About the third order expansion that describes the path of a light beam outside the plane of the axis through an optical system of refracting elements

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There is no doubt that Seidel aberrations are essential to be understood by anybody doing classical optics and therefore many optics books introduce them. From that point of view there is no need to translate the original paper. However, this translation targets people who are interested in the optics history and the way authors were writing, formulating their chain of thoughts, in a time where standard terminology in a specific area had not been set yet and it was unfeasible to use figures due to reproduction techniques available. In that sense the translation stays as close as possible to the original formulation of Dr. Seidel.

Having Dr. Matthew Novak, with opposite linguistic background than me (he's a native English speaker knowing German) review the translation helped make it possible.

The original publication from Dr. Seidel was split and published over three consecutive issues (1027, 1028 & 1029) of Astronomische Nachrichten. This translation recombines the three parts into one document.

Tucson,AZ, November 19, 2007

While deriving the equations used to calculate the so called deviation due to spherical shapes in given optical systems and the elimination of these deviations in to be built instruments, one usually limits concern to those rays that are parallel to or intersect the axis which passes through the centers of curvature of all refracting (or reflecting) surfaces of the system. The analysis therefore only completely describes those light cones for points on the axis: Light cones that start from points outside the center of the field of view (along which the axis is oriented), can only be represented by rays that lay in the plane defined by the peak of the light cone and the axis; a plane, which is apparently not left by the rays throughout their whole travel. Even limiting the theory to such a plane is sufficient for the existing demand of technology. The elimination of further errors beside those that are removed by current calculation methods would require increasing the number of available media; in other words the number of refracting or reflecting surfaces. Our practical optics alone, too early bereaved from its great master who measured science by its applicability, still has great difficulties to give surfaces the exact spherical shape and required radius even when the most skilled craftsmen are doing the work and so it's better to restrict to limited theoretical effects rather than giving rise to more space for fabrication errors. Therefore one can say that from this perspective an extension of the theory is not an urgent need. However there are cases where the extension of the analysis into space can not be omitted, even more often, ignoring any practical applicability, one finds himself with uncomfortable feelings being narrowed down to just the axis plane, leaving out the richness of effects around it. There is no requirement to limit oneself that much ; Since Gauss showed that by only including the first order elements, rays outside the mentioned plane can be traced as easy as those inside, nobody would expect that the third order case (i.e. here the next higher order) would cause much greater difficulty than the special case.

At first glance the expansion into space seems to make the expressions much more complicated which is the only opponent one faces in the given exercise. The position of a line in space, e.g. a light beam after any given number of refractions or reflections, is given by four constants, which can be chosen to have different meanings. Each time the ray changes direction those four constants will change value in such a way that the four new values may be derived from the old ones by the laws of refraction. Distinguishing two parts in each parameter of which the first part represents the approximation that ignores the deviation from the spherical shape and the second part corrects it to third order leaving the fifth order errors, it can be said that the first parts are known and only the four correction parts are subject to further analysis. In general one would expect that the value of each parameter after refraction would depend on the four values prior to it and in fact each new value will depend in a linear function on the four prior values since all powers and products of this small correction will fall into higher orders. The whole expression for a value of a correction element of any of the four parameters after e.g. n refractions will therefore appear as a sum of four members. Each single one has the four parameters defining the position of the ray after n-1 refractions as factor to which a fifth element of same order adds that is newly generated at the nth refracting surface. Considering that each of those four members by it self is to be substituted for by a similar sum of five members following a recursive procedure etc. it appears that after very few refractions any overview is going to be lost. There is, however, an excellent means to avoid that, which is to choose the four parameters that define the position of a ray after each refraction in the proper way. The geometrical meaning of these can be defined such that the expression for the correction part that represents the position of the ray after the nth refraction does not depend on all four previous correction part but only on one of them.

§1

Imagine the position of a ray before its nth refraction defined by two pairs of coordinate points in which two fixed plains A and B, that are perpendicular to the axis, are intersected. Its position after refraction can be related to the two transverse planes A' and B' in a similar way. The position of the four planes is arbitrary; therefore it is possible to obtain a relation between A and A' and also between B and B'. One can assume that e.g. A' is at the position where a luminous object (real or just virtual) at A, after the rays leaving being changed in their direction by the refracting surface, finds its true or virtual image according to the dioptric approximations. In the same way the planes B and B' are related. Following this convention it is obvious that the approximations of the coordinates of the point in which the plane A' is intersected by the refracted beam only dependent on the coordinates that are related to plane A and are completely independent of the points valid for plane B. This is because all beams that cross the same point A are going to meet again at point A' according to the approximating equations independent of what the directions were. One can go one step further and say: if the coordinate system, that one uses to fix the points in the different planes in which the rays intersect them are perpendicular, which are parallel with respect to each other in all those planes and origin on the axis, - or if they are polar coordinates where all poles are lying on the optical axis and their angle makes them parallel, the similarity between object and image and their similar positions it will be true that the approximated value for each of the two coordinates belonging to the plane A' can be determined as soon as the corresponding coordinates in the plane A are known independent of the second coordinate. Therefore it follows that the special way the meaning of the four parameters that define the position of the ray in space are chosen the approximated value of each of the four parameters, regarding the refracted ray, will not depend on all four of the values prior to refraction, but only on one of them. [1] Crossing over to the correction element, that needs to be added to the approximated value in order extend the accuracies into the order of spherical aberration, the same statement will not be applicable as it was for the main element. Since the value which each of the four parameters after refraction has is found approximately without any knowledge of the values that are assigned to three of them prior to refraction, only a coarse knowledge of those three values, combined with the more exact knowledge of the fourth one, will be sufficient to find that value very accurately. In other words: in order to find the correction element for each one of the four parameters defining the position of the ray, the knowledge of all four parameters prior to refraction but for three of them the approximated value is sufficient. Only the correction of one, which has the analogous meaning for the position of the non refracted ray as the one for which the correction is requested to be known for the refracted ray, influences the demanded value in the same order, the others only influence higher orders. Through this approach one may successfully separate the four unknowns (the four correction elements for the position of the ray anywhere through its travel through the optical system) from each other and from that it follows that each single one of them can be found by a simple summation over the refracting surfaces following each other. Most conveniently the coordinates for the expansion are chosen to be perpendicular, since in that case all four parameters have similar meaning and through simple exchange derive from each other, such that one derivation contains everything. It is conveniently possible to cross over to polar coordinates at the end, because for an instrument where everything is the same around the axis they are most naturally choose themselves. One recognizes that two systems of planes perpendicular to the axis are made the foundation of the whole analysis of the path of the ray. One plane of each system belongs to each medium which the ray enters. The original position of the ray was initially determined by defining the coordinates of its intersection with two arbitrary defined planes A and B, which I'll be calling the base planes of the first and second system. The position after one refraction will be determined by the coordinates of the intersection points with A' and B', after two refractions with A" and B", etc. finally its last position by the intersection points with A* and B*, whereas on one hand the planes A', A", A"', A*, build one plane system and likewise do similarly B, B', B",...B*, which constitute the 2nd system, are chained up such that always the following plane in a system is positioned where the dioptric approximation relation after refraction of the light on the boundary surface, the true or virtual image of the preceding plane of the same system created.

§2

In two earlier essays, printed in issue 835 and issue 871 of A.N. (Astronomische Nachrichten), I highlighted the advantages that simplify the dioptric equations by replacing the natural elements of the optical system (meaning that the radii of curvature ρ of the surfaces and the separation d between them are replaced) through the other parameters h and σ. One easily realizes that for the purpose of the current analysis in which two systems of transverse planes play the major role, the introduction of those parameters is induced almost automatically. Indeed one could in contrast to the absolute elements of the optical instrument, call the parameters h and σ "The el-

[1] It is possible to achieve the same effect utilizing a different setup, but the proposed one is the most natural one

ements of a given system of image planes", because if the distance of the base plane A to the refracting surface is set to be $= \frac{h_0}{\sigma_{-1}}$, the ratios $\frac{h_0}{\sigma_1}$, $\frac{h_2}{\sigma_1}$; $\frac{h_2}{\sigma_3}$, $\frac{h_4}{\sigma_3}$ etc. will directly give the distances from the first image plane A' to the first and second refracting surface and then the distances from the second image plane A" to the second and third refracting surface etc. In doing so, the parameters h are being proportional to the approximations for the distances from the axis of those points in which the different refracting surfaces are intersected by a ray,[2] which originally intersected the transverse plane A and after its refractions intersects A', A" etc.; the various σ are proportional to the approximated values of the angles which the very same ray is going to take with respect to the axis. The strict definition both classes of parameters is included in the equation which shows the absolute relation between the elements ρ and d of the optical instrument [3]

$$\rho_{2i} = \frac{N_{2i}\, h_{2i}}{\nu_{2i-1}\,\sigma_{2i+1} - \nu_{2i+1}\,\sigma_{2i-1}};\ \sigma_{2i+1}\, d_{2i+1} = h_{2i} - h_{2i+2}, \tag{1}$$

if the relation, $\frac{h_0}{\sigma_{-1}}$ representing the distance of the base plane A to the refracting surface is used.

For the current analysis the two plane systems A and B are of equal importance, therefore it seems appropriate to use the two systems of elements, which are h, σ that correspond to the planes in A and h', σ' corresponding to the planes in B, in parallel. However that causes twice as many parameters to be introduced in parallel into the expressions as when they existed independent of each other; only in the case where such a pleonasm of labeling would be disadvantageous, the case where the calculations have to be done for an instrument that has to be built according to certain criteria, one can without great effort eliminate the parameters of the system B(h',σ') with the parameters of the system A(h,σ) utilizing the equations that give the relation between the two groups containing the analogous parameters and which are published in Issue 871.[4] In the other main application, when the equations are used to study the effects of an already built system, one can always calculate h, σ as well as h', σ' from the given values ρ and d using the simple following algorithm that is originated in equation 1:

One builds up the constants α_0, α_1, α_2 ... following the equations

$$\alpha_{2i} = -\frac{n_{2i-1} - n_{2i+1}}{\rho_{2i}} = +n_{2i-1}n_{2i+1}\frac{N_{2i}}{\rho_{2i}}$$
$$\alpha_{2i+1} = -\nu_{2i+1}d_{2i+1}$$

and chooses h_0 and σ_{-1} such that $\frac{h_0}{\sigma_{-1}}$ represents the distance of the base plane A to the first refracting surface[5](and analogous h'_0 and σ'_{-1} have the same meaning for base plane B); one defines $x_{-1} = n_{-1}\sigma_{-1}$, $x_0 = h_0$ and calculates with these initial values all later values of x according to the equation

$$x_{m+1} = \alpha_m x_m + x_{m-1} \tag{2}$$

Thereupon one gets in general:

$$h_{2i} = x_{2i}$$
$$\sigma_{2i+1} = \upsilon_{2i+1}x_{2i+1},$$

and h', σ' arise in a similar manner so that all supporting values are known.

§3

The complete expansion for the third order elements following the plan just explained is going to be a simple calculation. Of course one has to start with the equations that will support finding the position of a refracted beam from a beam that is known in space by its position and which hits a refracting (or reflecting) spherical surface. I'll be using here the equation in the same form as Gauss defined at the beginning of his "Dioptric Analysis" (Dioptrischen Untersuchungen). There, the coordinates x, y, z were assumed to be perpendicular with origin on the optical axes of the instrument: The x growing along the direction of the light, the y and z being perpendicular to it. The four parameters for the position of the ray are in this context the two pairs of constants (ξ, b and γ, c), which exist in the equations of the ray projection in the planes x y and x z. For an incident beam those two equations are given the following form:

$$y = \frac{\xi}{n}x + b, \qquad z = \frac{\gamma}{n}x + c;$$

And for the refracted beam

$$y = \frac{\xi'}{n'}x + b' \qquad z = \frac{\gamma'}{n'}x + c'$$

. Whereas the x origins at the point in which the refracting surface is intersected by the optical axis and whereas $\frac{1}{n}:\frac{1}{n'}$ represents the refraction ratio at the transition from the previous to the following medium.[6]The relationship

[2] In Issue 871 called the ordinary ray

[3] In accordance with the from my continuously used labeling of the refracting surfaces with even and the intermediate media odd indices, further all here not specially explained parameters like ρ,d,ν,η,N etc. I permit myself, in order not to be too excessive, to refer to earlier essays of mine in issue 835 and 871.

[4] The analogous parameters to h, σ that are called h', σ' in this document are called l, τ in the essay in issue 871

[5] The direction for which that distance is positive in the calculation was explained in earlier essays

[6] The n are the reciprocal values to the ν as labeled by me

between b, ξ, c, γ on one hand and b', ξ', c', γ' on the other are then strictly expressed by the equations: (3 and 4 below)

$$\left.\begin{array}{c} \dfrac{\xi}{n}\rho\,(1-\cos\Theta)+b=\dfrac{\xi'}{n'}\rho\,(1-\cos\Theta)+b' \\ \dfrac{\gamma}{n}\rho\,(1-\cos\Theta)+c=\dfrac{\gamma'}{n'}\rho\,(1-\cos\Theta)+c' \\ (\xi\rho+nb)\sin\lambda=(\xi'\rho+n'b')\sin\lambda' \\ (\gamma\rho+nc)\sin\lambda=(\gamma'\rho+n'c')\sin\lambda' \end{array}\right\} \qquad (3)$$

in which ρ, as mentioned above, labels the radius of curvature of the refracting surface, while Θ represents the (small) angle between incident plummet and axis. λ and λ' on the other hand represent the angles that deviate little from the quadrant, and are enclosed by the incident and refracted beam with the straight line intersecting both rays, being perpendicular to the axis and going through the center of the sphere, respectively. The quantities b, ξ, c, γ, b', ξ', c', γ', $\sin\Theta$, $\cos\lambda$, $\cos\lambda'$ are going to be used as small quantities of the same order (the first order) as consequence of restricting the analysis to small field of view and small aperture. If one replaces based on that restriction the exact equations (3) by such equations that only contain members of first and third order, the resultant errors are apparently only of fifth order, or they behave with respect to the most significant members of these equations as being of fourth order. The members of even order do not appear at all in these equations.

Thus by replacing in equations (3) by the following:

$$\left.\begin{array}{ll} b=b_0+\Delta b & b'=b'_0+\Delta b' \\ c=c_0+\Delta c & c'=c'_0+\Delta c' \\ \xi=\xi_0+\Delta\xi & \xi'=\xi'_0+\Delta\xi' \\ \gamma=\gamma_0+\Delta\gamma & \gamma'=\gamma'_0+\Delta\gamma' \end{array}\right\} \qquad (4)$$

By denoting the subscript 0 to quantities of first order, the denotation of Δ marks the quantities containing third order and simultaneously ignoring everything that is of other than third order one gets instead of the exact equations one gets approximated ones that are accurate except for the error of fifth order. Each of the new equations may be separated into two new ones because the first order members and the third order members on both sides must be equal. One therefore obtains between the approximations b_0, c_0 etc. of the parameters of the position of the ray the four equations (compare with *Gauss* p.3 and 8)

$$b'_0=b_0;\qquad c'_0=c_0;\qquad \xi'_0+\frac{n'\,b'_0}{\rho}=\xi_0+\frac{n\,b_0}{\rho};\qquad \gamma'_0+\frac{n'\,c'_0}{\rho}=\gamma_0+\frac{n\,c_0}{\rho}; \qquad (5)$$

In addition one obtains the following equations for the correction members' Δb, Δc, etc. (Equation (5) has already been used for their simplification):

$$\left.\begin{array}{c} \Delta b'-\Delta b=2\rho\sin\tfrac{1}{2}\Theta^2\,\left(\dfrac{\xi_0}{n}-\dfrac{\xi'_0}{n'}\right) \\ \left(\Delta\xi'+\dfrac{n'\Delta b'}{\rho}\right)-\left(\Delta\xi+\dfrac{n\,\Delta b}{\rho}\right)=\tfrac{1}{2}\left(\xi_0+\dfrac{n\ b_0}{\rho}\right)\left(\cos\lambda'^2-\cos\lambda^2\right) \\ \Delta c'-\Delta c=2\rho\sin\tfrac{1}{2}\Theta^2\left(\dfrac{\gamma_0}{n}-\dfrac{\gamma'_0}{n'}\right) \\ \left(\Delta\gamma'+\dfrac{n'\Delta c'}{\rho}\right)-\left(\Delta\gamma+\dfrac{n\,\Delta c}{\rho}\right)=\tfrac{1}{2}\left(\gamma_0+\dfrac{n\ c_0}{\rho}\right)\left(\cos\lambda'^2-\cos\lambda^2\right) \end{array}\right\} \qquad (6)$$

One recognizes that it is sufficient to do the further derivation for the quantities b, ξ, b', ξ' which describe the projection of the beam in the xy plane, through simple exchange of those quantities with c, γ, c', γ' one can then obtain these quantities for the xy plane.

§4

In accordance with the plan explained in the introduction the Gaussian parameters b, ξ, c, γ for the position of the ray before refraction and b', ξ', c', γ' for the position of the ray after refraction need to be replaced by the four corresponding coordinate points in the transverse planes A and B that describe our system in the medium through which the ray currently travels. If one assumes that the refracting surface, whose effect needs to be analyzed, is labeled by the index 2i (meaning that it is the i+ $1^{st.}$ surface), then $\frac{h_{2i}}{\sigma_{2i-1}}$ and $\frac{h'_{2i}}{\sigma'_{2i-1}}$ constitute the distances of the transverse planes A and B for the preceding medium 2i-1 to this surface. $\frac{h_{2i}}{\sigma_{2i+1}}$ and $\frac{h'_{2i}}{\sigma'_{2i+1}}$ constitute the distances between the transverse planes A' and B' in the succeeding medium 2i+1 to the same surface. However, In order to simplify the labeling I'll allow myself, as long as only the effect of a single surface is analyzed, to write the quantities (like h, h', ρ) that carry the index 2i without an index, as well as those that carry the index 2i-1 (referencing the medium just preceding the refracting surface) will be represented by a '-' underneath and analogously, the quantities referenced to the succeeding medium 2i+1 are marked by a '+' sign underneath. So that they are written as

$$\underset{-}{\nu},\ \underset{+}{\nu},\ h,\ h',\ N,\ \underset{-}{\sigma},\ \underset{+}{\sigma},\ \underset{-}{\sigma'},\ \underset{+}{\sigma'}$$

instead of

$$\nu_{2i-1},\ \nu_{2i+1},\ h_{2i},\ h'_{2i},\ N_{2i},\ \sigma_{2i-1},\ \sigma_{2i+1},\ \sigma'_{2i-1},\ \sigma'_{2i+1}$$

The right-angle coordinates of the point in which the ray intersects the transverse plane A before being refracted are labeled accordingly with

$$\underset{-}{\eta} + \Delta\underset{-}{\eta}, \qquad \underset{-}{\zeta} + \Delta\underset{-}{\zeta};$$

The ones of the point in which the same ray intersects plane B are labeled

$$\underset{-}{\eta'} + \Delta\underset{-}{\eta'}, \qquad \underset{-}{\zeta'} + \Delta\underset{-}{\zeta'};$$

And likewise for the refracted ray the coordinates of the point in plane A' at which it points to are called

$$\underset{+}{\eta} + \Delta\underset{+}{\eta}, \qquad \underset{+}{\zeta} + \Delta\underset{+}{\zeta};$$

And the ones in the corresponding point B'

$$\underset{+}{\eta'} + \Delta\underset{+}{\eta'}, \qquad \underset{-}{\zeta'} + \Delta\underset{+}{\zeta'};$$

The first members of these twopart expressions always represent their approximations which one would obtain by neglecting the members of third order and the second members represent the correction of third order. It is adequate to immediately define the polar coordinates of the mentioned four points. The pole of them is assumed to be in each of the four planes A, B, A', B', where they are intersected by the optical axis. The radial vectors labeled as $r + \Delta r$ and the position angles labeled as $\nu + \Delta\nu$ are measured from here. These quantities receive the same indices and accents as η and ζ. The direction from which ν is measured and its sense of rotation is defined such that one obtains

$$\eta = r\ \cos\,\nu, \qquad \zeta = r\ \sin\,\nu.$$

It is easy to derive the equations through which the transition from the quantities b,ξ, c, γ, to η, η', ζ, ζ' etc. is achieved. Namely in the Gaussian equations of the non refracted beam y must become $\underset{-}{\eta} + \Delta\underset{-}{\eta}$ and z takes the value of $\underset{-}{\zeta} + \Delta\underset{-}{\zeta}$ for the case where x is $\frac{h}{\underset{-}{\sigma}}$; in contrast y must be $\underset{-}{\eta'} + \Delta\underset{-}{\eta'}$ and z becomes $\underset{-}{\zeta'} + \Delta\underset{-}{\zeta'}$ if $x = \frac{h'}{\underset{-}{\sigma'}}$, and analogous conditions apply for the refracted beam for which the '+' labeled quantities have the same meaning as the ones with '-' label in the first case. That way the bs', ξs' can be expressed by the ηs' and the cs', γs' by the ζs'. Initially one gets equations for which the quantity $\frac{h}{\underset{-}{\sigma}} - \frac{h'}{\underset{-}{\sigma'}}$ (i.e. the separation of the two transverse planes A and B) or if talking about the refracted beam the quantity $\frac{h}{\underset{+}{\sigma}} - \frac{h'}{\underset{+}{\sigma'}}$ (Separation between A' and B') appear in the denominator. E.g. one gets:

$$\frac{\xi_0}{n} = \frac{\underset{-}{\eta} - \underset{-}{\eta'}}{\frac{h}{\underset{-}{\sigma}} - \frac{h'}{\underset{-}{\sigma'}}}; \qquad \frac{\Delta\xi}{n} = \frac{\Delta\underset{-}{\eta} - \Delta\underset{-}{\eta'}}{\frac{h}{\underset{-}{\sigma}} - \frac{h'}{\underset{-}{\sigma'}}}$$

$$b_0 = \frac{\frac{h}{\underset{-}{\sigma}}\underset{-}{\eta'} - \frac{h'}{\underset{-}{\sigma'}}\underset{-}{\eta}}{\frac{h}{\underset{-}{\sigma}} - \frac{h'}{\underset{-}{\sigma'}}}; \qquad \Delta b = \frac{\frac{h}{\underset{-}{\sigma}}\Delta\underset{-}{\eta'} - \frac{h'}{\underset{-}{\sigma'}}\Delta\underset{-}{\eta}}{\frac{h}{\underset{-}{\sigma}} - \frac{h'}{\underset{-}{\sigma'}}}$$

If one multiplies numerator and denominator with $\underset{-}{\sigma}\ \underset{-}{\sigma'}$ the denominator becomes $h\underset{-}{\sigma'} - h'\underset{-}{\sigma}$ and similar for the refracted beam $h\underset{+}{\sigma'} - h'\underset{+}{\sigma}$. But one proofs that the two quantities

$$\frac{1}{\underset{-}{\nu}}\left(h\underset{-}{\sigma'} - h'\underset{-}{\sigma}\right) \text{ and } \frac{1}{\underset{+}{\nu}}\left(h\underset{+}{\sigma'} - h'\underset{+}{\sigma}\right)$$

or more explicitly written as

$$\frac{1}{\nu_{2i-1}}\left(h_{2i}\,\sigma'_{2i-1} - h'_{2i}\sigma_{2i-1}\right) \text{and} \frac{1}{\nu_{2i+1}}\left(h_{2i}\,\sigma'_{2i+1} - h'_{2i}\sigma_{2i+1}\right)$$

have the same value that is independent of the index 2i of the refracting surface or in other words a constant value through all media and surfaces of the optical system. This simply results from the fact that the quantities ρ and d in accordance to equation (1) must have the same values and can be represented through h and σ or through h' and σ'. This theorem has already been derived in issue 871 equation I. I'll denote that value with the letter T and therefore

$$T = \frac{h_0\sigma'_{-1} - h'_0\sigma_{-1}}{\nu_{-1}} = \frac{h_0\sigma'_1 - h'_0\sigma_1}{\nu_1} = \frac{h_2\sigma'_1 - h'_2\sigma_1}{\nu_1} = \frac{h_2\sigma'_3 - h'_2\sigma_3}{\nu_3} = \cdots = \frac{h\underset{-}{\sigma'} - h'\underset{-}{\sigma}}{\underset{-}{\nu}} = \frac{h\underset{+}{\sigma'} - h'\underset{+}{\sigma}}{\underset{+}{\nu}} = \cdots \qquad (7)$$

One therefore gets:

$$\frac{\xi_0}{n}=\frac{\underset{-}{\sigma}\,\underset{-}{\sigma}'}{\underset{-}{\nu}}\,\frac{\underset{-}{\eta}-\underset{-}{\eta}'}{T};\qquad \frac{\Delta\xi}{n}=\frac{\underset{-}{\sigma}\underset{-}{\sigma}'}{\underset{-}{\nu}}\,\frac{\Delta\underset{-}{\eta}-\Delta\underset{-}{\eta}'}{T}$$

$$b_0=\frac{1}{\underset{-}{\nu}}\,\frac{h\underset{-}{\sigma}'\,\underset{-}{\eta}'-h'\underset{-}{\sigma}\underset{-}{\eta}}{T};\qquad \Delta b=\frac{1}{\underset{-}{\nu}}\,\frac{h\underset{-}{\sigma}'\,\Delta\underset{-}{\eta}'-h'\underset{-}{\sigma}\,\Delta\underset{-}{\eta}}{T}$$

and very similar equations result for $\frac{\xi_0'}{n'}$, b_0' etc. if one replaces $\underset{-}{\nu}$, $\underset{-}{\sigma}$, $\underset{-}{\sigma}'$, $\underset{-}{\eta}$, $\underset{-}{\eta}'$ with $\underset{+}{\nu}$, $\underset{+}{\sigma}$, $\underset{+}{\sigma}'$, $\underset{+}{\eta}$, $\underset{+}{\eta}'$.

If one substitutes the values of the quantities ξ_0, b_0, $\Delta\xi$, Δb, in the corresponding equations (5) and (6) they transfer into equations which give the relations between the approximations $\underset{-}{\eta}$, $\underset{-}{\eta}'$, $\underset{+}{\eta}$, $\underset{+}{\eta}'$ of our new parameters and the relation between correction members Δ. The first equations take an especially simple form. In accordance to (5) the values of b_0 and $\xi_0+\frac{n\,b_0}{\rho}$ must remain unchanged if the quantities that represent the position of the ray before refraction, are exchanged with the ones that are valid after refraction, that is in our nomenclature replacing the quantities marked with a '-' sign with the quantities marked by a '+' sign. Now the value for $\xi_0+\frac{n\,b_0}{\rho}$ can be found in the new quantities:

$$=\frac{1}{\underset{-}{\nu}\,\underset{-}{\nu}\,T}\left\{\underset{-}{\sigma}\,\underset{-}{\sigma}'\left(\underset{-}{\eta}-\underset{-}{\eta}'\right)+\underset{-}{\sigma}'\,\underset{-}{\eta}'\frac{h}{\rho}-\underset{-}{\sigma}\,\underset{-}{\eta}\frac{h'}{\rho}\right\};$$

If one replaces $\frac{h}{\rho}$ in accordance to equation (1) with the value $\frac{\underset{-}{\nu}\underset{+}{\sigma}-\underset{+}{\nu}\underset{-}{\sigma}}{N}$ and likewise $\frac{h'}{\rho}$ with the value $\frac{\underset{-}{\nu}\underset{+}{\sigma}-\underset{+}{\nu}\underset{-}{\sigma}}{N}$ (where $N=\underset{-}{\nu}-\underset{+}{\nu}$) and combines the members that contain $\underset{-}{\eta}$ as well as $\underset{-}{\eta}'$ one obtains:

$$\frac{1}{T}\left\{\frac{\underset{-}{\sigma}\,\underset{-}{\eta}}{\underset{-}{\nu}}\,\frac{\underset{-}{\sigma}'-\underset{+}{\sigma}'}{N}-\frac{\underset{-}{\sigma}'\,\underset{-}{\eta}'}{\underset{-}{\nu}}\,\frac{\underset{-}{\sigma}-\underset{+}{\sigma}}{N}\right\}$$

b_0 that is the expression

$$\frac{1}{T}\left\{\frac{\underset{-}{\sigma}'\,\underset{-}{\eta}'}{\underset{-}{\nu}}h-\frac{\underset{-}{\sigma}\,\underset{-}{\eta}}{\underset{-}{\nu}}h'\right\}$$

must remain unchanged if one exchanges the labels '-' and '+'. Since the quantities $\frac{\underset{-}{\sigma}-\underset{+}{\sigma}}{N}=\frac{\underset{-}{\sigma}-\underset{+}{\sigma}}{\underset{-}{\nu}-\underset{+}{\nu}}$ and $\frac{\underset{-}{\sigma}'-\underset{+}{\sigma}'}{N}$ as well as h, h' and T maintain their values the following relations must be true:

$$\frac{\underset{-}{\sigma}\,\underset{-}{\eta}}{\underset{-}{\nu}}=\frac{\underset{+}{\sigma}\,\underset{+}{\eta}}{\underset{+}{\nu}};\qquad \frac{\underset{-}{\sigma}'\,\underset{-}{\eta}'}{\underset{-}{\nu}}=\frac{\underset{+}{\sigma}'\,\underset{+}{\eta}'}{\underset{+}{\nu}}.$$

That is the products of the form $\frac{\sigma}{\nu}\eta$ stay constant through all consecutive media of the optical system. [7] It will be appropriate to introduce the constant values that are proportional to our approximated members η, η' (as well as ζ, ζ')of our coordinates, instead of the variable parameters by themselves. Therefore one gets:

$$\left.\begin{aligned}
H&=\frac{\sigma_{-1}}{\nu_{-1}}\eta_{-1}=\frac{\sigma_1}{\nu_1}\eta_1=\frac{\sigma_3}{\nu_3}\eta_3=\dots=\frac{\underset{-}{\sigma}}{\underset{-}{\nu}}\,\underset{-}{\eta}=\frac{\underset{+}{\sigma}}{\underset{+}{\nu}}\underset{+}{\eta}=\dots\\
Z&=\frac{\sigma_{-1}}{\nu_{-1}}\zeta_{-1}=\frac{\sigma_1}{\nu_1}\zeta_1=\frac{\sigma_3}{\nu_3}\zeta_3=\dots=\frac{\underset{-}{\sigma}}{\underset{-}{\nu}}\underset{-}{\zeta}=\frac{\underset{+}{\sigma}}{\underset{+}{\nu}}\underset{+}{\zeta}=\dots\\
H'&=\frac{\sigma_{-1}'}{\nu_{-1}}\eta_{-1}'=\frac{\sigma_1'}{\nu_1}\eta_1'=\frac{\sigma_3'}{\nu_3}\eta_3'=\dots=\frac{\underset{-}{\sigma}'}{\underset{-}{\nu}}\underset{-}{\eta}'=\frac{\underset{+}{\sigma}'}{\underset{+}{\nu}}\underset{+}{\eta}'=\dots\\
Z'&=\frac{\sigma_{-1}'}{\nu_{-1}}\zeta_{-1}'=\frac{\sigma_1'}{\nu_1}\zeta_1'=\frac{\sigma_3'}{\nu_3}\zeta_3'=\dots=\frac{\underset{-}{\sigma}'}{\underset{-}{\nu}}\underset{-}{\zeta}'=\frac{\underset{+}{\sigma}'}{\underset{+}{\nu}}\underset{+}{\zeta}'=\dots
\end{aligned}\right\}\qquad(8)$$

The quantities H and Z could be considered as the reduced right-angle coordinates of the intersection points between ray and transverse planes. If one would use different scales in the different planes of the systems A and B to measure the coordinates such that their units behave as the $\frac{\nu}{\sigma}$ related to those planes, then the coordinates in all planes A would be expressed by the quantities H, Z and accordingly in the planes B by H' and Z'. The consequence is to express the correction members $\Delta\nu$, $\Delta\zeta$ the

[7] Certainly this result can be derived directly from the laws of refraction without having them rewritten in the form of equation (5) between b and ξ

same way so that in general

$$\left.\begin{aligned} \frac{\nu_{2i-1}}{\sigma_{2i-1}}\,(H+\Delta H_{2i-1}) &= \eta_{2i-1}+\Delta\eta_{2i-1}\\ \frac{\nu_{2i-1}}{\sigma_{2i-1}}\,(Z+\Delta Z_{2i-1}) &= \zeta_{2i-1}+\Delta\zeta_{2i-1}\\ \frac{\nu_{2i-1}}{\sigma_{2i-1}}\left(H'+\Delta H'_{2i-1}\right) &= \eta'_{2i-1}+\Delta\eta'_{2i-1}\\ \frac{\nu_{2i-1}}{\sigma_{2i-1}}\left(Z'+\Delta Z'_{2i-1}\right) &= \zeta'_{2i-1}+\Delta\zeta'_{2i-1} \end{aligned}\right\} \tag{9}$$

represent the right-angle coordinates of the intersection of the ray with the different planes that are perpendicular to the axis in our two systems. Apparently the quantities ΔH, ΔZ etc. must get indices as above although the quantities H, Z themselves don't carry them, since first mentioned quantities will not be constant throughout the media. By applying polar coordinates it will be natural to use the same scale for the radius vector so that

$$\left.\begin{aligned} \frac{\nu_{2i-1}}{\sigma_{2i-1}}\,(R+\Delta R_{2i-1}) &= r_{2i-1}+\Delta r_{2i-1}\\ \frac{\nu_{2i-1}}{\sigma'_{2i-1}}\left(R'+\Delta R'_{2i-1}\right) &= r'_{2i-1}+\Delta r'_{2i-1} \end{aligned}\right\} \tag{10}$$

represent the length of one and the same in the different planes in which

$$\left.\begin{aligned} \nu+\Delta\nu_{2i-1}\\ \nu'+\Delta\nu'_{2i-1} \end{aligned}\right\} \tag{11}$$

are the different position angles. From last mentioned quantities it becomes clear that the approximated value is constant for all image planes of system A as well as ν' is constant for all image planes of system B. Likewise in the equations (10) R and R' are going to be constant from medium to medium due to the relation between the right-angle and polar coordinates that are again

$$\left.\begin{aligned} H = R\cos\nu; &\quad H' = R'\cos\nu'\\ Z = R\cos\nu; &\quad Z' = R'\cos\nu' \end{aligned}\right\} \tag{12}$$

Although the comment needs to be made that in accordance to equations (10) the radius vectors r, r' can't be considered positive for all transverse planes since the signs of R and R' are being fixed r, r' can become positive or negative dependent on the sign of the quantity $\frac{\nu}{\sigma}$: That is, in accordance to equations (8) for such planes of system A for which one contains the inverted image of the figure drawn in the other, the rs' will be had opposite signs. The same signs when the images have similar orientation. The same behavior is valid for the signs of the r's in the two planes of system B. This arrangement is advantageous in the way that ν maintains its value for all planes in one given system. If one would assign all r to be of positive value (which is in itself arbitrary) two different values would have to be assigned to ν which are the original one and (in the planes that give the inverted image of the base planes of system A) one rotated by a semi for the other.

The Gaussian parameters articulate themselves through the newly introduced reduced coordinates as:

$$\frac{\xi_0}{n} = \frac{H\underset{-}{\sigma'} - H'\underset{-}{\sigma}}{T}; \qquad \frac{\Delta\xi}{n} = \frac{\underset{-}{\sigma'}\,\Delta\underset{-}{H} - \underset{-}{\sigma}\,\Delta\underset{-}{H'}}{T}$$

$$b_0 = \frac{-H\,h' + H'\,h}{T}; \qquad \Delta b = \frac{-h'\,\Delta\underset{-}{H} + h\,\underset{-}{H'}}{T}$$

and in the same sense the quantities $\frac{\xi'}{n'}$, b' and their corrections result by replacing the index '-' with the index '+'. Further γ results instead of ξ and c instead of b if H and H' are replaced by Z and Z'.

Now one can replace the quantities b and ξ and their correction members in the first two equations of (6) by the derived relations. Therefore the binomial $\xi_0 + \frac{n\,b_0}{\rho}$, that was already discussed earlier, is assigned to the expression $\frac{1}{T\,N}\left(H(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - H'(\underset{-}{\sigma} - \underset{+}{\sigma})\right)$ and its correction $\Delta\xi + \frac{n\,\Delta b}{\rho}$ becomes $\frac{1}{T\,N}\left(\Delta\underset{-}{H}(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - \Delta\underset{-}{H'}(\underset{-}{\sigma} - \underset{+}{\sigma})\right)$ Analogously, one gets $\Delta\xi' + \frac{n'\,\Delta b'}{\rho} = \frac{1}{T\,N}\left(\Delta\underset{+}{H}(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - \Delta\underset{+}{H'}(\underset{-}{\sigma} - \underset{+}{\sigma})\right)$ and the expression $\frac{\xi_0}{n} - \frac{\xi'_0}{n'}$ similarly becomes $\frac{1}{T}\left(H(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - H'(\underset{-}{\sigma} - \underset{+}{\sigma})\right)$.

In the equations that result from the substitution the four quantities ΔH do not appear individually any more but only the two differences $\Delta\underset{+}{H} - \Delta\underset{-}{H}$ and $\Delta\underset{+}{H'} - \Delta\underset{-}{H'}$. Therefore one can derive two other equations where in one of them only the first difference appears and while in the second one only the other difference appears. The first of these equations arises if the first equation of equations (6) that is rewritten using the new variables is multiplied by $\underset{-}{\sigma} - \underset{+}{\sigma}$ and added to the second equation that is multiplied by h. That will produce the factor $h(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - h'(\underset{-}{\sigma} - \underset{+}{\sigma})$ which, in accordance to (7), can be written as $(\underset{-}{\nu} - \underset{+}{\nu})T = N\,T$ The result therefore becomes:

$$\Delta\underset{+}{H} - \Delta\underset{-}{H} = \frac{H(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - H'(\underset{-}{\sigma} - \underset{+}{\sigma})}{N\,T}\left((\underset{-}{\sigma} - \underset{+}{\sigma})2\rho\sin\tfrac{1}{2}\Theta^2 + \frac{h}{2}(\cos\lambda'^2 - \cos\lambda^2)\right)$$

A similar equation results for $\Delta \underset{+}{H'} - \Delta \underset{-}{H'}$ if the quantities $\underset{-}{\sigma}$, $\underset{+}{\sigma}$, h within the bracket that follows the fraction are being replaced by $\underset{-}{\sigma'}$, $\underset{+}{\sigma'}$, h' . The quantity before the bracket namely stays unaltered by the letter exchange since T changes its sign simultaneously with the numerator.

§5

The small quantities $\sin\frac{1}{2}\Theta$, $\cos\lambda$ and $\cos\lambda'$, remain to be considered whereas only the first order members are required since only those have an influence onto the third order value of $\Delta\underset{+}{H} - \Delta\underset{-}{H}$. Initially, the meaning of the angle Θ relates to $\rho\sin\Theta =$ the distance from the axis to the point where the ray intersects the refracting sphere. Instead of this point one can choose the point in which the ray intersects a plane that touches the sphere at the axis. This plane is our y-z coordinate plane. The equations of the incoming beam, by ignoring the third order members and replacing b_0, ξ_0, c_0, γ_0 with their values found above, now become:

$$\left.\begin{aligned} y\,T &= -\,(H\,h' - H'\,h) + x\,(H\underset{-}{\sigma'} - H'\underset{-}{\sigma}) \\ z\,T &= -\,(Z\,h' - Z'\,h) + x\,(Z\underset{-}{\sigma'} - Z'\underset{-}{\sigma}) \end{aligned}\right\} \qquad (13)$$

(The ones for the refracted beam result through the simple change of $\underset{-}{\sigma}$, $\underset{-}{\sigma'}$, to $\underset{+}{\sigma}$, $\underset{+}{\sigma'}$). Therefore one can write:

$$2\rho\sin\tfrac{1}{2}\Theta^2 = \frac{1}{2\rho}\frac{1}{T^2}\left\{(H\,h' - H'\,h)^2 + (Z\,h' - Z'\,h)^2\right\} = \frac{1}{2\rho}\frac{1}{T^2}\left\{R^2h'^2 + R'^2h^2 - 2R\,R'\,h\,h'\,\cos(\nu' - \nu)\right\}$$

In order to find λ one must first know the coordinates y and z of the point in which the ray intersects the plane perpendicular to the axis placed in the center of the refracting sphere. These are obtained if one sets $x = \rho$ in equation (13). For the intersection point we get:

$$\frac{y\,T}{\rho} = H(\underset{-}{\sigma'} - \frac{h'}{\rho}) - H'\,(\underset{-}{\sigma} - \frac{h}{\rho})$$

Applying equation (1) the same way as already done above one get.

$$\underset{-}{\sigma} - \frac{h}{\rho} = \underset{-}{\sigma} - \frac{\underset{-}{\nu}\,\underset{+}{\sigma} - \underset{+}{\nu}\,\underset{-}{\sigma}}{\underset{-}{\nu} - \underset{+}{\nu}} = \frac{\underset{-}{\nu}}{N}(\underset{-}{\sigma} - \underset{+}{\sigma})$$

and

$$\underset{-}{\sigma'} - \frac{h'}{\rho} = \frac{\underset{-}{\nu}}{N}(\underset{-}{\sigma'} - \underset{+}{\sigma'})$$

therefore it becomes

$$y\frac{T\,N}{\rho\,\underset{-}{\nu}} = H(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - H'\,(\underset{-}{\sigma} - \underset{+}{\sigma})$$

as well as

$$z\frac{T\,N}{\rho\,\underset{-}{\nu}} = Z\,(\underset{-}{\sigma'} - \underset{+}{\sigma'}) - Z'\,(\underset{-}{\sigma} - \underset{+}{\sigma})$$

The equations of the straight lines that connect the center of the sphere with the found intersection points therefore become:

$$x = \rho$$
$$\frac{y}{\underset{-}{P} - \underset{+}{P}} = \frac{z}{\underset{-}{Q} - \underset{+}{Q}}$$

Where the used abbreviations are

$$\underset{-}{P} = H\;\underset{-}{\sigma'} - H'\;\underset{-}{\sigma}; \qquad \underset{-}{Q} = Z\;\underset{-}{\sigma'} - Z'\;\underset{-}{\sigma}$$
$$\underset{+}{P} = H\;\underset{+}{\sigma'} - H'\;\underset{+}{\sigma}; \qquad \underset{+}{Q} = Z\;\underset{+}{\sigma'} - Z'\;\underset{+}{\sigma}$$

One can see that the straight line (as it must be) remains unchanged if the incoming ray is replaced by the refracted ray. The angles constructed by this line with the incoming and refracted beam are λ and λ'. In accordance to the general rules of analytical stereometry with which angles between two straight lines can be found one therefore can find[8]:

$$\cos\;\lambda^2 = \frac{\left\{\underset{-}{P}(\underset{-}{P} - \underset{+}{P}) + \underset{-}{Q}(\underset{-}{Q} - \underset{+}{Q})\right\}^2}{TT\left\{(\underset{-}{P} - \underset{+}{P})^2 + (\underset{-}{Q} - \underset{+}{Q})^2\right\}}$$

$$\cos\;\lambda'^2 = \frac{\left\{\underset{+}{P}(\underset{-}{P} - \underset{+}{P}) + \underset{+}{Q}(\underset{-}{Q} - \underset{+}{Q})\right\}^2}{TT\left\{(\underset{-}{P} - \underset{+}{P})^2 + (\underset{-}{Q} - \underset{+}{Q})^2\right\}}$$

If one subtracts the two equations from each other and applies on the difference of the squares in the numerator $M^2 - N^2 = (M - N)(M + N)$ a simple relation results

$$\cos\;\lambda'^2 - \cos\;\lambda^2 = -\frac{1}{T\,T}\left\{\underset{-}{P}^2 + \underset{-}{Q}^2 - \underset{+}{P}^2 - \underset{+}{Q}^2\right\}$$

[8] To comment that H and Z are small quantities in first order compared to T

If P and Q are rewritten as their values and following equation (12) one transfers back to polar coordinates the equation represents itself as.

$$\cos\lambda'^2 - \cos\lambda^2 = -\frac{1}{T^2}\left\{R^2(\underset{-}{\sigma}'^2 - \underset{+}{\sigma}'^2) + R'^2(\underset{-}{\sigma}^2 - \underset{+}{\sigma}^2) - 2RR'\cos(\nu'-\nu)(\underset{-}{\sigma}\,\underset{-}{\sigma}' - \underset{+}{\sigma}\,\underset{+}{\sigma}')\right\}$$

Having found Θ, λ and λ' the previous equation for $\Delta\underset{+}{H} - \Delta\underset{-}{H}$ now becomes

$$\Delta\underset{+}{H} - \Delta\underset{-}{H} = \tfrac{1}{2}\frac{H(\underset{-}{\sigma}' - \underset{+}{\sigma}') - H'(\underset{-}{\sigma} - \underset{+}{\sigma})}{N\,T^3}\left\{\begin{array}{l} R^2\frac{h'^2}{\rho}(\underset{-}{\sigma} - \underset{+}{\sigma}) + R'^2\frac{h^2}{\rho}(\underset{-}{\sigma} - \underset{+}{\sigma}) - 2RR'\cos(\nu'-\nu)\frac{h\,h'}{\rho}(\underset{-}{\sigma} - \underset{+}{\sigma}) \\ -R^2h(\underset{-}{\sigma}'^2 - \underset{+}{\sigma}'^2) - R'^2h(\underset{-}{\sigma}^2 - \underset{+}{\sigma}^2) + 2RR'\cos(\nu'-\nu)h(\underset{-}{\sigma}\,\underset{-}{\sigma}' - \underset{+}{\sigma}\,\underset{+}{\sigma}') \end{array}\right\} \tag{14}$$

The expression for $\Delta\underset{+}{Z} - \Delta\underset{-}{Z}$ is obtained in the bracket preceding factor H and H' are being replaced with Z and Z' the quantity in the bracket remains unchanged. This quantity can be somewhat simplified by paired combination of the members that are multiplied with R and R' of identical power. If one namely writes instead of $\frac{h'}{\rho}$ its value (from 1.) $\frac{\underset{-}{\nu}\,\underset{+}{\sigma}' - \underset{+}{\nu}\,\underset{-}{\sigma}'}{N}$, and instead of $h'(\underset{-}{\sigma} - \underset{+}{\sigma})$ the value $h(\underset{-}{\sigma}' - \underset{+}{\sigma}') - N\,T$, which resulted from equation (7), the h' may be eliminated and the expression in the brackets therefore becomes.

$$\left.\begin{array}{l} -\,R^2\left(h\frac{\underset{-}{\sigma}' - \underset{+}{\sigma}'}{N}(\underset{-}{\nu}\,\underset{-}{\sigma}' - \underset{+}{\nu}\,\underset{+}{\sigma}') + T(\underset{-}{\nu}\,\underset{+}{\sigma}' - \underset{+}{\nu}\,\underset{-}{\sigma}')\right) \\ -\,R'^2h\frac{\underset{-}{\sigma} - \underset{+}{\sigma}}{N}(\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \\ +\,2RR'\cos(\nu'-\nu)h\frac{\underset{-}{\sigma}' - \underset{+}{\sigma}}{N}(\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \end{array}\right\} \tag{15}$$

After the polar coordinates have somewhat introduced themselves in this expression instead of the right-angle coordinates it appears to be appropriate to leave the last mentioned and to search for ΔR and $\Delta\nu$ instead of ΔH and ΔZ. Apparently from 12 we have:

$$\Delta R = \Delta H\,\cos\nu + \Delta Z\,\sin\nu$$
$$R\,\Delta\nu = -\Delta H\,\sin\nu + \Delta Z\,\cos\nu$$

Whereas the indices of ΔR, $\Delta\nu$, ΔH and ΔZ are always the same. Therefore in order to obtain the expression for $\Delta\underset{+}{R} - \Delta\underset{-}{R}$ one will have to multiply the expression for $\Delta\underset{+}{H} - \Delta\underset{-}{H}$ by $\cos\nu$ and the similar expression for $\Delta\underset{+}{Z} - \Delta\underset{-}{Z}$ by $\sin\nu$ and then add both and do it very similar to obtain $R\Delta\underset{+}{\nu} - R\Delta\underset{-}{\nu}$. Because the quantities $\Delta\underset{+}{H} - \Delta\underset{-}{H}$ and $\Delta\underset{+}{Z} - \Delta\underset{-}{Z}$ have the factor in brackets in common apparently it will stay common for $\Delta\underset{+}{R} - \Delta\underset{-}{R}$ and $R\Delta\underset{+}{\nu} - R\Delta\underset{-}{\nu}$ and at the place of the fraction before the brackets in (14) substitute

$$\text{in the expr. for } \Delta R \cdots \tfrac{1}{2}\frac{R(\underset{-}{\sigma}' - \underset{+}{\sigma}') - R'(\underset{-}{\sigma} - \underset{+}{\sigma})\cos(\nu'-\nu)}{N\,T^3}$$

$$\text{in the expr. for } R\Delta\nu \cdots -\tfrac{1}{2}\frac{R'\sin(\nu'-\nu)(\underset{-}{\sigma} - \underset{+}{\sigma})}{N\,T^3}$$

The term in the brackets (see 15) will now be multiplied through by the numerator of the fractions in order to completely arrange for powers of R and R'. That way one obtains:

$$2\,T^3(\Delta \underset{+}{R} - \Delta \underset{-}{R}) = R'^3 h \left(\frac{\underset{-}{\sigma} - \underset{+}{\sigma}}{N}\right)^2 (\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \tag{I.}$$

$$-R'^2 R(1 + 2\cos(\nu' - \nu)^2) h \frac{(\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\sigma'} - \underset{+}{\sigma'})}{N\,N} (\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \tag{II.}$$

$$+R' R^2 \cos(\nu' - \nu) \left\{ \begin{array}{l} 2h \left(\frac{\underset{-}{\sigma'} - \underset{+}{\sigma'}}{N}\right)^2 (\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \quad \text{(IIIa.)} \\ +h \frac{(\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\sigma'} - \underset{+}{\sigma'})}{N\,N} (\underset{-}{\nu}\,\underset{-}{\sigma'} - \underset{+}{\nu}\,\underset{+}{\sigma'}) + \frac{T}{N} (\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\nu}\,\underset{+}{\sigma'} - \underset{+}{\nu}\,\underset{-}{\sigma'}) \quad \text{(IIIb.)} \end{array} \right\}$$

$$-R^3 \left(h \left(\frac{\underset{-}{\sigma'} - \underset{+}{\sigma'}}{N}\right)^2 (\underset{-}{\nu}\,\underset{-}{\sigma'} - \underset{+}{\nu}\,\underset{+}{\sigma'}) + \frac{T}{N} (\underset{-}{\sigma'} - \underset{+}{\sigma'})(\underset{-}{\nu}\,\underset{+}{\sigma'} - \underset{+}{\nu}\,\underset{-}{\sigma'}) \right) \tag{IV.}$$

$$2\,T^3 R(\Delta \underset{+}{\nu} - \Delta \underset{-}{\nu}) = R' \sin(\nu' - \nu) \times \text{ everything that follows}$$

$$R'^2 h \left(\frac{\underset{-}{\sigma} - \underset{+}{\sigma}}{N}\right)^2 (\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \tag{V.}$$

$$-R' R \cos(\nu' - \nu) h \frac{(\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\sigma'} - \underset{+}{\sigma'})}{N\,N} (\underset{-}{\nu}\,\underset{-}{\sigma} - \underset{+}{\nu}\,\underset{+}{\sigma}) \tag{VI.}$$

$$+R^2 \left(h \frac{(\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\sigma'} - \underset{+}{\sigma'})}{N\,N} (\underset{-}{\nu}\,\underset{-}{\sigma'} - \underset{+}{\nu}\,\underset{+}{\sigma'}) + \frac{T}{N} (\underset{-}{\sigma} - \underset{+}{\sigma})(\underset{-}{\nu}\,\underset{+}{\sigma'} - \underset{+}{\nu}\,\underset{-}{\sigma'}) \right) \tag{VII.}$$

§6

These equations yield directly the change the quantities ΔR and $\Delta \nu$ suffer caused by the deflection of the rays at a new refractive surface. To obtain the complete values of both quantities after an arbitrary number of refractions (or reflections) initially one would have to assign the quantities h and N the general index $2i$ of any surface then assign the index $2i-1$ to the quantities of the preceding medium of that surface that are marked with '-' and assign the index $2i+1$ to the surface currently labeled with '+'. Thereafter, if there are $k+1$ refracting surfaces present (where the endmost would get the index $2k$ according to our nomenclature), one would set $2i$ to be $0, 2, 4, ...2k$ and sum all equations obtained for ΔR and $\Delta \nu$ respectively.

It was shown that T, R, R' as well as ν and ν' are of constant value throughout the whole optical system. Therefore after summation the simply quantities $2\,T^3 \Delta R_{2k+1}$ and $2\,T^3 R\,\Delta \nu_{2k+1}$ will appear on the left side of the equations since the original values ΔR_{-1} and $\Delta \nu_{-1}$ are zero since errors in the image only arise due to refraction or reflection. On the right hand side all factors that depend on R, R', ν and ν' will be constant as well and the summation of the different equations of similar type on the right hand side will only change that each of these factors, instead of being multiplied with a single quantity that depends on h, σ and σ' , will be multiplied by a sum over all surfaces of such quantities. E.g., in the expression of $2\,T^3 \Delta R_{2k+1}$ the factor R'^3 is being multiplied with following sum:

$$h_0 \left(\frac{\sigma_{-1} - \sigma_1}{N_0}\right) (\nu_{-1}\sigma_{-1} - \nu_1 \sigma_1) + h^2 \left(\frac{\sigma_1 - \sigma_3}{N_2}\right) (\nu_1 \sigma_1 - \nu_3 \sigma_3) + \cdots + h_{2k} \left(\frac{\sigma_{2k-1} - \sigma_{2k+1}}{N_{2k}}\right)^2 (\nu_{2k-11}\sigma_{2k-1} - \nu_{2k+1}\sigma_{2k+1}) \tag{16}$$

And in the expression for $2\,T^3 R\,\Delta \nu_{2k+1}$ the mentioned sum will additionally be multiplied by the member from the factor R' $\sin(\nu' - \nu)$ that contains the quantity R'^2 One recognizes in general that not seven different sums of the members depending on R, R', ν and ν' appear in the terms for $2\,T^3\Delta R_{2k+1}$ and $2\,T^3 R\,\Delta\nu_{2k+1}$ but instead just five. Even though the general member for ΔR consists of four and the one for $\Delta\nu$ consist of three parts respectively the quantities dependent on h, σ and σ' in row I. and V. as well as in row II. and VI. are identical[9]. In order to extinguish all errors of third order in the image plane, that means to extinguish all combinations of the values R, R', ν and ν', only five equations need to be resolved that result by setting the sums of the general members to 0 that are multiplied by the factors depending on R, R', ν and ν' that appear in the rows:

I or V
II or VI
IIIb or VII
IIIa
IV

If one can not satisfy all of these five conditions, one will have to choose the most important ones amongst them by firstly considering the coefficients of the members that would cause the biggest error in the image. Depending on some special requirements, one might request by oneself, it could be adequate not to set some selection of the five sums to zero but instead certain combinations of them such that one looks for a complete extinction of the third order errors, if it's not possible to achieve for all values of R, R', ν and ν', of those specific combinations of values that are needed for the special purpose of the most important rays of the instrument.

The quantity ΔR_{2k+1} multiplied with $\frac{\nu_{2k+1}}{\sigma_{2k+1}}$ (following equation 10) represents the deviation of the ray along the direction of the radius vector in the last plane of our system of planes A. $R\,\Delta\nu_{2k+1}$, multiplied with the same factor, is the deviation of the ray in the direction perpendicular to the radius vector in the mentioned plane. If only the knowledge of the quantities ΔR and $\Delta\nu$ (that is without the simultaneous calculation of $\Delta R'$ and $R'\Delta\nu'$) by themselves is required to be sufficient to describe the errors in the image (as it was just assumed to be true), it must be presumed that the image plane coincides with the last plane of our system A, or in other words one has to place the initial plane of the system to the position of the object for which the analysis or the extinction of the imaging errors is of interest. The convenience of use makes this the prime choice beside any other. The behavior is different for the initial plane of the system B (that is represented by the quantities having an apostrophe). The choice of position last mentioned is of no meaning to the quantities ΔR and $\Delta\nu$, therefore it can be arbitrarily placed if one only looks at the correction of these quantities. However applying it to specific instruments it's often the case that the opening of the effective cone of light is primarily limited through the width of certain diaphragms (Where lens mounts and under certain circumstances the iris of the eye are concerned). Thereupon, if one preferably places the base plane of the system B at the position of the limiting diaphragm[10], therefore one has the advantage to determine the maximum value the quantity R' can take in an easier manner than in any other case. The knowledge of this limiting value will be important since the relative quantity of the different members listed above in rows I through VII depend on it and the selection of the most important of those members must be based on that knowledge, too.

In all cases, the maximum of the values of R' being present will increase with the aperture of the instrument, while the maximum value of R is fixed by the limit of the requested field of view. Instrument for which one does not request a big field of view but a relatively large aperture (in order to expect a significant magnification from them) the importance of the different members in the expressions will stepwise decrease from row I through IV and V through VII respectively. Among other examples, this case appears in astronomical telescopes. It appears that for example the radius of the free aperture of the Knigsberger Heliometers, viewed from the center of curvature of the 2^{nd} surface, is as big as 362 minutes [11], while the radius of the simultaneously visible peace of sky is assumed by Bessel to be at most 48 minutes. The angular dimension of the aperture is therefore 7.5 times bigger than the one of the field of view and from that it follows that the third power of the quantity proportional to last mentioned would have less influence than the (by us neglected) 5^{th} power of the quantity proportional to the aperture. Therefore in instruments of similar kind, the most important of the five constraints of an error free image will be the one that simultaneously eliminates the two expressions for ΔR and $R\Delta\nu$ that are multiplied with the third power of R' or the one that sets the extensive sum as given in (16) with which they are multiplied equal to zero. The complex of these members that are present for the center of the field of view ($R = 0$) builds the measure of the so called "deviation due to the spherical shape" and one can see that the equations for the elimination of them arises here as it was derived in issue 835 I. The four new expressions that accompany the already known show a specific affinity to it.

The case can happen and it will actually occur often, that our equation (10) defining the quantity R requires a little modification. We namely set $R = r_{2i-1}\frac{\sigma_{2i-1}}{\nu_{2i-1}}$ or, since this quantity is constant through all radii, $R = r_{-1}\frac{\sigma_{-1}}{\nu_{-1}}$. If the object and therefore the base plane

[9] Further the terms in row IIIb and VII are identical but the first mentioned is not independent and needs to be added to IIIa

[10] If this diaphragm doesn't belong to the first media, one has to take the position of its preimage that is the place where an object in the first medium would have to be so that the mentioned diaphragm would be at an image position of that object.

[11] According values given by *Bessel*. Astronomische Untersuchungen volume I. p.101 and following

of the system A happen to be at infinite distance r_{-1} will become infinity for all points outside the center of the field of view, in contrast to that $\sigma_{-1} = 0$ makes it necessary to write $R = \frac{h_0}{r_{-1}} \left(\frac{r_{-1}\sigma_{-1}}{h_0} \right)$ where the quantity $\frac{r_{-1}\sigma_{-1}}{h_0}$ = the tangent of the apparent distance to the center of the field of view of the point the ray comes from, since r_{-1} is the linear displacement and $\frac{h_0}{\sigma_{-1}}$ is the distance from the object to the first surface of the optical system.

In order to completely know the position of the ray leaving the instrument one would have to write the expressions for $\Delta R'$ and $\Delta\nu'$ which represent the corrections of the coordinates of the intersection points of these rays with the last plane of system B. Due to the very similar meaning that our two plane systems inhere, it is clear that one must only exchange the following in the rows I through VII of the above expressions in order to obtain these quantities.

R and ΔR with R', $\Delta R'$
ν and $\Delta\nu$ with ν', $\Delta\nu'$
h with h'
$\underset{-}{\sigma}, \underset{+}{\sigma}$ with $\underset{-}{\sigma'}, \underset{+}{\sigma'}$

Through which equation 7 transfers

T Into -T.

§7

If the derived equations are supposed to be applied not to analyze the error of an already existing instrument but rather to calculate the dimensions of a newly to be fabricated one following certain conditions, one has to recognize the grievance (that was already mentioned at the beginning) that in our expressions the quantities h, σ and h', σ' appear simultaneously since they are not independent from each other. Therefore one will have to eliminate one using the other. In a special case that's unnecessary. The case where all refracting surfaces are that close to each other that the separation d from each other can be ignored, it will in general be appropriate to set the base plane of our plane system B at the firs surface. Thereby $h'_0 = 0$ and therefore, in accordance to 2^{nd} equation of (1), all other $h' = 0$ too, since the ρ can not be zero the denominator $\nu_{2i-1}\sigma'_{2i+1} - \nu_{2i+1}\sigma'_{2i-1}$ must disappear in accordance to the first equation of (1). That is that the quantities σ' must be proportional to the ν that carry the same index and could be (since the units of σ' and h are arbitrarily) set equal. Through that the quantities h' and σ' are known and only h and σ are kept unknown. Therefore the expressions themselves, I through VII, simplify significantly and initiate some general conclusions according the types of instruments being discussed here and will be mentioned at the end. In all other cases it remains necessary to eliminate the quantities h' and σ' through h and σ. The equations that relate both quantities are derived in issue 871 I. II. III [12]. Where there l is our h' and τ is our σ' respectively. Establishing T as defined in (7) and the further abbreviations

[12] They simply source by recognizing that ρ and d must have the same value whether they are represented by h and σ or by h' and σ'.

$$\frac{h'_0}{h_0} = \chi; \qquad \nu_{2i-1} + \nu_{2i+1} = 2\mu_{2i} \tag{17}$$

$$\sum^{i} = \sum_{p=1}^{p=i} \frac{\nu_{2p-1} d_{2p-1}}{h_{2p-2} h_{2p}} \tag{18}$$

one obtains the following equation that can be immediately applied to our expressions.

$$\left.\begin{aligned}
\sigma'_{2i-1} - \sigma'_{2i+1} &= (\sigma_{2i-1} - \sigma_{2i+1}) \left(\chi - T \sum^{i} \right) + \frac{TN_{2i}}{h_{2i}} \\
\nu_{2i-1}\sigma'_{2i-1} - \nu_{2i+1}\sigma'_{2i+1} &= (\nu_{2i-1}\sigma_{2i-1} - \nu_{2i+1}\sigma_{2i+1}) \left(\chi - T \sum^{i} \right) + \frac{2TN_{2i}\mu_{2i}}{h_{2i}} \\
\nu_{2i-1}\sigma'_{2i+1} - \nu_{2i+1}\sigma'_{2i-1} &= (\nu_{2i-1}\sigma_{2i+1} - \nu_{2i+1}\sigma_{2i-1}) \left(\chi - T \sum^{i} \right) \\
h'_{2i} &= h_{2i} \left(\chi - T \sum^{i} \right)
\end{aligned}\right\} \tag{19}$$

(The sums $\sum$ to the right disapear for $i = 0$)

These values can be placed into the equations for ΔR and $R\Delta\nu$ and then order them for the powers of T. It's clear by it self that any power of T in all of the 7 rows can't be multiplied by more than five different factors since we have already seen that the satisfaction of five equations is sufficient to cancel the errors of an image in a given plane and once they are canceled they must stay that way independent on the value of T. Since it's clear that the extinction of the errors that occur in the last image plane of our system A can only depend on the first plane of the system (that means from the position of the object) and from the values of ρ and d. According to equation (7) the quantity T may change its value without changing the just mentioned members by a change of h'_0 and σ'_{-1} as e.g. by shifting the base plane of plane system B (change of $\frac{h'_0}{\sigma'_{-1}}$). Satisfying the five conditions therefore can't depend on T. This point of view gives us a way to control the conditions our expressions need to satisfy after the elimination of h' and σ' After doing the substitutions, as given in (13), in the expressions I through VII, doing the proper ordering and reduce them as much as possible one gets:

$$\left.\begin{array}{l} 2T^3\left(\underset{+}{\Delta R} - \underset{-}{\Delta R}\right) \\ \text{or general member of} \\ \text{the expression of } 2T^3\Delta R_{2k+1} \end{array}\right\} = \left\{\begin{array}{l} R'^3\cos(\nu'-\nu)\,① \\ -R'^2R[1+2\cos(\nu'-\nu)^2]\,\{\chi\,① + T\,②\} \\ +R'R^2\cos(\nu'-\nu)\,\{3\chi^2\,① + 6\chi T\,② + T^2(2\,③ + ④)\} \\ -R^3\,\{\chi^3\,① + 3\chi^2 T\,② + \chi T^2(2\,③ + ④) + T^3\,⑤\} \end{array}\right\}(VIII.)$$

$$\left.\begin{array}{l} 2T^3(\underset{+}{\Delta\nu} - \underset{-}{\nu}) \\ \text{or general member of} \\ \text{the expression of } 2T^3R\Delta\nu_{2k+1} \end{array}\right\} = R'\sin(\nu'-\nu)\times\left\{\begin{array}{l} R'^2\,① \\ -2R'R\cos(\nu'-\nu)\,\{\chi\,① + T\,②\} \\ +R^2\,\{\chi^2\,① + 2T\,② + T^2\,④\} \end{array}\right\} \qquad (IX.)$$

Where the quantities labeled with ① ... ⑤ are defined in the following manner.

$$U_{2i} = \frac{1}{h_{2i}}\frac{N_{2i}}{\sigma_{2i-1}-\sigma_{2i+1}} - \sum_{p=1}^{p=i}\frac{\nu_{2p-1}d_{2p-1}}{h_{2p-2}h_{2p}} \qquad (X.)$$

(the sum on the right disappears for $i = 0$)

$$\left.\begin{array}{l} ① = h_{2i}\left(\frac{\sigma_{2i-1}-\sigma_{2i+1}}{N_{2i}}\right)^2 \\ \qquad \times(\nu_{2i-1}\sigma_{2i-1} - \upsilon_{2i+1}\sigma_{2i+1}) \\ ② = ①\,U_{2i} \\ ③ = ②\,U_{2i} \\ ④ = ③ - \frac{N_{2i}}{\rho_{2i}} \\ ④ = ④\,U_{2i} \end{array}\right\} \qquad (XI.)$$

It must be mentioned, that the letters d and ρ are used to simplify notation and therefore are present beside σ and h. In the case where the quantities just mentioned are not already known they have to be viewed as abbreviations for the expressions in σ and h in equation (1)[13].

[13] The expressions in (VIII) and V (IX) can be written shorter if one replaces the abbreviation U_{2i} by $V_{2i} = \chi + TU_{2i}$. It is more convenient to do the remaining derivations based on the last abbreviation, however I keep previous one here since the five conditional equations of an error free image are most visible.

The complete expressions for the errors arising in the image plane, $2T^3\Delta R_{2k+1}$ and $2T^3R\Delta\nu_{2k+1}$ will only differ such that in the general terms in VIII. and IX. the quantities ① , ② , ② , ④ and ⑤ are replaced by the sum of these quantities over all surfaces (that is $i = 0$ through $i = k$) S① , S② , etc. The elimination of the four sums S① , S② ,2S③ +S④ and S⑤ are going to be the conditions for the extinction of all errors of third order in the direction of the radius vector and the elimination of the three sums S① , S② and S④ extinct the errors of same order being perpendicular to the radius vector. Finally the image becomes, except for the errors of fifth order, very precise if and only if all of the five sums S① , S② ,S③ , S④ and S⑤ disappear. The condition S① $= 0$ is, as already mentioned, again the one for the extinction of the so called spherical deviation. One can see that the harmony of all expressions, in accordance to the relation of the general member of that equation and the other four given in (XI), of which the errors of third order in general depend is very pleasant and that they can also be used to numerically calculate the errors of a given optical instrument very conveniently.

§8

Although the equations last derived at first only describe the deviation of the rays in a given plane, namely the last one of our system A, they are sufficient for all analysis corresponding to the distribution of the light in different planes that are direct neighbors to it. Because if the distance of such a new plane E is very small with respect to its distance to the last plane of the system B the calculation of the deviations in plane E are apparently influenced very little by the quantities $\Delta R'$ and $\Delta\nu$ and fall into a higher order. Therefore, the intersection points of the rays leaving the instrument with the plane E can be obtain as accurately as our analysis allows by assuming that the position of the rays are defined by $R + \Delta R_{2k+1}$, $\nu + \Delta\nu_{2k+1}$ on one hand and R' and ν' on the other hand.

If one for example assumes that the errors ΔR and $\Delta\nu$ are not completely eliminated that is that the object lying on the base plane of system A is not perfectly precisely imaged onto the last plane of that system, and if one wants to analyze (what *Bessel* similarly calculated on the Koenigsberger Heliometer using trigonometric methods) in which plane the magnitude of the errors in the image are as small as possible the derivation just made will be of use, since it is clear ahead of time that the target plane must be close to the plane to which our equations relate. It behaves the same as if one asks under which conditions within third order representation a perfect image is produced on an axis symmetric surface different from our plane. For that to become true, it is required that all rays coming from one point, that means having the same R and ν, have a single intersection point after refraction or if one applies right-angle coordinates x, y and z as done earlier, the projection of the rays in the plane xy intersect in one point and the projection of the rays in the plane xz intersect in a second point and simultaneously the coordinate x is the same for both points. We think of x the same way as earlier as being measured parallel to the optical axis and since the directions of this axis and the self perpendicular y and z axes, are arbitrary, one can obtain the right-angle coordinates from polar coordinates most conveniently by defining then such that $\nu = 0$ and from that it immediately follows that $\frac{\nu_{2k+1}}{\sigma_{2k+1}}\Delta R_{2k+1}$ is the deviation parallel to y in the last plane of the system A' and $\frac{\nu_{2k+1}}{\sigma_{2k+1}}R\Delta\nu_{2k+1}$ is the deviation parallel to z in the same plane. If one does the analysis one will find that:

a) A plane E perpendicular to the axis, in which all rays, that belong to the same R and $\nu = 0$ but also belong to all other possible R' and ν', intersect in the same $z = 0$, only exists if the equations S ① $= 0$ and S ② $= 0$ must be satisfied. Under this conditions there will exist a plane E' in which the rays of the same complex will intersect in the same y.

b) In order for these two planes E and E' to superimpose, all rays R, ν must have x, y and z simultaneously in common. That is, they intersect in one point which means that S ③ $= 0$ must be satisfied additionally.

If one recognizes that here y is chosen to be parallel to the radius vector from which the light originates while z is perpendicular to selfsame the following arises: If only condition a) is satisfied but not condition b), two planes different from each other will exist in such a sense that in the first plane (E) a luminous point is imaged as a short line pointing to the center of the field of view having no width while in the second plane (E') it is imaged as a line perpendicular to the line pointing to the center but without width as well. A more accurate examination shows how the two images transfer into one other. Namely, if one continuously shifts the plane perpendicular to the axis from the position of E to the position of E' the first straight line is at the beginning replaced with an elongated narrow ellipse that continuously becomes shorter and wider until, after going through a circle, the short axis falls into the direction of the previously long one and vice versa and after a continuous increase of the first line and a decrease of the second one, the second line ends the process. The two planes E and E' will shift their position as the luminous point changes its distance to the axis. Therefore a single plane on which all points of an object drawn as line going through the axis are imaged without width does not exist but only a curved axis symmetric surface F can satisfy that as well as a second axis symmetric surface F' will have the property that objects drawn as circles concentric to the axis are imaged without width onto F'.[14] Both surfaces F and F' touch each other in their vertex which also represents the precise image of the center of the object. They coincide in one surface if [in addition to S ① and S ②] also S ③ $= 0$ and therefore, this surface represents the place where an extended object is imaged exactly except for the fifth order errors, since on that surface all errors are eliminated that are pointing into the direction of the radius vector and perpendicular to it.

If additionally S ④ $= 0$ the surface just noted surface becomes a plane since from VIII and IX it's clear that if S ① through S ④ vanish simultaneously all rays that have R and ν in common will intersect at the same point in the last surface of system A. Therefore one gets in this plane an image that is free of any indistinctness (except of fifth order quantities). The only third order error that it will still contain is distortion of the outer parts for their elimination S ⑤ $= 0$ must be satisfied. If this quantity is not extinguished the distances of the images of different luminous points from the axis will not be proportional to distances of the luminous points themselves from the axis since the first mention are proportional to $R + \Delta R$ while the second mentioned are proportional R.[15]

Therefore one can only talk about a distinct position of the image if the three equations S ① $= 0$, S ② $= 0$ and S ③ $= 0$ are simultaneously satisfied. Thereupon this position is a rotationally symmetric surface with the

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[14] The surface F must have the position of the planes E in common for which they are capable to image a point as a line. The same relation must hold for F' with all E'.

[15] If the sums S ① through S ④ are not made to be $= 0$ the quantity S ⑤ by itself can not be used as a measure for the distortion that the eye really perceives in the image. Since in that case not all rays that source in one luminous point will intersect the last plane in the same point the eye will place the image of that point at different positions of the plane dependent on the direction of the rays it is receiving with its current position and if it is not receiving all of them in all positions its motion will cause motions of the image that have their source in an imperfect precision of the image itself.

axis being the optical axis. One can view the very same as a sphere by replacing it with the sphere that it sources at its vertex. Since the difference between them falls into the orders that are neglected in our analysis. It is easy to determine the radius of that sphere or the curvature of the image because the center of the sphere must lie on the axis as well and since it is touching the last plane of our system A on the axis it is only required to calculate one lateral displaced point for a complete description of the sphere. Its radius shall be labeled as g_{2k+1} for g positive under the same conditions as defined for ρ. Following the same path that was previously mentioned to lead to the conditions in a) and b), g can easily be determined by the simple equation

$$\text{XII} \qquad -\frac{\nu_{2k+1}}{g_{2k+1}} = S\frac{N}{\rho},$$

from which it results again that the image is going to be flat if $S\frac{N}{\rho}$ is equal to zero. That is, if beside S ① , S ② and S ③ also S ④ vanishes, since XI following equation must hold $\frac{N}{\rho}$ = ③ − ④ .

Therefore in order to determine the radius of curvature of the last image of a flat object, beside the refraction ratios one only needs to know the radius of curvature of the different surfaces but not the distances between them and not the distance of the object to the instrument and not even the order the different curved surfaces following each other as long as it doesn't influence N. However one would be wrong if one would conclude from that, that an instrument for which the ρ stay constant would image all plane objects onto a surface of the same curvature independent of how one would vary the distances between the surfaces and object distances. Since a change in those quantities would in general cause the equations S ① = 0, S ② = 0 and S ③ = 0 to no longer stay satisfied, so that no precise image would be present any more and therefore a declaration of the curvature of the surface on which the image should be is impossible.[16]

In passing it shall be mentioned here that according to equation XII it will be very rarely possible to produce a precise image of a plane object. Because if one would only use one type of glass in an optical instrument the condition for a flat image $S\frac{N}{\rho} = 0$ would result if all surfaces are thought to be shifted next to each other the focal length must become infinity it therefore would behave as a thin plane parallel glass. Such a set up of its surfaces would rarely be tolerated by the purposes for which it is intended. The situation is not improved if two types of glasses are required for achromatic instrument. Since then this condition contradicts with the more important one of eliminating the dispersion of color while namely the later dictates that the less dispersive lenses defines the sign of the total focal length (if all surfaces are next to each other) and the first dictates that the more refractive (that is in fact more dispersive) would take priority. Only in cases where it is possible to apply relative large glasses thicknesses, as with small aperture as in oculars and maybe in microscope objectives one can hope to avoid the contradiction.

The equation XII that determines the radius of curvature of the image, if that really exists, can easily be extended for the case in which the object itself is not plane but spherical with the radius g_{-1}, whereas it is required that the center of its curvature lies on the axis. One could namely think that such an object itself coincides with a precise image, precise within third order, that is produced from a plane object by a number of fictitious refracting surfaces of radii P in order to assume that

$$-\frac{\nu_{-1}}{g_{-1}} = S\frac{N}{P}$$

The last image for which the curvature is being determined, therefore must coincide with the image that is caused by a (fictitious) plane object after the rays coming from it have passed all surfaces of radii P and radii ρ, so that following equation XII one must get

$$-\frac{\nu_{2k+1}}{g_{2k+1}} = S\frac{N}{P} + S\frac{N}{\rho}$$

and therefore

$$\frac{\nu_{-1}}{g_{-1}} - \frac{\nu_{2k+1}}{g_{2k+1}} = S\frac{N}{\rho} \qquad \text{(XIII.)}$$

§9

It was already pointed out that the second most important of the five conditions of an accurate flat image after the extinction of the members multiplied by R'^3 in I. and V. or in VIII. and IX. is the one that requires that the members multiplied by R'^2R (II. and VI. or VIII. and IX.) must be eliminated, thus besides S ① also S ②=0. On a common double objective assembled from two lenses only the four radii are available as variable quantities since the thicknesses are fixed to be very small ahead of time for irrefutable reasons. Therefore one can only satisfy four equations of which one is defined by

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16 The theorem in equation XII was published by Petzval in 1843 in his dioptric work (upon which to my knowledge no further publications followed) but he did not give the preconditions that must be satisfied so that the theorem has a meaning.- There is a method to achieve that theorem very easily by assuming a spherical curved object, just considering the effect of one refracting surface and then searching for the osculating sphere of that rotational symmetric surface on which the peaks of all focal planes lie that are illuminated by the individual points of the object. If one then views the image on that sphere as new object from which the rays reach a second surface and so on the same theorem results. This derivation by itself is largely deficient since it is not clear under which circumstances the peaks of the focal planes can be used instead of setting new objects that is ignoring some part of the spherical deviation which is of same order as the calculated deviation itself. Finding this equation that way seems to be by chance and was certified by the elaborated analysis under the conditions given in the text.

the total focal length, a second moves the different colored images of a distant object in the same plane, while the third eliminates the deviation from the sphere in the center of the field of view or equivalently sets S ① = 0. Different suggestions were made on the selection of the fourth one. Our analysis suggests to set S ② = 0 which will cause not only the center of the field of view but also its close neighborhood to be imaged as precisely as possible. A closer examination of that condition therefore makes sense since there are references available in which *Utzschneider* states that his great companion *Fraunhofer* was lead to the known peculiar construction of the telescope objective by following the goal to minimize the error in the image over the entire field of view.[17] One can do the test on the Koenigsberger Heliometer that's calculated based on *Fraunhofers'* method and *Bessel* published its constants (Astronomische Untersuchungen Bd. I. p.101). If one first calculates the σ and h following equation (2) having set $\sigma_{-1} = 0$ (due to the infinite distance of the object) and sets h_0, which is arbitrary, equal to the focal length +1131,4548 (in order to get $\sigma_7 = 1$) one gets

$$\begin{aligned} &\sigma_{-1} = 0 \\ &\sigma_1 = +0,4671176 \qquad \log h_0 = 3,0536373 \\ &\sigma_3 = +2,5035619 \qquad \log h_2 = 3,0525602 \\ &\sigma_5 = +0,2350630 \qquad \log h_4 = 3,0525602 \\ &\sigma_7 = +1 \qquad \log h_6 = 3,0521982 \end{aligned}$$

With these numbers I find the distinct member of S① to be

-629,848
-85923,723
+90164,021
-3716,744

S ① = -106,274

One recognizes that, since this sum does not completely vanish, a small spherical deviation remains for the center of the field of view as was demonstrated by *Bessel* in a.a.O. p. 103 and it does not only source in higher order terms (that were included in *Bessels'* trigonometric calculations) but also in residuals of third order.[18] If one also calculates the members of S ② one will find.[19]

+0,412
-12,672
+13,454
-1,662

S ② = -0,468

Although this sum does not vanish completely but rather about $\frac{1}{30}$ of its maximum value remains, I believe that the nearly achieved compensation of the positive and negative members did not happen by chance but undoubtedly, as the above referenced comments from *Utzschneider* imply, *Fraunhofer* truly intended to have the spherical deviation as constant as possible over the entire field of view. It is unlikely that he had an analytical expression that would have helped him but rather achieved his goal through repeated trigonometric analysis and therefore it appears to be resolved that the value of S ② remains bigger, compared to the biggest member of the sum, than for S① where it is just $\frac{1}{900}$. Since not as much of that error could appear due to the small field of view as for the one that's influenced only depends on the aperture. For the same reason it's also the less harmful one.

I therefore believe that the condition

S ② = 0

can be considered as the fourth condition of *Fraunhofer* for the double objective and will allow myself to call it after his name.

This condition shows a peculiar relation with the equation from *Herschel*, that as is generally known produces a minimal change in the spherical deviation as the distance of the object varies. By following our path the expression for that condition (while strictly considering the thicknesses of the media) is easily derived. If the deviation from the sphere is not only to be eliminated in the image plane to which h and σ relate to but also for the ones for which h' and σ' are valid so not only

$$\text{S ①} = S\, h \left(\frac{\underset{-}{\sigma} - \underset{+}{\sigma}}{N}\right)^2 (\underset{-}{\nu}\, \underset{-}{\sigma} - \underset{+}{\nu}\, \underset{+}{\sigma}),$$

but also

$$S\, h' \left(\frac{\underset{-}{\sigma'} - \underset{+}{\sigma'}}{N}\right)^2 (\underset{-}{\nu}\, \underset{-}{\sigma'} - \underset{+}{\nu}\, \underset{+}{\sigma'})$$

must disappear. One can here eliminate the h' and σ', as it was done earlier in the other expression, using equation (19) and will therefore obtain an expression ordered by the powers of T where the maximum member contains T^4. The member without a T is again S① ; the member multiplied by $\chi^3 T^1$, if it only arises from one surface, is

$$= h\ \text{②} - \underset{-}{\sigma}^2 + \underset{+}{\sigma}^2$$

[17] Among other places one can find an indication of this in a letter of *Utzschneider* to the 'Konferentzrath' *Schumacher* as printed in Astronominsche Nachrichten.

[18] This residual causes an even slightly bigger longitudinal deviation with trigonometric calculations. With our equations one would find that the incoming rays parallel to the axis at the edge of the objective would have their point of reunion 0,05114 lines closer to the objective than the central rays while in accordance to *Bessel* the difference in the same sense is 0,0461.

[19] The logarithms of the quantities U of the four surfaces are, 6,8160542; 6,1687300; 6,1738214; 6,6503272

and therefore the sum over all surfaces becomes

$$= 4\mathrm{S}\,\textcircled{2}\; - \sigma_{-1}^2 + \sigma_{2k+1}^2$$

This member must primarily be extinguished if the deviation from the sphere is supposed to be eliminated for objects of distance $\frac{h_0'}{\sigma_{-1}'}$ that are close to the $\frac{h_0}{\sigma_{-1}}$, since one has $T = \frac{\sigma_{-1}\sigma_{-1}'}{\nu_{-1}}\left(\frac{h_0}{\sigma_{-1}} - \frac{h_0'}{\sigma_{-1}'}\right)$ so that the condition from Herschel in our terminology becomes

$$0 = \mathrm{S}\,\textcircled{2}\; - \tfrac{1}{4}(\sigma_{-1}^2 - \sigma_{2k-1}^2) \qquad \text{(XIV)}.$$

It can only coexist with the *Fraunhofer* condition[20]if one has

$$\sigma_{-1}^2 = \sigma_{2k+1}^2,$$

that can be obtain in three different ways:

α) It can be that $\frac{\sigma_{2k+1}}{\sigma_{-1}} = +1$. This case happens if the planes perpendicular to the axis are placed on the (from *Listing* so called) nodal points of the instrument. (see the note to p. 110 in Issue 871).

β) It can be that $\frac{\sigma_{2k+1}}{\sigma_{-1}} = -1$. In this case the object and image are at the anti points of the nodal points, that is at the points that are as far from the corresponding focal points as the nodal points but on the opposite side.

If the same medium e.g. atmospheric air is present at the beginning and the end so that $\nu_{-1} = \nu_{2k+1}$ then the nodal points coincide with the Gaussian principal points and the object and image in the cases α and β are of equal size and have in the first case similar and in the second opposite orientation. Only case β will be of some practical importance, which occurs in a camera obscura when objects are imaged to their natural size.

γ) The examination of the possibility of σ_{-1}^2 and σ_{2k+1}^2 becoming equal by being equal to zero is of more interest. In that case the object as well as the image are at infinite distance; it therefore occurs in a telescope in a setting where it shows the stars clearly to a far sighted observer. A telescope in that setting and viewed as a whole is therefore a preferred instrument that has the property of fulfilling the *Fraunhofer* and *Herschel* condition simultaneously if it fulfills one of them; or for the sameself the two advantages that on one hand imaging an extended object at very far distance as free of errors as possible and on the other hand always image the center of the object precisely when it is moved closer are inseparable.

The telescope objective can, strictly speaking, not have that property since $\sigma_{-1} = 0$ but σ_{2k+1} will be different from zero. Only if it happens to be e.g. for the *Fraunhofer* objective, that the member $-\frac{1}{4}(\sigma_{-1}^2 - \sigma_{2k+1}^2)$ of which the two expressions are different is much smaller than the quantities within the single members of S $\textcircled{2}$ they have in common; e.g. for the discussed *Heliometer* objective $-\frac{1}{4}(\sigma_{-1}^2 - \sigma_{2k+1}^2) = +0,250$ or being not quite $\frac{1}{50}$ of the biggest member of S $\textcircled{2}$ so that the two errors of one and the same (*Fraunhofer* or *Herschel*) construction of the objective are reduced to a small fraction of the deviations that are caused by the refraction on just one surface. The Heliometer-objective leaves an even smaller error in equation (XIV) than it does in equation S $\textcircled{2}$ $= 0$; however one obtains noticeably bigger errors in both for objectives that are calculated for other conditions. One can say that the *Fraunhofer* construction principle will produce the most perfect double objective; whereas one must admit that it is difficult to apply it with the accuracy that its advantage would become noticeable; an analysis of the influence of imperfect fabrication of the distinct radii has onto the rise of errors in the image proves that especially the two most curved surfaces must be ground to high precision if a very noticeable spherical deviation should be avoided.

§10

One would have an instrument that would image objects at all distances without third order errors if it would be possible to eliminate all errors of that order in two different image planes. Since the position of a straight line is completely fixed by declaring its intersection points with two planes, any ray that can arrive at the instrument would leave the optical system accordingly (except for the 5^{th} order errors) as described with the approximated equations from which the existence of the images of the objects at all distances are calculated. If one therefore wanted to calculate such an instrument, one would not have to satisfy 10 distinct equations but rather 9; since the equation S $\textcircled{3}$ $-$ S $\textcircled{4}$ $= 0 = S\frac{N}{\rho}$ contains nothing that relates to a specific position of an image, it would therefore be the unaltered one within the five equations that are used to eliminate the errors in a second plane. But an instrument that is really doing something can not satisfy the 9 equations. One of the four conditions that would be added to our five is namely the one from *Herschel* , equation (XIV), that would have to coexist with the one from *Fraunhofer* S $\textcircled{2}$ $= 0$ and since the instrument would have to produce images of the same quality for objects at all distances, $\sigma_{-1}^2 = \sigma_{2k+1}^2$ needed always to be true, independent of the value of σ_{-1}; that is, from the geometrical meaning of σ all the rays coming from any point on the axis after leaving the optical system would have to have the same angle with respect to the axis as before. Therefore the effect of the instrument would mainly be reduced to that of a plane glass or (if $\sigma_{-1} = -\sigma_{2k+1}$) to that of a plane mirror; if one requires functionality that goes beyond this simple one,

[20] For the error of color dispersion I already showed in the essay in Issue 871 that if it is eliminated for the center of the image for an object at specific distance the two conditions coincide, namely 1) that the error is eliminated for the outer parts of the field of view too and 2) that the deviation of distance remains eliminated as the object changes distance. One can see that for the error of the deviation of from the sphere these two conditions only become identical in very specific cases but strange to say that a simple relation amongst them exists.

one needs to abandon the requirement to obtain images in two planes that only leave errors of fifth order.

One cannot even achieve that level of accuracy in a single image plane if one doesn't assign significant separation to some of them, as well as proper determination of the radii of the surfaces. For the special case where all $d = 0$ it is simple to write the five expressions. The first of them S $\textcircled{1}$ suffers no change other than the simplification that results from the comment that all h become identical and therefore can be set = 1.

The expression S $\textcircled{2}$, that becomes equal to $S\left(\frac{\underset{-}{\sigma}-\underset{+}{\sigma}}{N}\right)(\underset{-}{\nu}\ \underset{-}{\sigma} - \underset{+}{\nu}\ \underset{+}{\sigma})$ can be simplified if one writes $\mu(\underset{-}{\sigma} - \underset{+}{\sigma}) + \frac{N}{2}(\underset{-}{\sigma} - \underset{+}{\sigma})$ instead of $(\underset{-}{\nu}\ \underset{-}{\sigma} - \underset{+}{\nu}\ \underset{+}{\sigma})$ whereas μ, as above, is the arithmetic mean from $\underset{-}{\nu}$ and $\underset{+}{\nu}$; therefore the sum becomes

$$\text{S}\ \textcircled{2} = S\frac{\mu}{N}(\underset{-}{\sigma} - \underset{+}{\sigma})^2 + \tfrac{1}{2}(\sigma_{-1}^2 - \sigma_{2k+1}^2) \qquad \text{(XV)}.$$

and similarly the *Herschel* expression (see XIV)

$$\text{S}\ \textcircled{2} - \frac{1}{4}(\sigma_{-1}^2 - \sigma_{2k+1}^2) = S\frac{\mu}{N}(\underset{-}{\sigma} - \underset{+}{\sigma})^2 + \tfrac{1}{4}(\sigma_{-1}^2 - \sigma_{2k+1}^2) \qquad \text{(XVI)}.$$

Further results $\textcircled{3} = (\underset{-}{\nu}\ \underset{-}{\sigma} - \underset{+}{\nu}\ \underset{+}{\sigma})$ and

$$\text{S}\ \textcircled{3} = \nu_{-1}\sigma_{-1} - \nu_{2k+1}\sigma_{2k+1} \qquad \text{(XVII)}.$$

Since $\textcircled{4} = \textcircled{3} - \frac{N}{\rho}$ or here following equation (1) $= \underset{-}{\nu}\ \underset{-}{\sigma} - \underset{+}{\nu}\ \underset{+}{\sigma} - \underset{-}{\nu}\ \underset{+}{\sigma} + \underset{+}{\nu}\underset{-}{\sigma} = 2\mu(\underset{-}{\sigma} - \underset{+}{\sigma})$ given one gets

$$\text{S}\ \textcircled{4} = 2S\mu(\underset{-}{\sigma} - \underset{+}{\sigma}) \qquad \text{(XVIII)}.$$

and finally $\textcircled{5} = 2\mu N = \underset{-}{\nu}^2 - \underset{+}{\nu}^2$, therefore

$$\text{S}\ \textcircled{5} = \nu_{-1}^2 - \nu_{2k+1}^2 \qquad \text{(XIX)}.$$

This last expression will in most applications become 0 automatically since the whole instrument is immersed in one single medium so that $\nu_{-1} = \nu_{2k+1}$ if no or an even number of reflections occur whereas $\nu_{-1} = -\nu_{2k+1}$ for an odd number of reflections. But if $\nu_{-1}^2 = \nu_{2k+1}^2$ the expression S $\textcircled{3} = \nu_{-1}\sigma_{-1} - \nu_{2k+1}\sigma_{2k+1}$ can't be made to be zero without giving the whole instrument an infinite focal length (compare Issue 871 equation VI), that is (since with $d = 0$ it cant be a magnifying or demagnifying telescope) without reducing its functionality again to be equivalent to a mirror or a thin plane glass (that has no function). It was already denoted above, §7, that for the case of disappearing thicknesses the expressions of the different errors can be conveniently obtained from row (I) through (VII) by setting h'=0 and making σ' equal to ν. Indeed one obtains the same results as were just derived an other way; whereby a control of proper elimination is given with which the expressions (I) through (VII) were transformed into (VIII) and (IX).

The equations (VIII) and (IX) can be verified with some other controls too. For some portion they relate to the symmetry of the expressions as the unit of ν and N is arbitrary likewise h and σ can be assigned an arbitrary factor as well as -R can be written instead of +R if one simultaneously changes ν into $\nu + \pi$ and an analogous relation exists for R' and ν'. Some other checks are simple conclusions from the strict equations of refraction from which e.g. no $\Delta\nu$ can arise if $\nu' = \nu$. In accordance to the same equations the influence of a refracting surface can not produce a new member in the expressions of $\Delta\nu$ if two consecutive planes of our transverse system A (namely the one preceding the surface and the one in the succeeding medium) virtually coincide in the center of curvature of that surface. - If the ray is normal to the surface it does not suffer any deviation; - all rays that intersect with the normal in the same point and construct the same angles with it are strictly rotated by the same angle out of their direction and each of them will stay in the plane that coincides with the normal and its direction before refraction. Finally the errors ΔR and $\Delta\nu$, that are viewed in the last plane of system A, can not at all depend on the position of the planes of system B; therefore the first mentioned must remain unaltered if one changes T and simultaneously allows R' and ν' to vary so that the same ray is kept unchanged. - The equations pass all checks that result from these examinations. The confirmation of that, which is not hard to carry out, is hardly of enough general interest that it would authorize me to use up more space here.

Munich 1855 April 6. *Ludwig Seidel*