Investigations into geometrical optics. II

Theory of mirror telescopes.

by

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§ 1. Introduction.

1. In the competition between refractors and reflectors, the reflectors are at present gaining ground. Many and various of the former doubts concerning the precision and stability of large mirrors have been dismissed by the technical progress of recent years. At present glass mirrors with a silver-plated front side are commonly used. As the thickness of the silver layer turns out very even from experience, the exact form giving process used in the fabrication of lenses is also suitable for the fabrication of mirrors. Warping and temperature related strain can be reduced to a harmless measure by suitable mounting (Ritchey, Chicago). The low weather resistance (the silver layer quickly loses its high gloss) is compensated for by arranging the elements in such a way that the mirror can easily be taken out and can be freshly silvered during the course of a single day.

With this obstacle removed, the advantages of reflecting telescopes are plainly shown, two of which stand in first place. The first to emphasize is the economic advantage offered by reflectors. An ordinary achromatic lens has four polished surfaces, the reflecting telescope (apart from the small plane mirror) has only one and the quality demand of the glass mass of the mirror (although it must be good) is not the very highest. Consequently, the price ratio between lenses and mirrors of the same diameter can rise up to 10:1 with large dimensions.
In addition the mirror is free from all colour aberrations. While the secondary spectrum of the so-called achromatic lens is still it's worst defect, no colour separation at all occurs at the mirror.

As well has having zero dispersion, the reflection capability of silver reaches far into the ultra violet, which is very valuable for photographic and spectral recordings.
These advantages are opposed by one substantial disadvantage at least with the present reflecting telescopes: the restriction of the visual field. A parabolic mirror delivers a perfect image on the axis, but only half a degree from the axis with an aperture ratio of 1/4, a coma the size of 29" appears. In the following investigation the question is asked whether progress cannot be achieved in this point by using two mirrors of a suitably calculated shape instead of the commonly used parabolic mirrors with diagonal plane mirrors. The answer is a positive one. It is possible to design telescopes with 2 mirrors, that deliver the same expansion of the usable visual field (2°-3° diameter) at an aperture ratio of 1/3, corresponding, for example, to the refractors of the same diameter commonly used in the enterprise of producing photographic sky maps. With this, it seems another application area is opening up for reflecting telescopes.

2. Summary: Instead of beginning with the special task mentioned above, the universal theory of third order aberrations for mirror systems is developed. It is a simplified analogue to Seidel's aberration theory for lens systems, discussed in the previous paper I § 6. As an application result, familiar notes about the aberration of a single parabolic mirror and a full overview is given on practical usability of systems of two mirrors. A particularly favourable system of this kind is isolated. Finally the area of validity of the third order aberration theory is expanded, the previously received mirror shapes are followed up to higher aperture degrees. The problem is to define a system with two mirrors that not only has a sharp focal point, but simultaneously strictly follows the sine condition. Using Abbe's terminology such a system is described as an "aplanatic" system, free from spherical aberration and coma. The mirror meridians derive from a differential equation, which is in a strange way algebraically integratable. For these systems, which are recognised as particularly useful by the theory of third order aberrations in relation to remaining aberrations and by their general geometry, the exact mirror shapes are calculated from the integrals and compared to second order rotational surfaces which share a common pole.

§ 2. Third order aberrations of mirror systems

The entire development follows in close analogy with the derivation of the third order aberration theory of a lens system in the previous paper I. § 6. Even the terms stay almost the same throughout. I will therefore only carry out the calculation process without going into detailed reasoning.

3. The Eikonal of a single mirror. Let the mirror be a rotational surface. The x-Axis coincides with the rotational axis and will be counted positive according to the direction of light propagation. The overview of the ray path is simplified, if you think of the mirror itself and the entire system of reflected rays mirrored at the tangential plane on the vertex of the mirror, replacing Fig. 1 by Fig.2. The advantage is that now light propagation always follows one direction. The analogy of the concave mirror and the convex lens is immediately apparent.

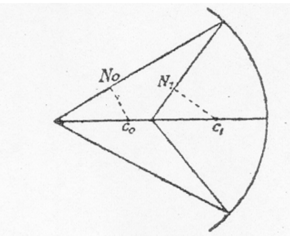


Fig. 1.

For spherical mirrors the equation is:

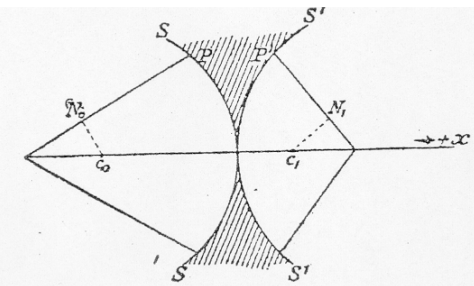


Fig. 2.

$$1)\qquad X - a = \sqrt{r^2 - Y^2 - Z^2} - r = -\frac{Y^2+Z^2}{2r} - \frac{(Y^2+Z^2)^2}{8r^3} - \dots$$

whereby ***a*** is the sagitta of the mirror***,*** ***r*** is the radius, which is set positive for a hollow mirror, ***X, Y, Z*** are the coordinates of a point ***P*** on the mirror surface ***S***. For the corresponding point ***P'*** on the surface ***S'*** the coordinates will be:

$$2)\qquad X' - a = a - X, \qquad Y' = Y,\ Z' = Z.$$

By ascribing to the mirror an arbitrary spherical shape, except for the fourth order terms, we set exactly:

$$3)\qquad X = a - \frac{Y^2+Z^2}{2r} - \frac{(Y^2+Z^2)^2}{8a^3}(1+b),$$

and define ***b*** as the deformation of the mirror. It is by the way immediately recognisable that the mirror surfaces within this accuracy can always be replaced by rotational ellipsoids or hyperboloids, whose equation reads

$$4)\qquad X = +\frac{1}{1+b}\left(\sqrt{1-\left(\frac{Y^2+Z^2}{r^2}\right)(1+b)}-1\right).$$

If $x = c_0,\ x = c_1,\ x = c_0 + M_0,\ x = c_1 + M_1$ are the equations for the object plane, image plane, entrance and exit pupils respectively and we insert the exact same terms as in I§6,

$$5)\qquad s = a - c_0, \qquad s' = a - c_1, \qquad t = a - c_0 - M_0, \qquad t' = a - c_1 - M_1,$$

we get as the expression of the conjugated location of both plane pairs within the accuracy of Gaussian optics:

$$6)\qquad \frac{1}{s} - \frac{1}{r} = \frac{1}{s'} + \frac{1}{r} = K, \qquad \frac{1}{t} - \frac{1}{r} = \frac{1}{t'} + \frac{1}{r} = L,$$

in which ***K*** and ***L*** again stand for Abbe's invariants.
The magnification ratio of both plane pairs becomes

$$7)\qquad \frac{l_1}{l_0} = \frac{s'+r}{s-r} = \frac{s'}{s} \qquad \frac{\lambda_1}{\lambda_0} = \frac{t'+r}{t-r} = \frac{t'}{t}.$$

Initially, the Angle Eikonal between the planes c_0 and c_1 is formed.

$$W = N_0 P + P' N_1$$
$$= (X - c_0) m_0 + Y p_0 + Z q_0 - (X' - c_1) m_1 - Y' p_1 - Z' q_1.$$

The factors m_0, p_0, q_0, m_1, p_1 and q_1 retain the meanings ascribed to them in I, namely they are the direction cosines of the entering and reflected rays.

Replacing m with $\sqrt{1 - p^2 - q^2}$, X with the expression (3), eliminating X', Y', Z' with the help of (2) and then expanding a series to the 4th order, leads to:

$$W = s - s' - \frac{Y^2 + Z^2}{r} - s \cdot \frac{p_0^2 + q_0^2}{2} + s' \frac{p_1^2 + q_1^2}{2} + Y(p_0 - p_1) + Z(q_0 - q_1)$$
$$- (1+b)\, \frac{(Y^2 + Z^2)^2}{4r^3} + \frac{Y^2 + Z^2}{4r}\, [p_0^2 + q_0^2 + p_1^2 + q_1^2] - \frac{s}{8}(p_0^2 + q_0^2)^2 + \frac{s'}{8}(p_1^2 + q_1^2)^2.$$

In this development, Y and Z can be replaced by their valid values within Gaussian theory

$$8)\qquad \frac{2Y}{r} = p_0 - p_1, \quad \frac{2Z}{r} = q_0 - q_1$$

Hence W is established as a function of p_0, q_0, p_1, q_1.
Next is to change to the Seidel variables and the Seidel Eikonal. The latter consists of 4th order elements of W if we omit elements with orders higher than the 4th order of S^4, and hence has the value:

$$S^4 = -(1+b)\, \frac{(Y^2 + Z^2)^2}{4r^3} + \frac{Y^2 + Z^2}{4r}\, [p_0^2 + q_0^2 + p_1^2 + q_1^2] - \frac{s}{8}(p_0^2 + q_0^2)^2 + \frac{s'}{8}(p_1^2 + q_1^2)^2,$$

or in regard to (6) by rearranging

$$9)\qquad S^4 = \frac{1}{8s'}\left[s'(p_1^2 + q_1^2) + \frac{Y^2 + Z^2}{r}\right]^2 - \frac{1}{8s}\left[s(p_0^2 + q_0^2) - \frac{Y^2 + Z^2}{r}\right]^2 - b\left(\frac{Y^2 + Z^2}{4r^3}\right)^2.$$

Introducing the Seidel variables themselves simplifies the expression from the equations in paper I. (eq. 48), that were valid for lens systems, due to the fact that now $n = n' = 1$.

Therefore the result is instead of I. (48),

$$\text{10)}\quad p_0 = \eta_1 \frac{\lambda_0}{M_0} - \frac{y_0}{\lambda_0} \qquad q_0 = \zeta_1 \frac{\lambda_0}{M_0} - \frac{z_0}{\lambda_0},$$
$$p_1 = \eta_1 \frac{\lambda_1}{M_1} - \frac{y_0}{\lambda_1} \qquad q_0 = \zeta_1 \frac{\lambda_1}{M_1} - \frac{z_0}{\lambda_1}.$$

With the introduction of the abbreviation:

$$\text{11)}\quad H = \frac{t}{\lambda_0} = \frac{t'}{\lambda_1} \qquad h = \frac{\lambda_0 s}{M_0} = \frac{\lambda_1 s'}{M_1},$$

this becomes:

$$\text{12)}\quad p_0 = \eta_1 \frac{h}{s} - y_0 \frac{H}{t} \qquad q_0 = \zeta_1 \frac{h}{s} - z_0 \frac{H}{t}$$
$$p_1 = \eta_1 \frac{h}{s'} - y_0 \frac{H}{t'} \qquad q_1 = \zeta_1 \frac{h}{s'} - z_0 \frac{H}{t'},$$

whereby (8) changes into

$$Y = \eta_1 h - y_0 H \qquad Z = \zeta_1 h - z_0 H.$$

using the terms

$$\text{13)}\quad R_0 = y_0^2 + z_0^2, \qquad \varrho_1 = \eta_1^2 + \zeta_1^2, \qquad \varkappa_{01} = y_0\eta_1 + z_0\zeta_1,$$

then follows with constant usage of equation (6) in close analogy to I (52):

$$Y^2 + Z^2 = H^2 R_0 + h^2 \varrho_1 - 2Hh\varkappa_{01}$$
$$s(p_0^2 + q_0^2) - \frac{Y^2+Z^2}{r} = H^2 R_0\left[L - (K-L)\frac{s}{t}\right] + h^2\varrho_1 K - 2Hh\varkappa_{01} L$$
$$s'(p_1^2 + q_1^2) + \frac{Y^2+Z^2}{r} = H^2 R_0\left[L - (K-L)\frac{s'}{t'}\right] + h^2\varrho_1 K - 2Hh\varkappa_{01} L.$$

Inserting these terms into S^4 (equation (9), results in the desired Eikonal development:

$$\begin{aligned}
\text{14)}\quad 4S^4 = \; & -R_0^2 H^4 \left\{\frac{b}{r^3} + L\left(\frac{3L-2K}{r}\right) + (K-L)^2\left(\frac{s}{t^2} - \frac{s'}{t'^2}\right)\right\} \\
& - \varrho_1^2 h_4 \left\{\frac{b}{r^3} + \frac{K^2}{r}\right\} \\
& - 4\varkappa_{01}^2 H^2 h^2 \left\{\frac{b}{r^3} + \frac{L^2}{r}\right\} \\
& - 2R_0\varrho_1 H^2 h^2 \left\{\frac{b}{r^3} + \frac{K(2L-K)}{r}\right\} \\
& + 4R_0\varkappa_{01} H^3 h \left\{\frac{b}{r^3} + \frac{L(2L-K)}{r}\right\} \\
& + 4\varrho_1\varkappa_{01} H h^3 \left\{\frac{b}{r^3} + \frac{KL}{r}\right\}.
\end{aligned}$$

We move on immediately and form

4. The aberrations of an any mirror system. According to the derivation of the equations in I. § 5 we can achieve expressions for the total system aberrations by simply adding like terms representing the aberrations arising at individual mirrors, which can be represented by the coefficients of the Eikonal developed above.

Differentiating the various mirrors placed one after the other by indices $\boldsymbol{i = 1}$ to $\boldsymbol{i = k}$, we find a complete analogy to I. § 6 (54):

$$15)\quad \begin{aligned} B &= \sum_{i=1}^{k} h_i^4 \left\{ \frac{b_i}{r_i^3} + \frac{K_i^2}{r_i} \right\} \\ C &= \sum_{i=1}^{k} h_i^2 H_i^2 \left\{ \frac{b_i}{r_i^3} + \frac{L_i^2}{r_i} \right\} \\ D &= \sum_{i=1}^{k} h_i^2 H_i^2 \left\{ \frac{b_i}{r_i^3} + \frac{K_i(2L_i - K_i)}{r_i} \right\} \\ E &= \sum_{i=1}^{k} h_i H_i^3 \left\{ \frac{b_i}{r_i^3} + \frac{L_i(2L_i - K_i)}{r_i} \right\} \\ F &= \sum_{i=1}^{k} h_i^3 H_i \left\{ \frac{b_i}{r_i^3} + \frac{K_i L_i}{r_i} \right\}. \end{aligned}$$

The radii of curvature ρ_S and ρ_T, of the sagittal and tangential focal surfaces are connected with $\boldsymbol{C}$ and $\boldsymbol{D}$ by the equation:

$$16)\quad \frac{1}{\varrho_t} = 2(D + 2C) \qquad \frac{1}{\varrho_s} = 2D.$$

All variables occurring here result from equations which are extracted from Gaussian theory and are analogous to the formulae (55), (56), (57), (58) of I. :

$$17)\quad \begin{gathered} \frac{1}{s_i} - \frac{1}{r_i} = \frac{1}{s_i'} + \frac{1}{r_i} = K_i, \qquad \frac{1}{t_i} - \frac{1}{r_i} = \frac{1}{t_i'} + \frac{1}{r_i} = L_i, \\ H_1 = t_1, \qquad h_1 = \frac{s_1}{s_1 - t_1}, \qquad \frac{H_{i+1}}{H_i} = \frac{t_{i+1}}{t_i'}, \qquad \frac{h_{i+1}}{h_i} = \frac{s_{i+1}}{s_i'}, \end{gathered}$$

$$18)\quad \begin{gathered} d_i = s_{i+1} - s_i' = t_{i+1} - t_i', \\ H_i h_i (L_i - K_i) = 1. \end{gathered}$$

The meaning of the symbols are explained once again:

$\boldsymbol{r_i}$ radius of curvature of the i^{th} mirror (positive for concave mirrors),

$\boldsymbol{b_i}$ the deformation of the i^{th} mirror (positive for increased mirror curvature at the marginal zone).

$\boldsymbol{s_i}$, $\boldsymbol{t_i}$, $\boldsymbol{s'_i}$, $\boldsymbol{t'_i}$ are the distances of the four planes to the vertex of the $\boldsymbol{i}^{th}$ mirror, namely these planes are in sequence:

The Gaussian image of the object plane, which is created by the (i-1)th mirror.

The Gaussian image of the entrance pupil, which is displayed by the (i-1) mirror.

The Gaussian image of the object plane, which is displayed by the i^th mirror.
The Gaussian image of the entrance pupil, which is displayed by the i^th mirror.

The easiest way to determine the sign of these distances is to mirror the entire later system at the tangential plane in the vertex of each mirror repeating the process defined in Fig 1 and 2. The distances are positive if, after completion of this construction, the plane in question lies in front of the i^th mirror (in the direction of light propagation).

The factors $\boldsymbol{h_i}$ are (by means of Gaussian theory) proportional to the perpendicular distances from the axis in which individual mirror planes are intersected by a ray originating from the centre of the object plane. The same is valid for the variables $\boldsymbol{H_i}$ referring to a ray originating from the centre of the entrance pupil. The factors $\boldsymbol{d}_i$ are the always positive separations of the poles of successive mirrors. $\boldsymbol{K_i}$ and $\boldsymbol{L_i}$ are Abbe's invariants.

To complete the overview, the formulae are repeated for the transformation to numerical aberrations according to I 21a) and 21b), in which we limit ourselves however to a infinitely distant object. Describing f as the focal length of the entire system and putting:

$$
19)\qquad
\begin{aligned}
B' &= -51.''566\, B^3 f^3\\
C' &= -56.654\, Cf\\
D' &= -56.654\, Df\\
E' &= 29.692\, E\\
F' &= 81.076\, Ff^2
\end{aligned}
$$

Then you get:

19a)

$B'v^3$	diameter of the blur circle caused by spherical aberration.
$E'g^3$	distortion.
$F'gv^2$	radial size of the coma.
$(2C'-D')g^2v$	the radial axis of the blur ellipses created by astigmatism and image curvature.
$D'g^2v$	the tangential axis of the blur ellipses created by astigmatism and image curvature.

Here $\boldsymbol{g}$ stands for the visual field diameter with a diameter of 6° as a unit, $\boldsymbol{v}$ is the aperture ratio of the instrument with an aperture ratio of 1/10 as a unit. Because of the importance of the sign please compare with I. No.11.

5. The Petzval condition for mirror systems. By subtracting the aberration $\boldsymbol{D}$ from **C** and considering (18), we obtain:

$$
20)\qquad C - D = \sum_{i=1}^{k} \frac{1}{r_i},
$$

or by introducing the radii of curvature of the tangential and sagittal focal surfaces:

21) $$\frac{1}{\varrho_t}-\frac{3}{\varrho_s}=4\sum_{i=1}^{k}\frac{1}{r_i},$$

This is Petzval's equation for mirror systems. For an aberration free mirror system the following requirement has to be fulfilled:

$$\Sigma\frac{1}{r_i}=0,$$

which means , that such a system can only be achieved by a combination of concave and convex mirrors (positive and negative ***r***).

§ 3. The single mirror

6. As a first application of the equations derived above we want to consider the aberrations of a single mirror and limit ourselves to a infinitely distant object (s = ∞) . If we leave out the index ***i*** everywhere, we obtain from the formulae (17):

$$s=\infty,\qquad K=-\frac{1}{r},\qquad L=\frac{1}{t}-\frac{1}{r},\qquad H=t,\qquad h=1,$$

and therefore:

$$B=\frac{b+1}{r^3},\qquad C=\frac{t^2}{r^3}\left[b+\left(\frac{r}{t}-1\right)\right]^2,\qquad C-D=\frac{1}{r},$$

$$E=\frac{t^3}{r^3}\left\{b+\left(\frac{r}{t}-1\right)\left(\frac{2r}{t}-1\right)\right\},\qquad F=\frac{t}{r^3}\left\{b-\frac{r}{t}+1\right\}.$$

By moving the entrance pupil to the mirror itself, therefore not attaching a special aperture in front of the mirror, then ***t = 0*** and it follows, if you directly turn to the numerical aberrations (f = r/2):

22) $$B'=-6''\!,4\,(1+b).\qquad C'=-28''\!,3,\qquad D'=0.0,$$
$$E'=0.0,\qquad F'=-20''\!,3.$$

The deformation ***b*** will be used to eliminate the spherical aberration, therefore ***b + 1 = 0*** which means according to (3) a parabolic profile of the mirror. For the parabolic mirror there are only two more aberrations, the curvature of the sagittal focal surface and the coma with numerical values of 20.3" and 28.3" respectively. What these aberrations amount to at different aperture ratios and different visual fields is shown in the following table calculated with (19a):

opening proportion	diameter of visual field	scatter caused by image curvature	scatter casued by coma
1/10	1/2°	0",4	1",7
	1°	1,6	3,4
	2°	6,3	6,8
	4°	25,2	13,6
1/5	1/2°	1",3	18",7
	1°	5,0	37,4
	2°	20,0	74,9
	4°	83,8	149,8.

2*

7. To evaluate these numbers, a comparison is recommended to the common two-element telescope objectives, the so called normal refractors, as used for example in photographic sky mapping. Using these objectives the spherical aberration, coma and distortion disappear, however sagittal and tangential focal surface curvature exists with the numerical values ***2C′+ D′ = – 104″*** and ***D′ = – 47***". The objectives have a aperture ratio of f/10 and they are regarded as useful for a visual field of 2.8° diameter (the maps are squares with a side length of 2°). Under these conditions, we get a dispersion of 23" in radial and 10" in tangential direction due to the image curvature. The much larger blur caused by the existence of secondary spectrum is not even considered here. Comparing these results with the table above, we see that the parabolic mirror with the aperture ratio of f/10 can well compete with the two-piece refracting objective of the same aperture ratio even with regard to the usable visual field. Comparing the light concentrations in the above table, one must not forget that the mirror will be even more superior than indicated by the monochromatic aberration results in the table, due to the missing secondary spectrum, but is less suitable for exact measurements because the coma produces an asymmetrical blur patch and can therefore cause a systematic shift of brighter against fainter stars.[1])

If you change over to a mirror with a aperture ratio of f/3 the table shows that here the useful visual field has only ½ º diameter, being limited by coma, which takes on critical values at small distances from the axis.

§ 4. Systems of two mirrors.

By proceeding to dealing with systems of two mirrors (in which none of them is to be flat naturally), the goal is set by the last remark of the previous paragraph. Specifically, our goal is to obtain an extended visual field even at small aperture ratios, that is to remove besides the spherical aberration in particular the coma. The consideration of distortion is forgone throughout, as it is irrelevant to astronomical applications.

8. Explicit aberration terms. The conditions required to make the spherical aberration and the coma disappear for 2 mirrors are given below:

23)
$$B = h_1^4\left(\frac{b_1}{r_1^3}+\frac{K_1^2}{r_1}\right)+h_2^4\left(\frac{b_2}{r_2^3}+\frac{K_2^2}{r_2}\right) = 0,$$
$$F = h_1^3 H_1\left(\frac{b_1}{r_1^3}+\frac{K_1 L_1}{r_1}\right)+h_2^3 H_2\left(\frac{b_2}{r_2^3}+\frac{K_2 L_2}{r_2}\right) = 0.$$

1) compare H. C. Plummer, Monthly Notices of the Roy. Astr. Soc. Vol. 62. page 352 and Vol. 63 page 16.

Astigmatism and image curvature are defined by

$$
24)\qquad \begin{aligned} D &= h_1^2 H_1^2 \left\{ \frac{b_1}{r_1^3} + \frac{K_1(2L_1 - K_1)}{r_1} \right\} + h_2^2 H_2^2 \left\{ \frac{b_2}{r_2^3} + \frac{K_2(2L_2 - K_2)}{r_2} \right\}, \\ C - D &= \frac{1}{r_1} + \frac{1}{r_2}. \end{aligned}
$$

From the first two equations one can see that with any arbitrary arrangement of the mirror system one can choose the deformations b_1 and b_2 so that spherical aberration and coma disappear. We ask, what image blurring aberrations then still remain?

One eliminates b_1 and b_2 from the first two equations and inserts the result into D. To achieve this we form the equation:

$$
B \,.\, H_1 H_2 - F(h_1 H_2 + h_2 H_1) + D \,.\, h_1 h_2 ,
$$

which reduces to $h_1\, h_2\, D$ with the elimination of B and F. this way one finds at once:

$$
\begin{aligned} h_1 h_2 \,.\, D = &\frac{h_1^3 H_1}{r_1} K_1(K_1 - L_1)(h_1 H_2 - h_2 H_1) \\ &+ \frac{h_2^3 H_2}{r_2} K_2(K_2 - L_2)(h_2 H_1 - h_1 H_2) \end{aligned}
$$

or according to (18):

$$
h_1 h_2 D = -(h_1 H_2 - h_2 H_1)\left(\frac{h_1^2}{r_1} K_1 - \frac{h_2^2}{r_2} K_2 \right).
$$

Now all the factors present above are to be expressed by mirror radii, the mirror separation d and by the distance to the entrance pupil determined by $t_1 = H_1$, whereby the objects must be at infinity $(s_1 = \infty)$. It then follows from equation (17):

$$
25)\qquad \begin{aligned} &K_1 = -\frac{1}{r_1}; \quad s_1' = -\frac{r_1}{2}, \quad s_2 = d - \frac{r_1}{2}, \quad K_2 = \frac{1}{d - \frac{r_1}{2}} - \frac{1}{r_2}, \\ &h_1 = 1, \quad H_1 = t_1, \quad h_2 = 1 - \frac{2d}{r_1}, \quad H_2 = t_1 \frac{t_2}{t_1'} = t_1 + d\frac{t_1}{t_1'} = t_1 + d\left(1 - \frac{2t_1}{r_1}\right) \end{aligned}
$$

and with that

$$
\begin{aligned} &h_1 H_2 - h_2 H_1 = d, \qquad \frac{h_1^2}{r_1} K_1 - \frac{h_2^2}{r_2} K_2 = -\frac{2}{r_1^3} + \left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2d}{r_1 r_2} \right]^2, \\ &D = \frac{d}{1 - \frac{2d}{r_1}} \left\{ -\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2d}{r_1 r_2} \right]^2 + \frac{2}{r_1^3} \right\}, \\ &C = D + \frac{1}{r_1} + \frac{1}{r_2}. \end{aligned}
$$

One realises that the position of the entrance pupil described by t_1 is no longer relevant, as predicted by the theorem of paper I. No. 11. Therefore, only the radii of curvature r_1, r_2 and the mirror distance d are accountable for the remaining image aberrations. Before further discussion, it is recommended to link these three factors to the condition that the distance of the focal length of the system should have a prescribed value of f.

By looking at the magnification between the object and image planes when the object is infinitely distant one sees immediately that the value of the focal length is:

$$f = -s_1' \cdot \frac{s_2'}{s_2}$$

From the derived relationship follows:

$$26) \qquad \frac{1}{f} = \frac{2}{r_1} + \frac{2}{r_2} - \frac{4d}{r_1 r_2}.$$

By eliminating r_2 from the terms C and D, then follows:

$$27) \qquad D = \frac{d}{1 - \frac{2d}{r_1}}\left(\frac{2}{r_1^2} - \frac{1}{4f^2}\right), \qquad C = \frac{1}{4f^2} \cdot \frac{2f - d}{1 - \frac{2d}{r_1}}.$$

Thus we arrive at the final terms for the remaining image aberrations. For these it is unnecessary to know the size of the deformations b_2 and b_1.

From the conditions $B = 0$ and $F = 0$ we have two linear equations for b_1 and b_2, which we now solve. Inserting the values given in (25) for all variables, and eliminating finally r_2 with the help of (26) again, one obtains:

$$28) \qquad b_1 = -1 - \frac{r_1^3}{4df^2}\left(1 - \frac{2d}{r_1}\right), \qquad b_2 = \frac{1}{d} \cdot \frac{2fr_1^3}{(r_1 - 2f)^3} - \left(\frac{r_1 + 2f}{r_1 - 2f}\right)^2.$$

We note the formula for the distance of the image plane behind the last mirror $(-s_2')$:

$$28a) \qquad -s_2' = f \cdot \frac{s_2}{s_1'} = f\left(1 - \frac{2d}{r_1}\right).$$

For the evaluation of the ray path in which a ray originating from an axial object intersects the two mirrors, the ratio h_2 / h_1 of the heights is important after all, because the same ratio determines the mirror radii. For abbreviation we want to name this variable λ. This is:

$$29) \qquad \lambda = \frac{h_2}{h} = \frac{s_2}{s_1'} = 1 - \frac{2d}{r_1}.$$

9. Overview of the systems and their aberrations. Now the first problem we consider will be to form an aberration free mirror system (apart from the distortion). There is such a system because ***C*** and ***D*** disappear if the following conditions are met:

$$d = 2f \qquad r_1 = \pm 2\sqrt{2}\,f$$

From the conditions that d has to be positive and the image has to be real, it follows that f is to be taken positive and r_1 negative. r_2 yields the value $r_2 = 2\sqrt{2}\,f$ according to the Petzval condition. Furthermore s_2 becomes $-s_2 = (1+\sqrt{2})\,f$. The system and the ray path in the mirror system is illustrated in figure 3.

It is clear that this is impractical due to the fact, that the mirror separation becomes double the focal length, and that the mutual obstruction of the mirrors does not allow a larger visual field no matter where you may place the pierced apertures on these mirrors.

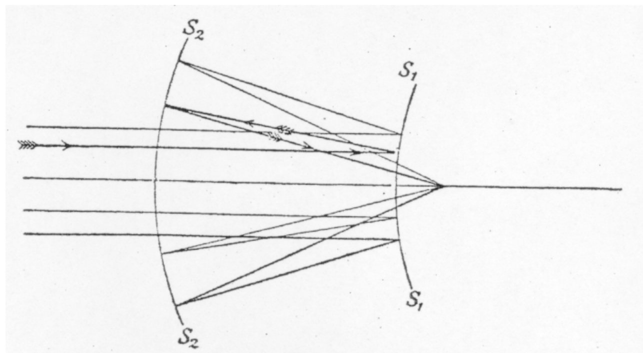


Fig. 3.

Therefore one shall abandon the thought of an aberration free system and instead restrict oneself to select those systems with practical configurations and with as little as possible aberration. The requirement for practical usability is that one mirror is not allowed to shadow the other mirror too much. Therefore a size ratio from the second to the first mirror must be sufficiently different from 1. Cases in which the second mirror (that is in reality the one closer to the object) is larger than first light receiving mirror are ruled out, because the size of the first mirror defines the aperture ratio and then you would need to give the second mirror an unseemly large diameter. Hence, the only remaining cases are where the second mirror is the smaller one. Here you need to distinguish two sub-cases: depending on whether the second mirror collects the light reflected from the first mirror before the focus of the first mirror (λ positive) or whether there lies a focal point between the first and second mirror (λ negative). The last sub-case proves itself in all respects unfavourable because the condition $\lambda = 1 - 2d / r_1 < 0$ results in too small values for r_1 and as a consequence remaining image aberrations ***C*** and ***D*** are too large. Thus, the first sub-case only remains in which the ray path essentially corresponds to that of the Cassegrain reflectors (ref. Figs. 8, 9).

The following table gives an overview of the proportions which occur. The focal length f (here always positive) is set to 1. The systems are sorted according to the size ratio λ of the second (smaller) to the first (larger) mirror. The diameter of the latter is $^1/_3$ of the focal length assuming $\nu = {}^{10}/_3$.

V is the remaining effective aperture ratio due to the small mirror in front.

$$\left(V = \frac{1}{3}\sqrt{1-\lambda^2}\right)$$

The variable $-s_2'$ gives the distance from the second mirror to the image (for $f = 1$ this is numerically identical to λ). Below that the separation d of the two mirrors and their radii of curvature r_1 and r_2 follows. The deformations b_1 and b_2 are given by equation (28) and have as unit – according to their definition – the difference of the paraboloid from parameter r_1 and r_2 respectively and its curvature sphere in the vertex .

The two most interesting variables finally follow, namely the radial and tangential scatter Δy and Δx due to the aberrations ***C*** and ***D***. These are calculated from the equations(19), (19a), (27), (29):

$$\Delta y = -5{,}''243\,\frac{1}{\lambda}\left[4 + (1-\lambda)\left(\frac{4}{r_1} - \frac{3}{2}r_1\right)\right] \qquad \Delta z = -5{,}''243\,\frac{d}{\lambda}\left(\frac{8}{r_1^2} - 1\right),$$

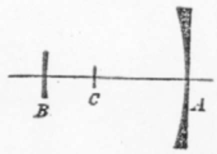
which yield for the aperture ratio of f/3 and a visual field of diameter of 2º (g=⅓), the values of (27) according to (19) and (19a):

		I.	II.	III.	IV.
mirror radius prorportion	λ:	0,3	0,4	0,5	0,6
active opening	V:	$^1/_{3,1}$	$^1/_{3,3}$	$^1/_{3,5}$	$^1/_{3,7}$
distance of the plate to the second mirror	$-s_2'$:	0,3	0,4	0,5	0,6

		I.			II.			III.				IV.			
distance of the		1,05	1,4	1,75	0,9	1,2	1,5	0,75	1,0	1,25	1,50	0,6	0,8	1,0	1,2
mirror radiuses	r_1	3,0	4,0	5,0	3,0	4,0	5,0	3,0	4,0	5,0	6,0	3,0	4,0	5,0	6,0
deformation	r_2	1,8	1,2	1,0	2,4	1,6	1,33	3	2	1,67	1,5	3,6	2,4	2,0	1,8
	b_1	− 2,9	−4,4	−6,4	− 4,0	−6,3	−9,3	−5,5	−9,0	−13,5	−19,0	− 7,7	−13,0	−19,7	−28,0
	b_2	+26,4	+2,4	−0,15	+35,0	+4,3	+0,7	+47,0	+7,0	+1,97	+ 0,5	+63,0	+11,0	+ 3,82	+ 1,63
radial scatter		− 31"	−9"	+12"	−27"	−13"	0"	−25"	−16"	−7"	− 2"	−24"	−17"	−10"	− 6"
tagential scatter		+ 2"	+12"	+21"	+1"	+8"	+13"	+1"	+5"	+9"	+12"	+ 1"	+ 3"	+ 6"	+ 8".

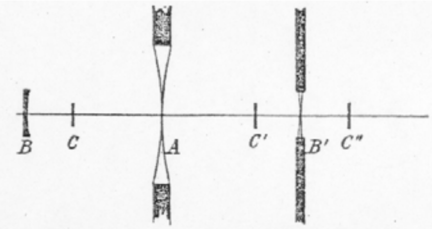
One recognises a number of systems in this table that deliver a useful visual field of 2° diameter and are permissible with their general layout. A better isolation of the best systems proceeds by the consideration of

10. Silhouetting in mirror systems. By this I mean the mutual shadowing of the two mirrors and the photographic plate situated in the image plane. You gain an overview of the quite complicated relationships by imagining the original system with the large mirror ***A***, the small mirror ***B*** and also the circular virtual plate ***C*** imaged by reflection through the vertex planes of the two mirrors with the method explained above. In this way Figure 5 results from Figure 4. The distances of the different apertures are noted.



$AB = d \qquad BC = \lambda$

Fig. 4.

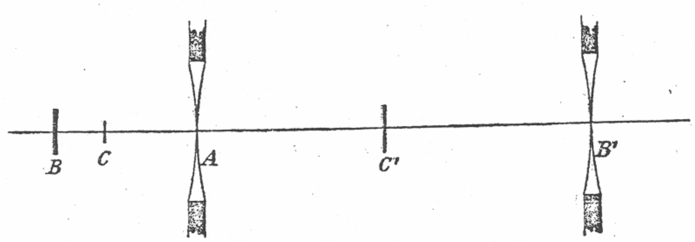


$AB = AB' = d \qquad BC = B'C' = B'C'' = \lambda$

Fig. 5.

Reflecting parts are distinguished from transparent parts by shading.

We proceed with a related principle from Abbe and Helmhotz, and image all apertures to the front through the parts of the optical system lying in front of it, seek in other words the image of ***B′*** and ***C′*** projected trough ***A*** as well as the image of ***C″*** projected through ***B′*** and ***A*** . After easy considerations one receives from Figure (5) the stated relations in Figure (6) according to Gaussian theory.



$$AB = d \qquad BC = \lambda \qquad AB' = \frac{d}{\lambda} \quad AC' = \frac{d}{\lambda}\left[1 - \frac{1}{d+1-\lambda}\right].$$

Fig. 6.

Naming the diameter of the aperture with the associated letter, yields:

$$B = \lambda A \qquad B' = A \qquad C' = \frac{C}{\lambda}\,\frac{1}{d+1-\lambda}.$$

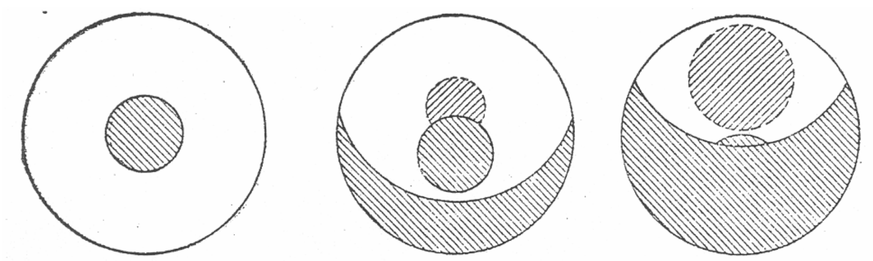
Besides the two equal sized outer apertures ***A*** and ***B′*** there are therefore three more inner apertures ***B, C*** and ***C′*** included in the ray path of which the latter two originate from the photographic plate itself. The image of C' falls into infinity and therefore does not come into consideration.

If we now let a pencil of parallel rays originate from an infinitely distant point, then those rays reach the plate, which can penetrate this aperture system with their undeviated straight path. Therefore we obtain the effective parts of the mirrors, by projecting the entire aperture system onto a plane perpendicular to the optical axis and by searching for the surface elements which are not covered by a projection of a non-transparent aperture. At a short distance of the object from the axis one may be allowed to replace the projected ellipses by circles and only consider the centre displacement thereof.

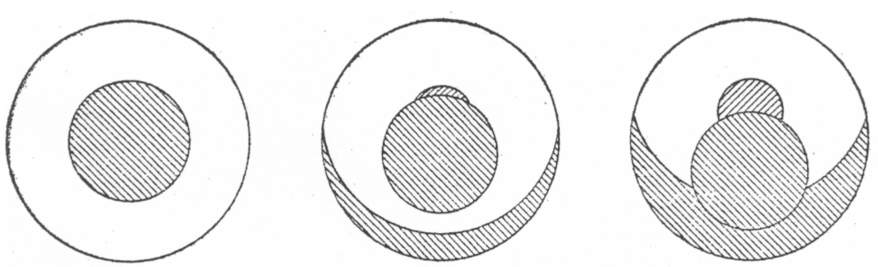
Instead of a general discussion with its case differentiations I will give a few examples. These are all made with regard to the aperture ratio of f/3. The figures 7a) – c) are valid for the system ***λ = 0.3***, ***d = 1.4***, the figures d) – f) for the system ***λ = 0.5, d = 1.25.*** Judging by the aberration scatters given above, both systems are useful for a visual field of 2.5° – 3° diameter; they are selected so that they are almost free of aberration and almost entirely afflicted with astigmatism only. On first sight one would think it more favourable to give the second mirror a radius of only ***0.3*** instead of ***0.5*** because there is a gain in light intensity associated in the ratio of

$$\frac{1-0.3^2}{1-0.5^2} = 1.21$$

and moreover one also reluctantly gives away the centre parts of the mirror as these are in practice easier to produce exactly. However, a look at the figures teaches you that this advantage is only valid for the axial pencil and if you are only a little off the axis, then the system with the smaller front mirror ***(λ=0.3)*** suffers larger silhouetting than the system ***(λ=0.5)***. Particularly troublesome with the first system is that the edge of the photographic plate strongly silhouettes rays travelling from the primary mirror to the secondary mirror, even with a diameter of ***2°***, while in practice you need space for the fitting of the plate and you would in general like to take a slightly larger plate than the actual usable visual field.



a) central object b) 1° displacement off the axis, plate diameter ***2°*** c) 2° displacement off the axis, plate diameter ***4°***



d) Central Object e) 1° axis displacement plate diameter 4° f) 2° axis displacement plate diameter 4°

The parts hatched from top left to bottom right are silhouetted from the photographic plate itself.

Figure 7

11. Selection of the best systems. It seems to me, that in practice the system ***λ = 0.5, d = 1.25*** shows the most favourable proportions, whereby it should not be said that for special purposes one of the neighbouring shapes would not be more favourable. I summarise again the results of its construction.

Focal distance ***f = 1*** mirror distance ***d = 1.25***

Diameter of the large mirror:		***0.33***	
Diameter of the small mirror:		***0.167***	
Effective aperture ratio:		**1/3.5**	
Radii of curvature:	r_1:	***5.0***	
	r_2:	***1.67***	
Deformation:	b_1:	***-13.5***	
	b_2:	***+1.97***	
Radial scatter with 1° sideways of axis:		Δy:	***-7″***
Tangential scatter with 1° sideways of axis:		Δz:	***+9″***

The system delivers nearly circular images of about ***16″*** diameter at ***1.4°*** from the axis at an aperture ratio of f/3.5 and has therefore approximately the same useful visual field as the previously mentioned normal refractors with an aperture ratio of $^1/_{10}$. If you would increase the aperture ratio to an abnormally high value of 1/1.2, then there would still remain a usable visual field diameter of 1.5° .

$ 5 The aplanatic mirror system

12. When one thinks of a construction of a mirror with an aperture ratio of 1/3 or even 1/1, one must not forget that the entire theory of third order aberrations discussed up to now is an approximation only valid for paraxial rays. Therefore we have to try to calculate a mirror system of arbitrary aperture, which is strictly free of spherical aberration and at the same time fulfils the strict sine condition because the absence of coma is secured with the last condition according to I No.7 and 13. The focal point of the desired system must be an aplanatic point according to Abbe's notation, therefore the entire system shall be called "aplanatic".

Because of the entirely new approach, the notation shall be selected independently from the way it was used previously as in figure (8). New relationships will have to be re-established later on. The smaller (formerly second) mirror may be termed as S_1 , the large one catching the first light termed with S' . The two requirements to the mirror system can be formulated in the following way

1) Rays from an infinitely distant point on the axis of the system (which we imagine to run horizontally) are to be united in the focal point F. Another expression of this condition is that the path length from this infinitely distant point to the focal point shall be the same for all rays according to the theorem which directly results from the minimal property of the Eikonal (which says that the optical path length of all those rays is identical which originate from one point and unite in a second point). By drawing the image plane perpendicular to the axis trough the focal point and by naming the distance on the incoming ray parallel to the axis from the point of intersection with the image

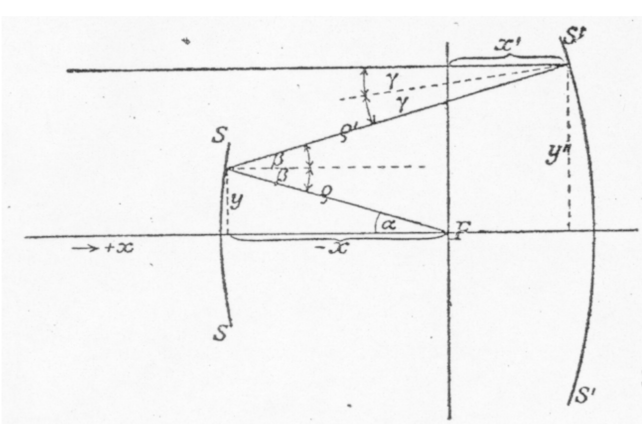


Fig. 8.

plane to the intersection with the mirror ***S'*** as ***x'*** and by also naming the length of the ray between ***S'*** and ***S*** as ρ' , and between ***S*** and the focal point as ***ρ*** , then our condition reads:

$$30) \qquad \varrho+\varrho'+x' = 2(e+1),$$

wherein ***e*** represents a constant.

2) The sine condition shall be fulfilled. If the object point moves further and further away, then the sines of the incoming angles of the rays will become more and more linearly proportional to the perpendicular distances ***y'*** of the rays from the axis where they meets the mirror. If we call the angle at the focal point α, then the sine condition for an object at infinity therefore reads:

$$\frac{y'}{\sin\alpha} = \text{const.}$$

We want to set this constant equal ***1***, and request:

$$31) \qquad y' = \sin\alpha.$$

By this we only determine the unit of measurement in which we want to measure the length specifically in such a way that the focal length of the entire system becomes 1, as can easily be seen from I. equation (16).

13. The task now consists of determining the equations of the meridian curves of both mirrors so that both conditions are fulfilled. The angles between ray and the normal onto the mirrors are named ***β*** and ***γ*** . One imagines the shape of the mirrors to be determined by initially defining ***ρ*** as a function of ***α*** – whereby the equation of the meridian intersection of the mirror ***S*** is specified in polar coordinates – and further by defining ***β*** and ***ρ'*** or ***x'*** as functions of ***α.*** This represents a special way of representing the meridian intersection for the mirror ***S*** in terms of a single parameter.

With these definitions, the slope ***β*** of the mirror normal from ***S*** toward the radius vector ***ρ*** is

$$32) \qquad \frac{1}{\varrho}\frac{\partial\varrho}{\partial\alpha} = \operatorname{tg}\beta.$$

Furthermore we can read the following relationships from the figure :

$$33) \qquad 2\beta = \alpha+2\gamma,$$

$$34) \qquad x'+\varrho\cos\alpha = \varrho'\cos 2\gamma,$$

$$35) \qquad y' = \varrho\sin\alpha+\varrho'\sin 2\gamma.$$

The equations (30) – (35) contain the mathematical expression of our task. By eliminating the angle ***γ*** with the help of (33), and the distance x_1 with the help of (34) and ***y'*** according to (31), the following system results:

$$\sin\alpha = \varrho \sin\alpha + \varrho' \sin(2\beta - \alpha). \tag{36}$$

$$\varrho + \varrho' + \varrho' \cos(2\beta - \alpha) - \varrho \cos\alpha = 2(e+1) \tag{37}$$

$$\frac{1}{\varrho}\frac{\partial\varrho}{\partial\alpha} = \mathrm{tg}\,\beta. \tag{38}$$

By further eliminating ***ρ'***, the equations become:

$$2(e+1) = \varrho(1-\cos\alpha) + (1-\varrho)\sin\alpha\,\mathrm{cotg}\left(\beta - \frac{\alpha}{2}\right)$$

$$\frac{1}{\varrho}\frac{\partial\varrho}{\partial\alpha} = \mathrm{tg}\,\beta.$$

The first equation solved for ***tan β*** gives:

$$\mathrm{tg}\,\beta = \mathrm{tg}\frac{\alpha}{2}\frac{e+1-\varrho+\cos^2\frac{\alpha}{2}}{e+\cos^2\frac{\alpha}{2}}.$$

This leads to the following first order differential equation for the meridian intersection of the mirror ***S***:

$$\frac{1}{\varrho}\frac{\partial\varrho}{\partial\alpha} = \mathrm{tg}\frac{\alpha}{2}\frac{e+1-\varrho+\cos^2\frac{\alpha}{2}}{e+\cos^2\frac{\alpha}{2}}.$$

This equation can be integrated in a very simple way. By setting firstly:

$$\xi = \frac{1}{\cos^2\frac{\alpha}{2}},$$

we obtain the equation in the algebraic form:

$$\frac{\xi}{\varrho}\frac{\partial\varrho}{\partial\xi} = \frac{(e+1-\varrho)\xi+1}{e\xi+1}. \tag{38}$$

Introducing further a new variable ***η*** by:

$$\frac{\xi}{\varrho} = \eta, \qquad \frac{\partial\eta}{\partial\xi} = \frac{1}{\varrho} - \frac{\partial\varrho}{\partial\xi}\frac{\xi}{\varrho^2},$$

then follows:

$$\frac{\partial\eta}{\partial\xi} = \frac{\xi-\eta}{1+e\xi}$$

or:

$$[1+e\xi]\frac{\partial\eta}{\partial\xi} + \eta = \xi.$$

The integrating factor of this differential equation is:

$$[1+e\xi]^{\frac{1}{e}-1}.$$

After multiplying the differential equation through by this and integrating we find:

$$\eta(1+e\xi)^{\frac{1}{e}} = \int d\xi\,\xi(1+e\xi)^{\frac{1}{e}-1}$$
$$= \int d\xi \frac{(1+e\xi)^{\frac{1}{e}} - (1+e\xi)^{\frac{1}{e}-1}}{e}$$
$$= \frac{(1+e\xi)^{\frac{1}{e}+1}}{e(e+1)} - \frac{(1+e\xi)^{\frac{1}{e}}}{e} + c,$$

whereby c is the integration constant, or in a slightly different form:

$$\eta = c(1+e\xi)^{-\frac{1}{e}} + \frac{\xi-1}{e+1}.$$

By reintroducing the original variables α and ρ, we obtain the polar equation for the mirror S:

$$39)\qquad \frac{1}{\varrho} = \frac{\sin^2\frac{\alpha}{2}}{e+1} + c\left(e+\cos^2\frac{\alpha}{2}\right)^{-\frac{1}{e}}\left(\cos^2\frac{\alpha}{2}\right)^{\frac{1+e}{e}}.$$

With that the task is basically solved. It remains to express x' as a function of α, in order to obtain the shape of the mirror S'.

According to 30) we have:

$$x' = 2(e+1) - \varrho - \varrho'.$$

From 36) and 37) it follows by eliminating β:

$$\varrho' = 1 + e - \varrho\sin^2\frac{\alpha}{2} + \frac{\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(1-\varrho)^2}{1+e-\varrho\sin^2\frac{\alpha}{2}}$$

and with that:

$$x' = e + 1 - \varrho\cos^2\frac{\alpha}{2} - \frac{\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(1-\varrho)^2}{1+e-\varrho\sin^2\frac{\alpha}{2}}$$

or:

$$x' = e + 1 - \cos^2\frac{\alpha}{2}\cdot\frac{\varrho\left(1+e-2\sin^2\frac{\alpha}{2}\right)+\sin^2\frac{\alpha}{2}}{1+e-\varrho\sin^2\frac{\alpha}{2}}.$$

Inserting ρ for the expression 39) and adding the sine condition 31), the formulae deliver:

$$40)\qquad x' = e + 1 - \frac{\sin^2\alpha}{4(e+1)} - \frac{1}{c(e+1)^2}\left(e+\cos^2\frac{\alpha}{2}\right)^{2+\frac{1}{e}}\left(\cos^2\frac{\alpha}{2}\right)^{-\frac{1}{e}}$$
$$y' = \sin\alpha.$$

the rectangular coordinates for the meridian intersection of the mirror S' as a function of the parameter α.

15. We now wish to visualise the relationship of the two constants c and e with the geometrical quantities of the mirror system. Also at this point we wish to establish the relationship to the previous notation. For an axial ray we get from Figure (8): $\boldsymbol{\rho = SF = \lambda}$, whereby $\boldsymbol{\lambda}$ as previously, stands for the distance of the focal point from the small mirror $\boldsymbol{S}$ with the focal length of $\boldsymbol{f = 1}$, and also $\boldsymbol{\rho + x' = \rho + \rho' = d}$ whereby $\boldsymbol{d}$ is the separation of the two mirrors. On the other hand, for $\boldsymbol{\alpha = 0}$ follows from the formulae 39) and 40):

$$\frac{1}{\varrho} = c(1+e)^{-\frac{1}{e}}, \qquad x' = e+1-\left(\frac{e+1}{c}\right)^{\frac{1}{e}}$$

and through comparison we have:

$$41) \qquad e+1 = d, \qquad c = \frac{d^{\frac{1}{d-1}}}{\lambda}$$

For further comparison with the previously approximated derivations the present expressions shall be developed in series expansion.
With 39) we receive directly the expansion of $\boldsymbol{\rho}$ with powers of $\boldsymbol{\sin^2(\alpha/2)}$ **:**

$$42) \qquad \frac{\varrho}{\lambda} = 1+\sin^2\frac{\alpha}{2}\left(1+\frac{1-\lambda}{d}\right)+\sin^4\frac{\alpha}{2}\left[\left(1+\frac{1-\lambda}{d}\right)^2-\frac{1}{2d}\right]+\cdots$$

Similarly we get from 40):

$$43) \qquad x' = d-\lambda-\frac{1-\lambda}{d}\sin^2\frac{\alpha}{2}+\frac{1}{d}\left(1-\frac{\lambda}{2}\right)\sin^4\frac{\alpha}{2}+\cdots$$

From here on we change over to rectangular coordinates by having:

$$44) \qquad \begin{array}{llll} \text{auf } S: & x = \varrho\cos\alpha & \text{und auf } S': & x' = x' \\ & y = \varrho\sin\alpha & & y' = \sin\alpha. \end{array}$$

By using 39) we can develop, from the equations of the last line, $\sin\frac{\alpha}{2}$ into a power series for $\boldsymbol{y}$ and $\boldsymbol{y'}$ respectively. The execution of the calculation leads to:

$$45) \qquad \begin{aligned} x &= -\lambda-\left(\frac{1-\lambda}{d}-1\right)\frac{y^2}{4\lambda}+\left\{\frac{1}{4d}-\frac{1-\lambda}{2d}+2\left(\frac{1-\lambda}{2d}\right)^2\right\}\frac{y^4}{8\lambda^3}+\cdots \\ x' &= d-\lambda-\frac{1-\lambda}{4d}y'^2+\frac{\lambda}{32d}y'^4+\cdots \end{aligned}$$

Of course both expansions are identical up to the terms of the fourth order that are received from the previous approach to the mirror meridian in equation 3), if those values are introduced which allow the elimination of spherical aberration and coma.

16. To find practical applications from the previous results, the mirror shapes of the system for which $\boldsymbol{\lambda = 0.5,\ d = 1.25}$ that was recognised above as a particularly useful system shall be calculated according to the strict formulae and compared with the curvature of spheres and the osculating rotational surfaces of second order.
With this selection of constants the mirror surfaces themselves are described by the following equations:

Mirror $\boldsymbol{S}$: $$x = -\varrho\cos\alpha, \quad y = \varrho\sin\alpha, \quad \frac{1}{\varrho} = \frac{4}{5}\sin^2\frac{\alpha}{2}+2\cdot\frac{\left(\cos\frac{\alpha}{2}\right)^{10}}{\left(1-\frac{4}{5}\sin^2\frac{\alpha}{2}\right)^4}$$

Mirror S': $$x' = \frac{5}{4} - \frac{\sin^2\alpha}{5} - \frac{1}{2}\frac{\left(1-\frac{4}{5}\sin^2\frac{\alpha}{2}\right)^5}{\left(\cos\frac{\alpha}{2}\right)^8}, \quad y' = \sin\alpha.$$

The equations of the meridians of the rotational surface of 2nd order, which have a fourth order touch with the mirror in the vertex, become according to (4):

Mirror S: $$x = +\frac{1}{16} - \frac{9}{16}\cdot\sqrt{1-\frac{16}{15}y^2} \quad \text{Ellipsoid},$$

Mirror S': $$x' = \frac{23}{20} - \frac{2}{5}\cdot\sqrt{1+\frac{y'^2}{2}} \quad \text{Hyperboloid.}$$

The first terms of the series expansion in rectangular coordinates are:

$$x = -\frac{1}{2} + \frac{3}{10}y^2 + \frac{2}{25}y^4$$

$$x' = \frac{3}{4} - \frac{1}{10}y'^2 + \frac{1}{80}y'^4.$$

The equations of the osculating spheres (osculating at the pole) are:

$$x = \frac{7}{6} - \sqrt{\left(\frac{5}{3}\right)^2 - y^2}, \quad x' = \sqrt{25 - y'^2} - \frac{17}{4}.$$

The general arrangement and optical path in the system is evident from figure 9. The exact dimensions are obtained from the following table.

The numerical values of the table refer to a focal length of the system of 1000 mm. The first column lists parameter α, the second column the effective aperture ratio $\sqrt{3}\sin\alpha$ in consideration of the smaller front mirror, the columns y and y' list the distances in mm off axis of the ray intersection points with the mirrors (these are the required mirror radii for the respective aperture ratio). Under the x and x' column, the x-coordinates are not listed in the sense used up to now, but are instead the distances of the mirror points from the contact surfaces in the mirror vertex. The same variables follow for the curvature of spheres and the contacting surfaces of 2nd order. Finally, the deviations of the surfaces from each other are formed, in which the unit of one thousandth of a millimetre is chosen.

f=1000 mm
small mirror (***S***)

α	opening	y	x	x (sphere)	x (Ellipsoid)	$x - x(K)$	$x - x(s)$
		mm	mm	mm	mm		
5°	1:6,6	43,694	0,573	0,573	0,573	0 μ	0 μ
10°	1:3,3	87,755	2,315	2,312	2,315	3	0
15°	1:2,2	132,555	5,296	5,280	5,296	16	0
20°	1:1,7	178,478	9,635	9,584	9,639	51	− 4
25°	1:1,4	225,922	15,508	15,383	15,525	125	−17
30°	1:1,2	275,304	23,160	22,895	23,217	265	−57

Large mirror (***S'***)

α	opening	y'	x'	x'(sphere)	x'(Hyperboloid)	x' − x'(K)	x' − x'(H)
		mm	mm	mm	mm		
5°	1:6,6	87,156	0,759	0,761	0,759	− 1 μ	0 μ
10°	1:3,3	173,648	3,004	3,015	3,004	− 11	0
15°	1:2,2	258,819	6,641	6,702	6,643	− 61	− 2
20°	1:1,7	342,020	11,518	11,714	11,532	− 196	− 14
25°	1:1,4	422,618	17,431	17,893	17,479	− 462	− 48
30°	1:1,2	500,000	24,128	25,063	24,264	− 935	− 136

We recognise that the mirror surfaces can practically be replaced by an ellipsoid or a hyperboloid up to an aperture ratio of 1/3. Beyond that down to an aperture ratio of approximately 1/1.4 the deviation of surfaces of 2nd order remains limited to a few hundredths of a millimetre.

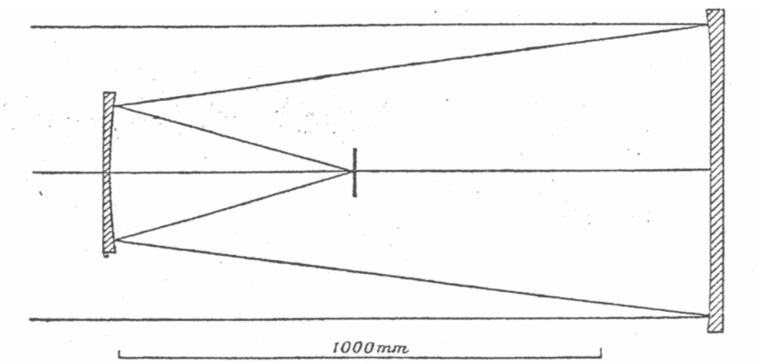


Fig. 9.

In practice the derivation and use of surfaces of 2nd order is of particular importance because the shape of their meridian intersections can be controlled by bringing a light source (virtual for the hyperboloid) into the one focal point and by examining the light union in the other focal point according to Foucault's knife-edge method.

Up to an aperture ratio of 1:2.8 the departure of such surfaces from spherical is so small that the production of such surfaces by gradually regrinding the original spherical mirrors presents no more difficulty than manufacturing a paraboloid mirror of the same aperture ratio.

As a last check-up for the usability of the mirror system, two marginal rays were traced trigonometrically through the system for an aperture ratio of 1/3.3 originating from an object point offset 1.5° from the axis. This resulted in a radial scatter of ***18″***. The conclusions on the usable visual field which were made at the end of the previous paragraph and which only took into consideration the theory of 3rd order aberrations corresponds herewith to the results of a strict calculation for aperture ratios of such magnitude in sufficient sharpness.