} = \bar{y}'\cdot\vec{H} \quad (x_s^2 + y_s^2)^{1/2} = y'\cdot\vec{\rho} \tag{25}$$

and

$$\bar{u}' = -\frac{\bar{y}'}{t'} \quad u' = \frac{y'}{t'}. \tag{26}$$

Here $\bar{y}'$ and $\bar{u}'$ are the chief ray height and slope at the image plane, respectively; $y'$ and $u'$ are the marginal ray height and slope at the aperture plane, respectively.

We substitute Eqs. (25) and (26) into Eq. (24) and write the first two terms of a Taylor series expression to obtain the fourth order approximation to the cosine-to-the-fourth-power of the ray angle as follows:

$$\begin{aligned}
&\cos^4[\theta'(\vec{H},\vec{\rho})] \\
&= \frac{1}{[1 + \bar{u}'^2(\vec{H}\cdot\vec{H}) + u'^2(\vec{\rho}\cdot\vec{\rho}) + 2\cdot u'\bar{u}'(\vec{H}\cdot\vec{\rho})]^2} \\
&= 1 - 2\cdot[\bar{u}'^2(\vec{H}\cdot\vec{H}) + u'^2(\vec{\rho}\cdot\vec{\rho}) + 2\cdot u'\bar{u}'(\vec{H}\cdot\vec{\rho})]\ldots \\
&\quad +3\cdot[\bar{u}'^2(\vec{H}\cdot\vec{H}) + u'^2(\vec{\rho}\cdot\vec{\rho}) + 2\cdot u'\bar{u}'(\vec{H}\cdot\vec{\rho})]^2 \\
&= 1 - 2\cdot\bar{u}'^2(\vec{H}\cdot\vec{H}) - 2\cdot u'^2(\vec{\rho}\cdot\vec{\rho}) - 4\cdot u'\bar{u}'(\vec{H}\cdot\vec{\rho})\ldots \\
&\quad +3\cdot\bar{u}'^4(\vec{H}\cdot\vec{H})^2 + 12\cdot u'^3\bar{u}'(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) \\
&\quad +12\cdot u'^2\bar{u}'^2(\vec{H}\cdot\vec{\rho})^2\ldots \\
&\quad +6\cdot u'^2\bar{u}'^2(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) \\
&\quad +12\cdot u'\bar{u}'^3(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) + 3\cdot u'^4(\vec{\rho}\cdot\vec{\rho})^2,
\end{aligned} \tag{27}$$

where $\theta'(\vec{H},\vec{\rho})$ in Eq. (27) indicates that the function is evaluated for a ray connecting a point defined by $\vec{\rho}$ on the aperture plane and a point defined by $\vec{H}$ on the image plane.

## Appendix B: Derivation of the Irradiance Function at the Image Plane of an Optical System with the Aperture Vector at the Exit Pupil

In an actual system, rays may not pass through the ideal image point due to aberrations, as shown in Fig. 9. The ray intersection with the image plane is determined by considering the transverse ray errors $\Delta\vec{H}$, and it is defined by the vector $\vec{H}+\Delta\vec{H}$.

In the presence of image aberrations, the cosine-to-the-fourth-power of the ray angle $\cos^4[\theta'(\vec{H}+\Delta\vec{H},\vec{\rho})]$ is evaluated by writing the differential as

$$\cos^4[\theta'(\vec{H}+\Delta\vec{H},\vec{\rho})] \simeq \cos^4[\theta'(\vec{H},\vec{\rho})] + \nabla_H \cos^4[\theta'(\vec{H},\vec{\rho})] \cdot \Delta\vec{H}, \tag{28}$$

where $\nabla_H \cos^4[\theta'(\vec{H},\vec{\rho})]$ is the gradient of the function in Eq. (27) with the respect to the field vector $\vec{H}$. With no second-order terms in the aberrations function, the terms $\nabla_H \cos^4[\theta'(\vec{H},\vec{\rho})]\cdot\Delta\vec{H}$ result in irradiance terms that are at least of the fourth order. Thompson has shown that the gradient operator is given by the derivative of the function with respect to the designated vector.[3] It follows that to the first order:

$$\nabla_H \cos^4[\theta'(\vec{H},\vec{\rho})] = [-4\cdot\bar{u}'^2\cdot H - 4\cdot u'\bar{u}'\cdot\vec{\rho})]. \tag{29}$$

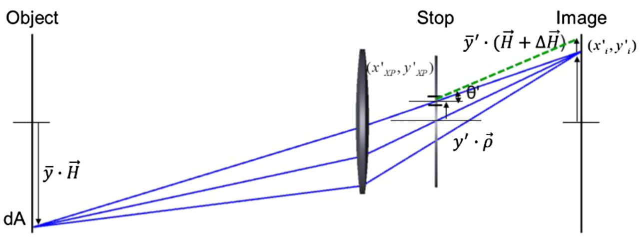


**Fig. 9** Geometrical variables involved in computing irradiance on the focal plane of an optical system with the aperture stop at the exit pupil. Real rays (solid lines) and first-order rays (dashed lines) coincide at the exit pupil. Real rays may not pass through the ideal image point due to aberrations.

Finally, the terms $\nabla_H \cos^4[\theta'(\vec{H},\vec{\rho})]\cdot\Delta\vec{H}$ are summarized in Table 11.

## Appendix C: Derivation of the Irradiance Function at the Exit Pupil of an Optical System with the Aperture Vector at the Exit Pupil

To obtain the Jacobian determinant, we express the transverse ray aberration vector $\Delta\vec{H}$ in orthogonal components along unit vectors $\hat{\rho}$ and $\hat{k}$ as follows:

$$\Delta\vec{H} = \Delta H_\rho \hat{\rho} + \Delta H_k \hat{k}, \tag{30}$$

where $\hat{\rho}$ is a unit vector along the aperture vector $\vec{\rho}$ and $\hat{k}$ is a unit vector perpendicular to the aperture vector $\vec{\rho}$, as shown in Table 12.

**Table 12** Third-order transverse ray aberrations in the orthogonal coordinates.

| Ray aberration $\Delta\vec{H}$ | $\hat{\rho}$ | $\hat{k}$ |
|---|---|---|
| $\Delta\vec{H}_{040}$ | $4\rho^3$ | |
| $\Delta\vec{H}_{131}$ | $3\rho^2 H_\rho$ | $\rho^2 H_k$ |
| $\Delta\vec{H}_{222}$ | $2\rho H_\rho^2$ | $2\rho H_\rho H_k$ |
| $\Delta\vec{H}_{220}$ | $2\rho(H_\rho^2+H_k^2)$ | |
| $\Delta\vec{H}_{311}$ | $(H_\rho^2+H_k^2)H_\rho$ | $(H_\rho^2+H_k^2)H_k$ |
| $\Delta\vec{H}_{400}$ | | |

**Table 13** Third-order transverse ray aberrations derivatives in orthogonal coordinates.

| Ray aberration $\Delta\vec{H}$ | $\partial_{H_\rho}\Delta\vec{H}_\rho$ | $\partial_{H_k}\Delta\vec{H}_k$ | $\partial_{H_k}\Delta\vec{H}_\rho$ | $\partial_{H_\rho}\Delta\vec{H}_k$ |
|---|---|---|---|---|
| $\Delta\vec{H}_{040}$ | | | | |
| $\Delta\vec{H}_{131}$ | $3\rho^2$ | $\rho^2$ | | |
| $\Delta\vec{H}_{222}$ | $4\rho H_\rho$ | $2\rho H_\rho$ | | $2\rho H_k$ |
| $\Delta\vec{H}_{220}$ | $4\rho H_\rho$ | | $4\rho H_k$ | |
| $\Delta\vec{H}_{311}$ | $(3H_\rho^2+H_k^2)$ | $(H_\rho^2+3H_k^2)$ | $2H_\rho H_k$ | $2H_\rho H_k$ |
| $\Delta\vec{H}_{400}$ | | | | |

**Table 11** Summary of contributions to irradiance from the transverse ray errors $\Delta\vec{H}$.

| Ray aberration $\Delta\vec{H}$ | Correction term $-4\bar{u}'^2(\vec{H}\cdot\Delta\vec{H})$ | Correction term $-4u'\bar{u}'(\Delta\vec{H}\cdot\vec{\rho})$ |
|---|---|---|
| $-\frac{4}{\text{Ж}}W_{040}(\vec{\rho}\cdot\vec{\rho})\vec{\rho}$ | $\frac{16}{\text{Ж}}W_{040}\bar{u}'^2(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ | $\frac{16}{\text{Ж}}W_{040}u'\bar{u}'(\vec{\rho}\cdot\vec{\rho})^2$ |
| $-\frac{2}{\text{Ж}}W_{131}(\vec{H}\cdot\vec{\rho})\vec{\rho}$ | $\frac{8}{\text{Ж}}W_{131}\bar{u}'^2(\vec{H}\cdot\vec{\rho})^2$ | $\frac{8}{\text{Ж}}W_{131}u'\bar{u}'(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ |
| $-\frac{1}{\text{Ж}}W_{131}(\vec{\rho}\cdot\vec{\rho})\vec{H}$ | $\frac{4}{\text{Ж}}W_{131}\bar{u}'^2(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ | $\frac{4}{\text{Ж}}W_{131}u'\bar{u}'(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ |
| $-\frac{2}{\text{Ж}}W_{222}(\vec{H}\cdot\vec{\rho})\vec{H}$ | $\frac{8}{\text{Ж}}W_{222}\bar{u}'^2(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | $\frac{8}{\text{Ж}}W_{222}u'\bar{u}'(\vec{H}\cdot\vec{\rho})^2$ |
| $-\frac{2}{\text{Ж}}W_{220}(\vec{H}\cdot\vec{H})\vec{\rho}$ | $\frac{8}{\text{Ж}}W_{220}\bar{u}'^2(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | $\frac{8}{\text{Ж}}W_{220}u'\bar{u}'(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ |
| $-\frac{1}{\text{Ж}}W_{311}(\vec{H}\cdot\vec{H})\vec{H}$ | $\frac{4}{\text{Ж}}W_{311}\bar{u}'^2(\vec{H}\cdot\vec{H})^2$ | $\frac{4}{\text{Ж}}W_{311}u'\bar{u}'(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ |

**Table 14** Fifth-order transverse ray aberrations for an optical system with the stop aperture at the exit pupil.

| Ray aberration $\Delta\vec{H}$ | $-\frac{1}{Ж}\vec{\nabla}_\rho W^6(\vec{H},\vec{\rho})$ | Terms $O^{(5)}$ |
|---|---|---|
| $\Delta\vec{H}_{060\rho}(\vec{\rho}\cdot\vec{\rho})^2\vec{\rho}$ | $-\frac{6}{Ж}W_{060}$ | $-\frac{6}{Ж}W_{040}u'^2$ |
| $\Delta\vec{H}_{151H}(\vec{\rho}\cdot\vec{\rho})^2\vec{H}$ | $-\frac{1}{Ж}W_{151}$ | $-\frac{1}{Ж}\left[\frac{1}{2}\cdot W_{131}u'^2+4\cdot W_{040}u'\bar{u}'\right]$ |
| $\Delta\vec{H}_{151\rho}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})\vec{\rho}$ | $-\frac{4}{Ж}W_{151}$ | $-\frac{1}{Ж}[4\cdot W_{131}u'^2+8\cdot W_{040}u'\bar{u}']$ |
| $\Delta\vec{H}_{333H}(\vec{H}\cdot\vec{\rho})^2\vec{H}$ | $-\frac{3}{Ж}W_{333}$ | $-\frac{1}{Ж}[2\cdot W_{131}\bar{u}'^2+4\cdot W_{222}u'\bar{u}']$ |
| $\Delta\vec{H}_{331H}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})\vec{H}$ | $-\frac{1}{Ж}W_{331}$ | $-\frac{1}{Ж}\left[2\cdot W_{220}u'\bar{u}'+\frac{1}{2}\cdot W_{311}u'^2+\frac{3}{2}\cdot W_{131}\bar{u}'^2\right]$ |
| $\Delta\vec{H}_{331\rho}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})\vec{\rho}$ | $-\frac{2}{Ж}W_{331}$ | $-\frac{1}{Ж}[2\cdot W_{222}u'\bar{u}'+W_{131}\bar{u}'^2+4\cdot W_{220}u'\bar{u}'+W_{311}u'^2]$; |
| $\Delta\vec{H}_{242\rho}(\vec{H}\cdot\vec{\rho})^2\vec{\rho}$ | $-\frac{2}{Ж}W_{242}$ | $-\frac{1}{Ж}[2\cdot W_{222}u'^2+4\cdot W_{131}u'\bar{u}']$ |
| $\Delta\vec{H}_{242H}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})\vec{H}$ | $-\frac{2}{Ж}W_{242}$ | $-\frac{1}{Ж}[4\cdot W_{040}\bar{u}'^2+4\cdot W_{131}u'\bar{u}'^2+W_{222}u'^2]$ |
| $\Delta\vec{H}_{240H}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})\vec{\rho}$ | $-\frac{4}{Ж}W_{240}$ | $-\frac{1}{Ж}[3\cdot W_{220}u'^2+W_{131}u'\bar{u}'+2\cdot W_{040}\bar{u}'^2]$ |
| $\Delta\vec{H}_{422H}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})\vec{H}$ | $-\frac{2}{Ж}W_{422}$ | $-\frac{1}{Ж}[3\cdot W_{222}\bar{u}'^2+2\cdot W_{220}\bar{u}'^2+2\cdot W_{311}u'\bar{u}']$ |
| $\Delta\vec{H}_{420\rho}(\vec{H}\cdot\vec{H})^2\vec{\rho}$ | $-\frac{2}{Ж}W_{420}$ | $-\frac{1}{Ж}[W_{220}\bar{u}'^2+W_{311}u'\bar{u}']$ |
| $\Delta\vec{H}_{511H}(\vec{H}\cdot\vec{H})^2\vec{H}$ | $-\frac{1}{Ж}W_{511}$ | $-\frac{1}{Ж}\frac{3}{2}\cdot W_{311}\bar{u}'^2$ |

**Table 15** Terms corresponding to the determinant of the Jacobian transformation between the object and image planes for an optical system with the stop aperture at the exit pupil.

| Irradiance coefficient $J_{k,l,m}$ | Fourth-order aberration contributions | Six-order aberration contributions | Orthogonal derivative contributions |
|---|---|---|---|
| $J_{020}(\vec{\rho}\cdot\vec{\rho})$ | $-\frac{4}{Ж}W_{131}$ | | |
| $J_{111}(\vec{H}\cdot\vec{\rho})$ | $-\frac{4}{Ж}W_{220}$ | | |
| | $-\frac{6}{Ж}W_{222}$ | | |
| $J_{200}(\vec{H}\cdot\vec{H})$ | $-\frac{4}{Ж}W_{311}$ | | |
| $J_{040}(\vec{\rho}\cdot\vec{\rho})^2$ | $-\frac{1}{Ж}[5\cdot W_{131}u'^2+16\cdot W_{040}u'\bar{u}']$ | $-\frac{6}{Ж}W_{151}$ | $\frac{3}{Ж^2}W_{131}W_{131}$ |
| $J_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})$ | $-\frac{1}{Ж}[7\cdot W_{222}u'^2+6\cdot W_{220}u'^2]$ | $-\frac{10}{Ж}W_{242}$ | $\frac{4}{Ж^2}W_{220}W_{131}$ |
| | $-\frac{1}{Ж}[22\cdot W_{131}u'\bar{u}'-16\cdot W_{040}\bar{u}'^2]$ | $-\frac{8}{Ж}W_{240}$ | $\frac{10}{Ж^2}W_{222}W_{131}$ |
| $J_{222}(\vec{H}\cdot\vec{\rho})^2$ | $-\frac{1}{Ж}[10\cdot W_{131}\bar{u}'^2+20\cdot W_{222}u'\bar{u}']$ | $-\frac{12}{Ж}W_{333}$ | $\frac{8}{Ж^2}W_{222}W_{222}$ |
| | $-\frac{1}{Ж}[8\cdot W_{220}u'\bar{u}'+2\cdot W_{311}u'^2]$ | $-\frac{4}{Ж}W_{331}$ | $\frac{16}{Ж^2}W_{222}W_{220}$ |
| | | | $-\frac{4}{Ж^2}W_{311}W_{131}$ |
| $J_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})$ | $-\frac{1}{Ж}[7\cdot W_{131}\bar{u}'^2+2\cdot W_{222}u'\bar{u}']$ | $-\frac{6}{Ж}W_{331}$ | $\frac{10}{Ж^2}W_{311}W_{131}$ |
| | $-\frac{1}{Ж}[12\cdot W_{220}u'\bar{u}'+3\cdot W_{311}u'^2]$ | | $-\frac{8}{Ж^2}W_{222}W_{220}$ |
| $J_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})$ | $-\frac{1}{Ж}[15\cdot W_{222}\bar{u}'^2+14\cdot W_{220}\bar{u}'^2]$ | $-\frac{10}{Ж}W_{422}$ | $\frac{4}{Ж^2}W_{220}W_{311}$ |
| | $-\frac{14}{Ж}W_{311}u'\bar{u}'$ | $-\frac{8}{Ж}W_{420}$ | $\frac{10}{Ж^2}W_{222}W_{311}$ |
| $J_{400}(\vec{H}\cdot\vec{H})^2$ | $-\frac{9}{Ж}W_{311}\bar{u}'^2$ | $-\frac{6}{Ж}W_{511}$ | $\frac{3}{Ж^2}W_{311}W_{311}$ |

Then, we consider the transformations $H'_\rho = H_\rho + \Delta H_\rho$ and $H'_k = H_k + \Delta H_k$, which gives the position of the given ray at the image pupil, so we find the Jacobian determinant:

$$J(\vec{H},\vec{\rho}) = \frac{\bar{y}'^2}{\bar{y}^2}\left(1+\nabla_H\Delta\vec{H}+\frac{\partial\Delta\vec{H}_\rho}{\partial\vec{H}_\rho}\frac{\partial\Delta\vec{H}_k}{\partial\vec{H}_k}-\frac{\partial\Delta\vec{H}_\rho}{\partial\vec{H}_k}\frac{\partial\Delta\vec{H}_k}{\partial\vec{H}_\rho}\right), \tag{31}$$

where $\nabla_H \Delta\vec{H}$ is the divergence of the transverse ray error vector, and $\frac{\partial \Delta\vec{H}_\rho}{\partial \vec{H}_\rho}$, $\frac{\partial \Delta\vec{H}_k}{\partial \vec{H}_k}$, $\frac{\partial \Delta\vec{H}_\rho}{\partial \vec{H}_k}$ and $\frac{\partial \Delta\vec{H}_k}{\partial \vec{H}_\rho}$ are derivatives of the transverse ray error vector in the orthogonal coordinates.[4,9]

With no second order terms in the aberration function, the terms $\frac{\partial \Delta\vec{H}_\rho}{\partial \vec{H}_\rho}\frac{\partial \Delta\vec{H}_k}{\partial \vec{H}_k}$ and $-\frac{\partial \Delta\vec{H}_\rho}{\partial \vec{H}_k}\frac{\partial \Delta\vec{H}_k}{\partial \vec{H}_\rho}$ result in irradiance terms that are at least of fourth order. The derivatives of the third-order transverse ray aberrations in orthogonal coordinates are summarized in Table 13.

The relationships between wavefront deformations and transverse ray aberrations to fifth order are given in Table 14. The fifth-order transverse ray errors are related to the wavefront deformation by the gradient of the aberration function and include additional terms that are products of the fourth-order pupil aberration coefficients and paraxial ray slopes $\bar{u}'$ and $u'$ in the image space. The subscripts $H$ and $\rho$ refer to the components along the field and aperture vectors, respectively.[1]

The divergence of the transverse ray errors $\nabla_H \Delta\vec{H}$ results in terms that are at least of the second order. Sasian has shown the procedure to calculate the divergence operator of the function with respect to the designated vector.[9,11] Following this procedure and taking the required derivatives, we find the terms in Table 15.

The Jacobian coefficients in Table 15 are the sum of all components. Thus, for example, the term $J_{400}(\vec{H}\cdot\vec{H})^2$ is given by

$$J_{400} = \left[-\frac{6}{\text{Ж}}W_{511} - \frac{9}{\text{Ж}}W_{311}\bar{u}'^2 + \frac{3}{\text{Ж}^2}W_{311}W_{311}\right](\vec{H}\cdot\vec{H})^2. \tag{32}$$

Finally, the coefficients in Tables 3 and 4 are obtained by combining equations in Tables 1, 2, and 15, and keeping only second- and fourth-order terms.

## Appendix D: Irradiance Coefficients $I_{k,l,m}$ at the Image Plane of an Optical System with the Field Vector at the Object Plane and the Aperture Vector at the Entrance Pupil Plane

Tables 16 and 17 give the image plane irradiance coefficients for the case of having the field vector at the object plane and the aperture vector at the entrance pupil plane.

**Table 16** Second-order irradiance coefficients $I^{(2)}_{k,l,m}$ at the image plane of an optical system with the aperture vector at the entrance pupil.

$$I_{020}(\vec{\rho}\cdot\vec{\rho}) = \left[-2u'^2 + \tfrac{4}{\text{Ж}}\overline{W}_{311}\right](\vec{\rho}\cdot\vec{\rho})$$

$$I_{111}(\vec{H}\cdot\vec{\rho}) = \left[-4u'\bar{u}' + \tfrac{4}{\text{Ж}}\overline{W}_{220} + \tfrac{6}{\text{Ж}}\overline{W}_{222}\right](\vec{H}\cdot\vec{\rho})$$

$$I_{200}(\vec{H}\cdot\vec{H}) = \left[-2\bar{u}'^2 + \tfrac{4}{\text{Ж}}\overline{W}_{131}\right](\vec{H}\cdot\vec{H})$$

**Table 17** Fourth-order irradiance coefficients $I^{(4)}_{k,l,m}$ at the image plane of an optical system with the aperture vector at the entrance pupil.

$$\begin{aligned}
I_{040}(\vec{\rho}\cdot\vec{\rho})^2 = \Big[&3u'^4 + \tfrac{16}{\text{Ж}}W_{040}u'\bar{u}' + \tfrac{6}{\text{Ж}}\overline{W}_{511} - \tfrac{3}{\text{Ж}}\overline{W}_{311}\cdot u'^2 \\
&- \tfrac{48}{\text{Ж}^2}\cdot\overline{W}_{222}W_{040} - \tfrac{48}{\text{Ж}^2}\cdot\overline{W}_{220}W_{040} + \tfrac{3}{\text{Ж}^2}\overline{W}_{311}\overline{W}_{311}\Big](\vec{\rho}\cdot\vec{\rho})^2
\end{aligned}$$

$$\begin{aligned}
I_{131}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) = \Big[&12u'^3\bar{u}' + \tfrac{16}{\text{Ж}}W_{040}\bar{u}'^2 + \tfrac{12}{\text{Ж}}W_{131}u'\bar{u}' \\
&+ \tfrac{10}{\text{Ж}}\overline{W}_{422} + \tfrac{8}{\text{Ж}}\overline{W}_{420} - \tfrac{5}{\text{Ж}}\overline{W}_{222}\cdot u'^2 - \tfrac{2}{\text{Ж}}\overline{W}_{220}\cdot u'^2 \\
&- \tfrac{6}{\text{Ж}}\overline{W}_{311}\cdot u'\bar{u}' - \tfrac{112}{\text{Ж}^2}\overline{W}_{131}W_{040} - \tfrac{28}{\text{Ж}}\overline{W}_{220}W_{131} \\
&- \tfrac{30}{\text{Ж}^2}\bar{W}_{222}W_{131} + \tfrac{4}{\text{Ж}^2}\overline{W}_{220}\overline{W}_{311} + \tfrac{10}{\text{Ж}^2}\overline{W}_{222}\overline{W}_{311}\Big](\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})
\end{aligned}$$

$$\begin{aligned}
I_{222}(\vec{H}\cdot\vec{\rho})^2 = \Big[&12u'^2\bar{u}'^2 + \tfrac{8}{\text{Ж}}W_{131}\bar{u}'^2 + \tfrac{8}{\text{Ж}}W_{222}u'\bar{u}' \\
&+ \tfrac{12}{\text{Ж}}\overline{W}_{333} + \tfrac{4}{\text{Ж}}\overline{W}_{331} + \tfrac{2}{\text{Ж}}\overline{W}_{131}\cdot u^2 - \tfrac{12}{\text{Ж}}\overline{W}_{222}\cdot u\bar{u} \\
&- \tfrac{8}{\text{Ж}}\overline{W}_{220}\cdot u\bar{u} + \tfrac{2}{\text{Ж}}\overline{W}_{311}\cdot\bar{u}^2 - \tfrac{48}{\text{Ж}^2}\overline{W}_{131}W_{131} \\
&- \tfrac{16}{\text{Ж}^2}\overline{W}_{222}W_{222} - \tfrac{8}{\text{Ж}^2}\overline{W}_{220}W_{222} - \tfrac{64}{\text{Ж}^2}\overline{W}_{040}W_{040} \\
&+ \tfrac{8}{\text{Ж}^2}\overline{W}_{222}\overline{W}_{222} + \tfrac{16}{\text{Ж}^2}\overline{W}_{222}\overline{W}_{220} - \tfrac{4}{\text{Ж}^2}\overline{W}_{311}\overline{W}_{131}\Big](\vec{H}\cdot\vec{\rho})^2
\end{aligned}$$

$$\begin{aligned}
I_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) = \Big[&6u'^2\bar{u}'^2 + \tfrac{4}{\text{Ж}}W_{131}\bar{u}'^2 + \tfrac{8}{\text{Ж}}W_{220}u'\bar{u}' \\
&+ \tfrac{6}{\text{Ж}}\overline{W}_{331} + \tfrac{2}{\text{Ж}}\overline{W}_{222}\cdot\bar{u}'^2 + \tfrac{4}{\text{Ж}}\overline{W}_{220}\cdot u'\bar{u}' - \tfrac{5}{\text{Ж}}\overline{W}_{311}\cdot\bar{u}'^2 \\
&- \tfrac{5}{\text{Ж}}\overline{W}_{131}\cdot u'^2 - \tfrac{96}{\text{Ж}^2}\overline{W}_{040}W_{040} - \tfrac{16}{\text{Ж}^2}\overline{W}_{131}W_{131} \\
&- \tfrac{16}{\text{Ж}^2}\overline{W}_{222}W_{220} - \tfrac{16}{\text{Ж}^2}\overline{W}_{220}W_{220} - \tfrac{4}{\text{Ж}^2}\overline{W}_{220}W_{222} \\
&+ \tfrac{10}{\text{Ж}^2}\overline{W}_{131}\overline{W}_{311} - \tfrac{8}{\text{Ж}^2}\overline{W}_{220}\overline{W}_{222}\Big](\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})
\end{aligned}$$

$$\begin{aligned}
I_{311}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) = \Big[&12u'\bar{u}'^3 + \tfrac{8}{\text{Ж}}W_{222}\bar{u}'^2 + \tfrac{8}{\text{Ж}}W_{220}\bar{u}'^2 \\
&+ \tfrac{4}{\text{Ж}}W_{311}u'\bar{u}' + \tfrac{10}{\text{Ж}}\overline{W}_{242} + \tfrac{8}{\text{Ж}}\overline{W}_{240} \\
&- \tfrac{5}{\text{Ж}}\overline{W}_{222}\cdot\bar{u}'^2 - \tfrac{2}{\text{Ж}}\overline{W}_{220}\cdot\bar{u}'^2 - \tfrac{6}{\text{Ж}}\overline{W}_{131}\cdot u'\bar{u}' \\
&- \tfrac{80}{\text{Ж}^2}\overline{W}_{040}W_{131} - \tfrac{28}{\text{Ж}^2}\overline{W}_{131}W_{222} - \tfrac{32}{\text{Ж}^2}\overline{W}_{131}W_{220} \\
&- \tfrac{6}{\text{Ж}^2}\overline{W}_{222}W_{311} - \tfrac{4}{\text{Ж}^2}\overline{W}_{220}W_{311} + \tfrac{4}{\text{Ж}^2}\overline{W}_{220}\overline{W}_{131} \\
&+ \tfrac{10}{\text{Ж}^2}\overline{W}_{222}\overline{W}_{131}\Big](\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})
\end{aligned}$$

$$\begin{aligned}
I_{400}(\vec{H}\cdot\vec{H})^2 = \Big[&3\bar{u}'^4 + \tfrac{4}{\text{Ж}}W_{311}\bar{u}'^2 + \tfrac{6}{\text{Ж}}\overline{W}_{151} - \tfrac{3}{\text{Ж}}\overline{W}_{131}\cdot\bar{u}'^2\ldots \\
&- \tfrac{32}{\text{Ж}^2}\overline{W}_{040}W_{220} - \tfrac{24}{\text{Ж}^2}\overline{W}_{040}W_{222} - \tfrac{8}{\text{Ж}^2}\overline{W}_{131}W_{311} \\
&+ \tfrac{3}{\text{Ж}^2}\overline{W}_{131}\overline{W}_{131}\Big](\vec{H}\cdot\vec{H})^2.
\end{aligned}$$

## Appendix E: Irradiance Coefficients $I_{k,l,m}$ at the Exit Pupil Plane of an Optical System with the Field Vector at the Object Plane and the Aperture Vector at the Entrance Pupil Plane

Tables 18 and 19 give the exit pupil plane irradiance coefficients for the case of having the field vector at the object plane and the aperture vector at the entrance pupil plane.

**Table 18** Second-order irradiance coefficients $\bar{I}^{(2)}_{k,l,m}$ at the exit pupil plane of an optical system with the aperture vector at the entrance pupil.

$$\bar{I}_{020}(\vec{H}\cdot\vec{H}) = \left[-2\bar{u}'^2 - \tfrac{4}{\text{Ж}}W_{311}\right](\vec{H}\cdot\vec{H})$$

$$\bar{I}_{111}(\vec{H}\cdot\vec{\rho}) = \left[-4u'\bar{u}' - \tfrac{4}{\text{Ж}}W_{220} - \tfrac{6}{\text{Ж}}W_{222}\right](\vec{H}\cdot\vec{\rho})$$

$$\bar{I}_{200}(\vec{\rho}\cdot\vec{\rho}) = \left[-2u'^2 - \tfrac{4}{\text{Ж}}W_{131}\right](\vec{\rho}\cdot\vec{\rho})$$

**Table 19** Fourth-order irradiance coefficients $\bar{I}^{(4)}_{k,l,m}$ at the exit pupil plane of an optical system with the aperture vector at the entrance pupil.

$$\begin{aligned}\bar{I}_{040}(\vec{H}\cdot\vec{H})^2 = \Big[&3\bar{u}^4 - \tfrac{6}{\text{Ж}}W_{511} + \tfrac{3}{\text{Ж}}W_{311}\bar{u}'^2 - \tfrac{16}{\text{Ж}}\overline{W}_{040}u'\bar{u}'\\ &- \tfrac{48}{\text{Ж}^2}\cdot W_{222}\overline{W}_{040} - \tfrac{48}{\text{Ж}^2}\cdot W_{220}\overline{W}_{040} + \tfrac{3}{\text{Ж}^2}W_{311}W_{311}\Big](\vec{H}\cdot\vec{H})^2\end{aligned}$$

$$\begin{aligned}\bar{I}_{131}(\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho}) = \Big[&12u'\bar{u}'^3 - \tfrac{10}{\text{Ж}}W_{422} - \tfrac{8}{\text{Ж}}W_{420}\\ &+ \tfrac{5}{\text{Ж}}W_{222}\bar{u}'^2 + \tfrac{2}{\text{Ж}}W_{220}\bar{u}'^2 + \tfrac{6}{\text{Ж}}W_{311}u'\bar{u}' - \tfrac{16}{\text{Ж}}\overline{W}_{040}u'^2\\ &- \tfrac{12}{\text{Ж}}\overline{W}_{131}u'\bar{u}' - \tfrac{112}{\text{Ж}^2}W_{131}\overline{W}_{040} - \tfrac{28}{\text{Ж}^2}W_{220}\overline{W}_{131} - \tfrac{30}{\text{Ж}^2}W_{222}\overline{W}_{131}\\ &+ \tfrac{4}{\text{Ж}^2}W_{220}W_{311} + \tfrac{10}{\text{Ж}^2}W_{222}W_{311}\Big](\vec{H}\cdot\vec{H})(\vec{H}\cdot\vec{\rho})\end{aligned}$$

$$\begin{aligned}\bar{I}_{222}(\vec{H}\cdot\vec{\rho})^2 = \Big[&12u'^2\bar{u}'^2 - \tfrac{12}{\text{Ж}}W_{333} - \tfrac{4}{\text{Ж}}W_{331} - \tfrac{2}{\text{Ж}}W_{131}\bar{u}'^2\\ &+ \tfrac{12}{\text{Ж}}W_{222}u'\bar{u}' + \tfrac{8}{\text{Ж}}W_{220}u'\bar{u}' - \tfrac{2}{\text{Ж}}W_{311}u'^2 - \tfrac{8}{\text{Ж}}\overline{W}_{131}u'^2\\ &- \tfrac{8}{\text{Ж}}\overline{W}_{222}u'\bar{u}' - \tfrac{48}{\text{Ж}^2}W_{131}\overline{W}_{131} - \tfrac{16}{\text{Ж}^2}W_{222}\overline{W}_{222}\\ &- \tfrac{8}{\text{Ж}^2}W_{220}\overline{W}_{222} - \tfrac{64}{\text{Ж}^2}W_{040}\overline{W}_{040} + \tfrac{8}{\text{Ж}^2}W_{222}W_{222}\\ &+ \tfrac{16}{\text{Ж}^2}W_{222}W_{220} - \tfrac{4}{\text{Ж}^2}W_{311}W_{131}\Big](\vec{H}\cdot\vec{\rho})^2\end{aligned}$$

$$\begin{aligned}\bar{I}_{220}(\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho}) = \Big[&6u'^2\bar{u}'^2 - \tfrac{6}{\text{Ж}}W_{331} + \tfrac{5}{\text{Ж}}W_{131}\bar{u}'^2\\ &- \tfrac{2}{\text{Ж}}W_{222}u'\bar{u}' - \tfrac{4}{\text{Ж}}W_{220}u'\bar{u}' + \tfrac{5}{\text{Ж}}W_{311}u'^2 - \tfrac{4}{\text{Ж}}\overline{W}_{131}u'^2\\ &- \tfrac{8}{\text{Ж}}\overline{W}_{220}u'\bar{u}' - \tfrac{96}{\text{Ж}^2}W_{040}\overline{W}_{040} - \tfrac{16}{\text{Ж}^2}W_{131}\overline{W}_{131}\\ &- \tfrac{16}{\text{Ж}^2}W_{222}\overline{W}_{220} - \tfrac{16}{\text{Ж}^2}W_{220}\overline{W}_{220} - \tfrac{4}{\text{Ж}^2}W_{220}\overline{W}_{222}\\ &+ \tfrac{10}{\text{Ж}^2}W_{311}W_{131} - \tfrac{8}{\text{Ж}^2}W_{222}W_{220}\Big](\vec{H}\cdot\vec{H})(\vec{\rho}\cdot\vec{\rho})\end{aligned}$$

$$\begin{aligned}\bar{I}_{311}(\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho}) = \Big[&12u'^3\bar{u}' - \tfrac{10}{\text{Ж}}W_{242}\\ &- \tfrac{8}{\text{Ж}}W_{240} + \tfrac{5}{\text{Ж}}W_{222}u'^2 + \tfrac{2}{\text{Ж}}W_{220}u'^2 + \tfrac{6}{\text{Ж}}W_{131}u'\bar{u}'\\ &- \tfrac{8}{\text{Ж}}\overline{W}_{222}u'^2 - \tfrac{8}{\text{Ж}}\overline{W}_{220}u'^2 - \tfrac{4}{\text{Ж}}\overline{W}_{311}u'\bar{u}'\\ &- \tfrac{80}{\text{Ж}^2}W_{040}\overline{W}_{131} - \tfrac{28}{\text{Ж}^2}W_{131}\overline{W}_{222} - \tfrac{32}{\text{Ж}^2}W_{131}\overline{W}_{220}\\ &- \tfrac{6}{\text{Ж}^2}W_{222}\overline{W}_{311} - \tfrac{4}{\text{Ж}^2}W_{220}\overline{W}_{311}\\ &+ \tfrac{4}{\text{Ж}^2}W_{220}W_{131} + \tfrac{10}{\text{Ж}^2}W_{222}W_{131}\Big](\vec{H}\cdot\vec{\rho})(\vec{\rho}\cdot\vec{\rho})\end{aligned}$$

$$\begin{aligned}\bar{I}_{400}(\vec{\rho}\cdot\vec{\rho})^2 = \Big[&3u'^4 - \tfrac{6}{\text{Ж}}W_{151} + \tfrac{3}{\text{Ж}}W_{131}u'^2\\ &- \tfrac{4}{\text{Ж}}\overline{W}_{311}u'^2 - \tfrac{32}{\text{Ж}^2}W_{040}\overline{W}_{220} - \tfrac{24}{\text{Ж}^2}W_{040}\overline{W}_{222}\\ &- \tfrac{8}{\text{Ж}^2}W_{131}\overline{W}_{311} + \tfrac{3}{\text{Ж}^2}W_{131}W_{131}\Big](\vec{\rho}\cdot\vec{\rho})^2\end{aligned}$$

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