§ 1. Introduction

1. The current report presents a general introduction to the aberration theory of optical instruments based on the concept of the “Eikonal”. The representation is based on Hamilton’s “Characteristic Function” which I will name together with Mr. Bruns as the “Eikonal”. I would like to show herewith that the practical calculating optician does not have to fear the Eikonal as something highly theoretical, that one can get very comfortably from the Eikonal to the practical laws and especially to Seidel’s formulae. Hamilton himself must have been very conscious on the applicability of his theorems, but completed (or at least published) investigations of only a few very simple cases of several lenses with axial object points. The derivation of the general calculation formulae directly from the Eikonal does not appear anywhere. This might have been caused by difficulties in the elimination of the so-called “intermediate variables”, which can be overcome simply by the introduction of the “Seidel Variables” and the “Seidel Eikonal” (below 5 & 6).

The advantage of the application of the Eikonal is no less significant in investigations of 5th order aberrations than in the theory of 3rd order aberrations (which Seidel’s formulae refer to).
The compilation of complete terms for the 5th order aberrations of a given optical system would not be too complicated following the formulae in § 5. The number of independent aberrations of the 5th order amounts, without more detail, to 9. Petzval, the calculator of the first “portrait lens” gave this number as 12, from which seems to follow that despite his calculations extending to aberration coefficients of the 9th order, he did not see through the relationship all too deeply.

Apart from the general outline on the 5th order aberrations of an optical system in 11 this report presents therefore only well-known facts in a changed form. New material will be developed in my subsequent investigations.

§2 Optical path length and Eikonal.

2. The notion of the Eikonal can be explained as follows: Given two points $P_0$ and $P_1$ with the orthogonal coordinates $x_0, y_0, z_0, x_1, y_1, z_1$ within an optical system, there is in general a light ray leading from the first point to the second. Let $s$ be the line distance covered by this ray within the individual media of refractive index $n$. Then $E = \Sigma ns$ the so-called “optical path length” of this ray, and this is a function of position of the two points $P_0$ and $P_1$. This function of the variables $x_0, y_0, z_0, x_1, y_1, z_1$ is called the Eikonal.

It is a well known law that the optical path length is a minimum for an actual ray (in relation to small stationary quantities of the 1st order) compared to all possible neighbouring paths connecting both the end points. From this follows directly an additional law: rays starting from $P_0$ are normal to surfaces of constant Eikonal about $P_0$ during their entire course. The surface constant Eikonals about $P_0$ are thereby defined by the equation:

$$\Sigma ns = E(x, y, z, x_0, y_0, z_0) = \text{const.}$$

in which $x_0, y_0, z_0$ are fixed while $x, y, z$ vary. These surfaces are nothing else but the wavefronts transiting the system.

If you draw the surface of constant Eikonal going through the point $P_1$ and chose a point $P_2$ on a normal to the surface at $P_1$, then search for a path of light from $P_0$ to $P_2$ that results in a minimum optical path length, then this is obviously the path through $P_1$ because the shortest light paths to all points $Q$ of the surface with constant Eikonal are equal in length, but the additional path $QP_2$ is shortest if $Q$ coincides with the base point $P_1$ of the normal to the surface of constant Eikonal at $P_1$. 
3. Considering further the change of the Eikonal with respect to an infinitely small displacement of the end point \( P_1 (x_1, y_1, z_1) \) to the point \( P_1' (x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1) \), then the displacement can be split into a length \( P_1 R \) on the surface of constant Eikonal up to the base point \( R \) of the normal from point \( P_1' \) to the surface of constant Eikonal and the length \( \delta N_1 \) from \( R \) to \( P_1' \) on the normal. The first displacement does not change the Eikonal, the second changes it by \( n_1 \delta N_1 \) if \( n_1 \) stands for the refractive index of the medium at this spot. Therefore combined:

\[
\delta E = n_1 \delta N_1.
\]

If the direction cosine \( m_1, p_1, q_1 \) is introduced from the normal of the surface with constant Eikonal in \( P_1 \), that is for the direction cosine of the ray through \( P_1 \), then one has got (taking the normal in the sense of direction of light propagation) apart from small quantities higher order:

\[
\delta N_1 = \delta x_1 m_1 + \delta y_1 p_1 + \delta z_1 q_1.
\]

Therefore:

\[
\delta E = n_1 (\delta x_1 m_1 + \delta y_1 p_1 + \delta z_1 q_1).
\]

Similarly if point \( P_1 \) is fixed and point \( P_0 \) is displaced, one obtains:

\[
\delta E = -n_0 (\delta x_0 m_0 + \delta y_0 p_0 + \delta z_0 q_0).
\]

Displacement of both points at the same time leads to:

1) \[
\delta E = n_1 (\delta x_1 m_1 + \delta y_1 p_1 + \delta z_1 q_1) - n_0 (\delta x_0 m_0 + \delta y_0 p_0 + \delta z_0 q_0)
\]

or in a different form:

2) \[
\frac{\partial E}{\partial x_1} = n_1 m_1, \quad \frac{\partial E}{\partial y_1} = n_1 p_1, \quad \frac{\partial E}{\partial z_1} = n_1 q_1,
\]

\[
\frac{\partial E}{\partial x_0} = -n_0 m_0, \quad \frac{\partial E}{\partial y_0} = -n_0 p_0, \quad \frac{\partial E}{\partial z_0} = -n_0 q_0.
\]

These equations present the practical meaning of the term Eikonal. Namely, if \( E \) is known as a function of \( x_0, y_0, z_0, x_1, y_1, z_1 \) and given the initial direction of a ray, then one can find the end point \( x_1, y_1, z_1 \) by solving the three last equations. It is understood that any one of three equations must be a logical consequence of the two others, because the endpoint can be any point on the ray given by \( x_0, y_0, z_0, m_0, p_0, q_0 \). The first three equations also deliver the ray direction for each point of the ray. So the one Eikonal function governs the entire optical picture.

4. The Angle Eikonal. The Eikonal defined here has in practice the inconvenience that obtains singularities as soon as point \( P_1 \) gets near a focus point conjugated with \( P_0 \). In this region many rays originating from \( P_0 \) will be intersecting each other.
To avoid this circumstance a quantity closely related to $E$ is to be introduced, which we will name the “Angle Eikonal”. If you set - with the introduction of two constants $c_0$ and $c_1$:

$$V = E - n_1 [(x_1 - c_1) m_1 + y_1 p_1 + z_1 q_1]$$

$$+ n_0 [(x_0 - c_0) m_0 + y_0 p_0 + z_0 q_0],$$

so that $V$ initially appears as a function of the points $x, y, z$ on the ray as well as a function of the ray directions $m, p, q$ and varies with all these variables, then follows according to equation (1) or (2):

$$\delta V = -n_1 [(x_1 - c_1) \delta m_1 + y_1 \delta p_1 + z_1 \delta q_1] + n_0 [(x_0 - c_0) \delta m_0 + y_0 \delta p_0 + z_0 \delta q_0].$$

This equation means that $V$ is in reality only a function of the initial and final direction of the ray and that for this equation is valid:

$$\frac{\partial V}{\partial m_0} = n_0 (x_0 - c_0), \quad \frac{\partial V}{\partial p_0} = n_0 y_0, \quad \frac{\partial V}{\partial q_0} = n_0 z_0.$$  

$V$ is according to its geometric meaning the optical path length between the base points of the normal constructed from the points $x = c_0$ and $x = c_1$ respectively lying on the $x$-axis. Because in general a specific ray is established with the declaration of the initial direction and from it the calculated final direction, from this follows that $V$ is only a function of the ray direction, of the quantities $m_0, p_0, q_0, m_1, p_1, q_1$.

The equations (4) enable one to calculate the final ray direction and the end coordinates in a similar fashion as with equation (2) for a given starting point and starting direction.

1) In fact, from the equation $m^2 + p^2 + q^2 = 1$ which describes the behaviour of the direction cosines, follows the two differential equations (which we by the way do not use again) for the Eikonal.

$$n_0^2 = \left(\frac{\partial E}{\partial x_0}\right)^2 + \left(\frac{\partial E}{\partial y_0}\right)^2 + \left(\frac{\partial E}{\partial z_0}\right)^2,$$

$$n_1^2 = \left(\frac{\partial E}{\partial x_1}\right)^2 + \left(\frac{\partial E}{\partial y_1}\right)^2 + \left(\frac{\partial E}{\partial z_1}\right)^2.$$
Singularities, such as $E$ has at the focal points, arise for $V$ if parallel rays are produced again as parallel rays, that is as with telescopic systems. However, as one frequently deals with focal points, very commonly converting parallel pencils of rays coming from infinity into convergent pencils, the use of $V$ is generally preferred over $E$.

A final simplification can follow from the consideration that the three direction cosines are not independent from each other but are connected by the condition

$$m^2 + p^2 + q^2 = 1$$

Eliminating $m$ from $V$ with the help of these conditions and naming the resulting function as $W$ then gives:

$$\frac{\partial W}{\partial p_1} = \frac{\partial V}{\partial p_1} - \frac{\partial V}{\partial m_1} \frac{p_1}{m_1}, \quad \frac{\partial W}{\partial q_1} = \frac{\partial V}{\partial q_1} - \frac{\partial V}{\partial m_1} \frac{q_1}{m_1}$$

Or according to (4):

$$\frac{\partial W}{\partial p_0} = -n_1 \left[ y_1 - (x_1 - c_1) \frac{p_1}{m_1} \right], \quad \frac{\partial W}{\partial q_0} = -n_1 \left[ z_1 - (x_1 - c_1) \frac{q_1}{m_1} \right]$$

and accordingly,

$$\frac{\partial W}{\partial p_0} = n_0 \left[ y_0 - (x_0 - c_0) \frac{p_0}{m_0} \right], \quad \frac{\partial W}{\partial q_0} = n_0 \left[ z_0 - (x_0 - c_0) \frac{q_0}{m_0} \right]$$

The quantities arising on the right are obviously the coordinates of the ray intersecting with the planes $x = c_0$ and $x = c_1$. If we call these $Y_0, Z_0, Y_1, Z_1$ then the equations can be written as:

$$\begin{align*}
\frac{\partial W}{\partial p_1} &= -n_1 Y_1, \quad \frac{\partial W}{\partial q_1} = -n_1 Z_1, \\
\frac{\partial W}{\partial p_0} &= n_0 Y_0, \quad \frac{\partial W}{\partial q_0} = n_0 Z_0.
\end{align*}$$

It is the function $W$ of the four variables $p_0, q_0, p_1, q_1$ which is to be named the Angle Eikonal. Its differential quotients deliver directly the intersecting coordinates of the ray with the two planes $x = c_0$ and $x = c_1$. These can be conveniently set as the object and image planes respectively. $W$, taken as the optical path length between the base points of those normals, has a very similar minimum property to $E$. If one changes over from a real light ray to any neighbouring path, then $E$ alters (with respect to small quantities of the 1st order) only as far as the initial and final coordinates of the ray are changed, because the remaining alterations of the path are inconsequential due to the minimum property. Therefore equation (1) is valid even with any changes of the entire path and the same follows then also for equation (3).
However the latter results in $\delta V = 0$ as soon as the initial and final direction of the ray is fixed, that is $V$ and as well as $W$ is a minimum for the genuine ray path compared with all other paths which have the same initial and final direction.

5. The Seidel variables. We want to turn to a third choice of variables. Similar to how in the celestial mechanics, where constant (unperturbed) path elements are introduced and the changes due to interference are calculated afterwards, we want to use variables which are constant at the calculation of a ray through an optical system (in the case where one limits oneself to Gaussian optics with only first order quantities $p, q, Y, Z$ which are regarded as small) and want to set up an equation for the alteration of the variables which correspond to the stringent application of Snell’s law. These variables are closely related to the those which L. Seidel introduced in his fundamental works (Astronomical News 1853 and 1856, volume 35, 37, 43), which is why they are to be named here the “Seidel variables”.

For simplicity we limit ourselves from now on to systems with rotational symmetry about one axis, which we take to coincide with the $x$ axis.

One obtains a first group of such variables, if one chooses $x = c_0$ and $x = c_1$ for the two conjugated planes (in the sense of Gaussian optics) – we will refer to these as the object plane and image planes respectively – and use the variables:

\[
\frac{Y_0}{l_0}, \frac{Y_1}{l_1}, \frac{Z_0}{l_0}, \frac{Z_1}{l_1},
\]

whereby $\frac{l_1}{l_0}$ stands for the magnification ratio after Gauss for the two planes $c_0$ and $c_1$.

These variables will obviously not be changed by calculations to the accuracy of Gaussian optics. Now it only remains for us to find the substitutes for the angle variables. To obtain these, consider a further pair of conjugated planes:

One obtains a first group of such variables, if one chooses $x = c_0$ and $x = c_1$ for the two conjugated planes (in the sense of Gaussian optics) – we will refer to these as the object plane and image planes respectively – and use the variables:

\[
6) \quad \frac{Y_0'}{\lambda_0}, \frac{Y_1'}{\lambda_1}, \frac{Z_0'}{\lambda_0}, \frac{Z_1'}{\lambda_1},
\]

These planes are to identified as the entrance pupil and exit pupil 1) of the optical systems to be considered. The intersection coordinates of the incoming and refracted ray with these planes are taken to be $Y_0', Z_0', Y_1', Z_1'$. If the magnification ratio is taken as $\frac{\lambda_1/\lambda_0}$ in this second conjugate pair we obtain a second group of variables in the following form:

\[
\frac{Y_0'}{\lambda_0}, \frac{Y_1'}{\lambda_1}, \frac{Z_0'}{\lambda_0}, \frac{Z_1'}{\lambda_1}.
\]

1) The entrance pupil is the real or virtual aperture occurring in object space, which limits the acceptance angle (or pencil waist in the case of an object at infinity) for all pencils of rays entering an optical system. The exit pupil is the Gaussian image of the entrance pupil in image space.
Geometrically, the relationship between \( Y', Z' \) and \( Y, Z \) is obtained with the angle quantities \( p, q \) :

\[
\begin{align*}
\frac{Y'_0 - Y'_0}{M'_0} &= \frac{p_0}{\sqrt{1 - p_0^2 - q_0^2}}, \quad \frac{Y'_1 - Y'_1}{M'_1} = \frac{p_1}{\sqrt{1 - p_1^2 - q_1^2}}, \\
\frac{Z'_0 - Z'_0}{M'_0} &= \frac{q_0}{\sqrt{1 - p_0^2 - q_0^2}}, \quad \frac{Z'_1 - Z'_1}{M'_1} = \frac{q_1}{\sqrt{1 - p_1^2 - q_1^2}}.
\end{align*}
\]

Here however a small change is recommended. If the square roots in the denominator on the right hand side are all set to one, and the variables \( Y', Z' \) are defined by the equation:

\[
\begin{align*}
\frac{Y'_0}{M'_0} &= p_0, \quad \frac{Z'_0}{M'_0} = q_0, \quad \frac{Y'_1}{M'_1} = p_1, \quad \frac{Z'_1}{M'_1} = q_1,
\end{align*}
\]

then the quotients \( \frac{Y'_0}{\lambda_0}, \frac{Y'_1}{\lambda_1} \) and so on will still stay constant within the accuracy of Gaussian optics. The advantage here is that the relationships between the old and the new variables will become linear.

Finally by adding a common constant factor to the variables (6), which simplifies later relationships, we get a system of the new variables \( y, z, \eta, \zeta \):

\[
\begin{align*}
y_0 &= \frac{Y'_0}{\lambda_0} \frac{n_0 \lambda_0}{\lambda_0 M'_0}, \quad z_0 = \frac{Z'_0}{\lambda_0} \frac{n_0 \lambda_0}{\lambda_0 M'_0}, \quad \eta_0 = \frac{Y'_0}{\lambda_0} + \frac{M'_0}{\lambda_0} p_0, \quad \xi_0 = \frac{Z'_0}{\lambda_0} + \frac{M'_0}{\lambda_0} q_0, \\
y_1 &= \frac{Y'_1}{\lambda_1} \frac{n_1 \lambda_1}{\lambda_1 M'_1}, \quad z_1 = \frac{Z'_1}{\lambda_1} \frac{n_1 \lambda_1}{\lambda_1 M'_1}, \quad \eta_1 = \frac{Y'_1}{\lambda_1} + \frac{M'_1}{\lambda_1} p_1, \quad \xi_1 = \frac{Z'_1}{\lambda_1} + \frac{M'_1}{\lambda_1} q_1.
\end{align*}
\]

It should be emphasised that the following relationship exists between the quantities \( M_0, M_1, \frac{\ell_1}{\lambda_0} \) and \( \frac{\lambda_1}{\lambda_0} \):

\[
\frac{n_0 \lambda_0 \ell_0}{\lambda_0 M_0} = \frac{n_1 \lambda_1 \ell_1}{\lambda_1 M_1},
\]

which contains the expression of the sine condition (refer to equation (16)) within the accuracy of Gaussian optics. The reversal of (7) gives the expression of the old variables through the new ones by way of the following relationships:

\[
\begin{align*}
p_0 &= \frac{\lambda_0}{\ell_0} \eta_0 - \frac{y_0}{n_0 \lambda_0}, \quad q_0 = \frac{\lambda_0}{M_0} \xi_0 - \frac{z_0}{n_0 \lambda_0}, \quad Y_0 = y_0 \frac{M_0}{n_0 \lambda_0}, \quad Z_0 = z_0 \frac{M_0}{n_0 \lambda_0}, \\
p_1 &= \frac{\lambda_1}{\ell_1} \eta_1 - \frac{y_1}{n_1 \lambda_1}, \quad q_1 = \frac{\lambda_1}{M_1} \xi_1 - \frac{z_1}{n_1 \lambda_1}, \quad Y_1 = y_1 \frac{M_1}{n_1 \lambda_1}, \quad Z_1 = z_1 \frac{M_1}{n_1 \lambda_1}.
\end{align*}
\]

6. The associated Eikonal. One function of Eikonal character for the new variables results in the following. First the relationships stated in equation (5) can be combined in the relationship:

\[
\delta W = -n_1 (Y_1 \delta p_1 + Z_1 \delta q_1) + n_0 (Y_0 \delta p_0 + Z_0 \delta q_0).
\]

If the new variables from equation (7a) are inserted here, then follows:
The terms on the right hand side are partially incomplete differentials. If instead of \( W \) we use a quantity \( S \):

\[
S = W + \frac{M_0}{n_0} \frac{y_0^2 + z_0^2}{\lambda_0^2} - \frac{M_1}{n_1} \frac{y_1^2 + z_1^2}{\lambda_1^2} + y_0 (\eta_1 - \eta_0) + z_0 (\xi_1 - \xi_0).
\]

With consideration of 9) the differential becomes:

\[
\delta S = \delta y_o (\eta_1 - \eta_0) + \delta z_o (\xi_1 - \xi_0) + \delta \eta_1 (y_0 - y_1) + \delta \xi_1 (z_0 - z_1).\]

Or in different form:

\[
\begin{align*}
\eta_1 - \eta_0 &= \frac{\delta S}{\delta y_o}, \\
y_1 - y_0 &= -\frac{\delta S}{\delta \eta_1}, \\
\xi_1 - \xi_0 &= \frac{\delta S}{\delta z_o}, \\
z_1 - z_0 &= -\frac{\delta S}{\delta \xi_1}.
\end{align*}
\]

\( S \) is therefore a function of four variables \( y_o, z_o, \eta_1, \xi_1 \). From its differential quotients we obtain directly the displacements of the intersecting coordinates of the ray against the values obtained using Gaussian optics. \( S \) will henceforth be named as the “Seidel Eikonal”.

\section{3. Stigmatic point pairs and the sine condition.}

7. As a consequence of the existence of the Eikonal there exists a certain reciprocity theorem between the displacements in our two plane pairs, whose important special case is expressed in the so-called sine condition.

If \( \eta_1, \xi_1 \) are fixed and \( y_o, z_o \) are variable, then from (12) it follows:

\[
\begin{align*}
\frac{\partial y_i}{\partial y_o} - 1 &= -\frac{\partial^2 S}{\partial \eta_1 \partial y_o}, \\
\frac{\partial y_i}{\partial z_o} &= -\frac{\partial^2 S}{\partial \eta_1 \partial z_o}, \\
\frac{\partial z_i}{\partial z_o} - 1 &= -\frac{\partial^2 S}{\partial \xi_1 \partial z_o}.
\end{align*}
\]

If on the other hand, \( y_o, z_o \) are fixed and \( \eta_1, \xi_1 \) are variable, then one obtains:

\[
\begin{align*}
1 - \frac{\partial \eta_0}{\partial \eta_1} &= \frac{\partial^2 S}{\partial y_o \partial \eta_1}, \\
-\frac{\partial \eta_0}{\partial \xi_1} &= \frac{\partial^2 S}{\partial z_o \partial \xi_1}, \\
1 - \frac{\partial \xi_0}{\partial \xi_1} &= \frac{\partial^2 S}{\partial y_o \partial \xi_1}, \\
-\frac{\partial \xi_0}{\partial \eta_1} &= \frac{\partial^2 S}{\partial z_o \partial \eta_1}.
\end{align*}
\]

Damit:

\[
\begin{align*}
\frac{\partial y_1}{\partial y_o} &= \frac{\partial \eta_1}{\partial \eta_1}, \\
\frac{\partial y_1}{\partial z_o} &= \frac{\partial \eta_1}{\partial \xi_1}, \\
\frac{\partial y_1}{\partial z_0} &= \frac{\partial \xi_1}{\partial \eta_1}, \\
\frac{\partial z_1}{\partial z_0} &= \frac{\partial \xi_1}{\partial \xi_1}.
\end{align*}
\]
Bearing in mind the reciprocity condition described above, consider a pair of stigmatic points, that is, points which have the characteristic that all rays starting out from the one, pass through the other. Particularly, if the centres of our object and image planes are to contain such a stigmatic point pair, then from \( y_0 = z_0 = 0 \), there must always follow \( y_1 = z_1 = 0 \) independent of the values of \( \eta \) and \( \zeta \) and independent of which path the ray from point \( y_0 = z_0 = 0 \) will take. For the Seidel Eikonal the condition for stigmatic point pairs comes to:

\[
\frac{\partial S}{\partial \eta_1} = \frac{\partial S}{\partial \zeta_1} = 0
\]

for \( y_0 = z_0 = 0 \) and any values of \( \eta \) and \( \zeta \). One further demands a sharp image of two infinitesimal surface elements perpendicular to the \( x \)-axis in the neighbourhood of the stigmatic points. That means that the relationships from Gaussian optics: \( y_1 = y_0, \ z_1 = z_0 \), for infinitely small \( y_0 \) and \( z_0 \), are fulfilled except for an infinitely small high order aberration, or more precisely, that:

\[
\frac{\partial y_1}{\partial y_0} = \frac{\partial z_1}{\partial z_0} = 1, \quad \frac{\partial y_1}{\partial z_0} = \frac{\partial z_1}{\partial y_0} = 0
\]

should be valid, namely for \( y_0 = z_0 = 0 \) and any value of \( \eta \) and \( \zeta \). With this, reciprocity delivers (13):

\[
\frac{\partial \eta_0}{\partial \eta_1} = \frac{\partial \zeta_0}{\partial \zeta_1} = 1, \quad \frac{\partial \eta_2}{\partial \zeta_1} = \frac{\partial \zeta_2}{\partial \eta_1} = 0
\]

again for all values of \( \eta \) and \( \zeta \) and for \( y_0 = z_0 = 0 \). That is, all rays starting off from our one stigmatic point will therefore converge in the other.

If you notice that due to rotational symmetry the incoming axial ray \( \eta_0 = 0 \) and \( \zeta_0 = 0 \) results in \( \eta_1 = \zeta_1 = 0 \) for the outgoing axial ray, then you get by integration of these equations:

\[
\eta_0 = \eta_1, \quad \zeta_0 = \zeta_1.
\]

In other words: the condition for a sharp image of two infinitely small surface elements perpendicular to the axis about the two stigmatic points \( y_0 = z_0 = 0 \) and \( y = z = 0 \) consists of the equality of the coordinates \( \eta \) and \( \zeta \) for the corresponding rays of the ray pencils starting out from the stigmatic points.

If you introduce the angle coordinates \( p, q \) as in (7a) with the consideration that if \( y = z = 0 \) then \( Y = Z = 0 \), then you find instead of (15) the equations:

\[
\frac{\eta_1}{\eta_0} = \frac{q_1}{q_0} = \frac{n_0 l_0}{n_1 l_1}.
\]

As \( p \) and \( q \) are equal to the sine of the ray inclinations against the \( Y \) and \( Z \) coordinate planes, this explains the demand for constant sine ratios in the ray pencils belonging to the stigmatic points. This is known as the “sine condition”.

A very useful feature of the sine condition is that it allows one to draw conclusions about the behaviour of rays in oblique pencils from the behaviour of the easier-to-follow rays of the axial pencil. A point pair which is stigmatic and also fulfils the sine condition is called an “aplanatic” point pair by Abbe.
§ 4. The series expansion of the Eikonal. The 3rd and 5th order aberrations of an optical system.

8. Because any general idea is lost by strictly pursuing a ray through an optical system even with only a few refracting surfaces, the aberration theory of optical systems is based almost entirely on series expansions, that is, sequences are developed following powers of the quantities $Y, Z, p, q$, or following $y, z, \eta, \zeta$, by assuming these to be small. The convergence of these series expansions are, in most practical cases, so good that a few elements provide a sufficiently accurate result, from which you can carry on using differential equations.

We stick to instruments which are rotationally-symmetrical about the x-axis.

Then you see immediately that the development of the Angle Eikonal, $W$, is using powers of $p$ and $q$ and that only whole numbers of the three terms

$$p^3 + q^3, \quad p^2 + q^1, \quad p_3 + q_3,$$

and in particular only elements of even power will occur. In the same way the Seidel Eikonal will proceed according to increasing powers of the quantities:

$$R_0 = y_0^3 + z_0^3, \quad q_1 = \eta_1^3 + \zeta_1^3, \quad \alpha_{01} = y_0 \eta_1 + z_0 \zeta_1$$

We remain initially with the Angle Eikonal and analyse this in parts which correspond to the various orders of the elements in relation to the powers of $p$ and $q$

$$W = W^2 + W^4 + W^6 + \ldots$$

In this way one can limit oneself firstly to $W^2$ and neglect the higher terms. Then from (5) one obtains the linear relationships between the quantities $p, q, Y, Z$. The content of these linear relationships forms the basis of Gaussian optics, whose derivation will not be repeated here. If one next takes $W^4$ into account then one obtains from (5) corrections of the 3rd order in $p, q$ to the Gaussian values of the coordinates. The theory of 3rd order aberrations emerging from this forms the next major chapter of optical system theory over and above the Gaussian theory. The consideration of $W^6$ results in aberrations of the 5th order etc.

If one changes now over to the Seidel Eikonal, then we find that second order elements vanish because within the accuracy in equation (12) according to Gauss $y_0 = y_1, \eta_0 = \eta_1$ etc. and therefore $S^2 = 0$ concludes. The development hence reads:

$$S = S^4 + S^6 + \ldots$$

The consideration of $S^4$ alone results again in the 3rd order aberration theory, $S^6$ in aberrations of the 5th order etc.

9. The 5 third order aberrations of a optical system.

With the use of the terms from (17) the general expression of $S^4$ reads:

$$S^4 = -\frac{A}{4} R^2_0 - \frac{B}{4} q_1^3 - C \chi_2^2 - \frac{D}{2} R_0 q_1 + E R_0 \chi_2 + F q_2 \chi_2$$

in which $A \ldots F$ are arbitrary constants and the signs and numerical factors are selected with regard to later simplicity.
If the differential equations according to (12) are carried out and if one places the object point in the \( x, y \) plane for simplicity, so that \( z_0 = 0 \), then one finds:

\[
y_1 - y_2 = y_0 \left[ 2Cy\eta_1 - Ey_0^2 - F(\eta_1^2 + \xi_1^2) + \eta_1 [B(\eta_1^2 + \xi_1^2) + Dy_1^2 - 2Ey_0\eta_1] \right]
\]

As the element containing \( A \) is eliminated, there are 5 possible different 3rd order aberrations left for optical system corresponding to the 5 coefficients \( B, C, D, E, F \) from the Eikonal development. We isolate the individual aberrations by assigning zero to each but one coefficient at a time. Here it is recommended to set

\[
\eta_1 = \sigma \cos \varphi, \quad \xi_1 = \sigma \sin \varphi
\]

and to view the so-called “aberration curves” at points \((y_1, z_1)\), obtained by holding \( \sigma \) constant and allowing \( \varphi \) to take on values between 0 and \( 2\pi \) radians. These are therefore the curves that are defined on the image plane by the pencil of rays emanating from the object point and after the calculation of the exit pupil (the plane in which we count \( \eta_1, \xi_1 \) ) in a circle with a radius \( \sigma \). As \( \eta_1 > \eta_0, \xi_1 > \xi_0 \) so are the curves defined by the intersections of the marginal rays of this pencil with the entrance pupil approximately circular.

Hence it occurs in turn by naming, for abbreviation, the displacement on \( y_1 \) and \( z_1 \) by the respective ray aberration components \( \Delta y_1 \) and \( \Delta z_1 \)

\[
20) \quad \Delta y_1 = 5\sigma \cos \varphi, \quad \Delta z_1 = 5\sigma \sin \varphi.
\]

These aberration curves form concentric circles about the Gaussian image point \((y_1 = y_0)\) and the radii of these circles grows as the third power of the aperture. This aberration is independent of field angle and is called “spherical aberration”.

\[
22) \quad \tilde{E} \geq 0 \quad \Delta y_1 = -Ey_0^2, \quad \Delta z_1 = 0.
\]

Since \( \eta_1 \) and \( \xi_1 \) disappear from the formula, the aberration has circular symmetry. Only the distances of the image points from the axis are not exactly proportional. A “distortion” takes place.

\[
c) \quad \tilde{F} \gtrsim 0 \quad \Delta y_1 = -Fy_0\sigma^2(1 + 2\cos^2 \varphi) = -Fy_0\sigma^2(2 + \cos 2\varphi), \quad \Delta z_1 = -Fy_0\sigma^2 \sin 2\varphi.
\]

These aberration curves, that occur for different incoming pencils of rays when the object point \( y_0 \) is fixed, are circles that touch two lines at 30° to the \( y \)-axis and originate from the Gaussian image point. This aberration is called “coma” because of the unsymmetrical tapering appearance.

---

**Fig. 8.**
d) The two aberrations \( C \) and \( D \) are best considered together.

\[
O \geq 0, \quad D \geq 0
\]

\[
y_1 - y_2 = (2C + D) y_1^2 \sigma \cos \varphi,
\]

\[
z_1 = D y_1^2 \sigma \sin \varphi.
\]

They are attributed to astigmatism and image curvature. The entering pencil of rays, which we want to view as very narrow, has two focal lines, of which one is directed radially or as we say, sagittal to the axis of the instrument, while the other focal line is tangential to a circle which is centred on the optical axis and lies in a plane perpendicular to the optical axis. The two surfaces which run through the focal lines if the object is displaced in the object plane, are called “tagential” and “sagittal” focal surfaces. One can represent both surfaces in a first approximation as spherical surfaces with a common pole on the axis and having the radii \( \rho_t \) and \( \rho_s \). One counts \( \rho_t \) and \( \rho_s \) positive if their centre of curvature lies in front of the image plane in sense of the light propagation direction. Implicitly one can see that such a curvature of the image planes corresponds to the following displacements of the ray intersections in the image plane:

\[
\Delta X_i = \frac{Y_i^2}{2 \rho_t}, \quad \Delta Z_i = \frac{Z_i^2}{2 \rho_t}.
\]

Introducing the Seidel variables, one finds:

\[
\Delta y_1 = \frac{y_1^2 \eta_1}{2 \rho_t \eta_1}, \quad \Delta z_1 = \frac{y_1^2 \zeta_1}{2 \rho_t \zeta_1}.
\]

Allowing oneself here to replace \( y_1 \) with \( y_2 \), then one obtains through comparison with (24):

\[
\frac{1}{\varphi_t} = 2n_1 (2C + D), \quad \frac{1}{\varphi_s} = 2n_1 D.
\]

One shall therefore describe accordingly \( 2C + D \) as tangential and \( D \) as sagittal focal surface curvature. Half of the difference of the two curvatures:

\[
\frac{1}{\varphi} - \frac{1}{\varphi_t} = 2n_1 C
\]

is described as astigmatism. Half the sum:

\[
\frac{1}{\varphi} = \frac{1}{2} \left( \frac{1}{\varphi_t} + \frac{1}{\varphi_s} \right) = 2n_1 (C + D)
\]

is called simply image curvature. In fact with the elimination of all other aberrations a spherical surface touching the image plane in the axis, of radius \( \rho \), will coincide with the sharp image of the object.
10. The numerical value of the aberrations. If you set the arbitrary size $\lambda_0 = 1$ and assuming further that the medium in object space and in image space has the refractive index $n_0 = n_1 = 1$, then one gets according to (7):

\[
y_0 = \frac{Y_0}{M_0}, \quad \eta_i = \frac{Y_i}{\lambda_i} = \sigma \cos \varphi_i,
\]

\[
z_0 = \frac{Z_0}{M_0}, \quad \xi_i = \frac{Z_i}{\lambda_i} = \sigma \sin \varphi_i.
\]

One can see, that $y_0$ and $z_0$ become nothing else than the angle distances of the object from the axis, seen from the centre of the entrance pupil, that is more exactly the tangents of the distance angles, that on the other side $\eta_i$ and $\xi_i$ match the coordinates in the exit pupil multiplied by the factor $1/\lambda_i$.

By setting $z_0 = 0$ as before, we want to introduce the terms:

\[
20a) \quad y_0 = g \cdot \tan 3^\circ, \quad \frac{\lambda_i \sigma}{M_i} = \frac{v}{20}.
\]

Thereby $g$ shall represent the size of the visual field as long as the visual field is defined as twice the maximum permissible distance of the object from the axis. In the same way, $v$ represents an “aperture ratio”. Namely as $M_i$ represents the distance between the exit pupil and the image plane, then $\frac{\lambda_i \sigma}{M_i}$ is the tangent of the apex angle of the exiting pencil of rays. If one inserts for $\sigma$ the maximum permissible value, that is the radius of the exit pupil, and calls twice this value the “aperture ratio” then the factors $\tan 3^\circ$ and $1/20$ are added so that the in practice the most frequently occurring values of aperture ratio and visual field $g$ and $v$ remain manageable.

After setting $\lambda_0 = 1$ the corrections $Ay$ and $Ax$ derived above are very near to the alterations of the coordinates of the intersection between the rays and the image plane, expressed in radial units, which correspond to the object seen from the centre of the entrance pupil. Dividing the latter therefore by $\text{arc } 1''$, we obtain the ray aberrations directly in arc seconds from their value re-projected onto the object.

By inserting the terms (20a) into the formulae (21) to (24) and dividing these by $\text{arc } 1''$ the following derivation is recommended. One sets:

\[
21a) \quad D' = \frac{\frac{9}{2} B M_i^2}{\lambda_i^2 \cdot \frac{1}{20^2 \cdot \text{arc } 1''}} = \frac{B}{\lambda_i^2} \cdot 1,71237 = 51,556 \cdot \frac{M_i^2}{\lambda_i^2}
\]

\[
C' = \frac{2 C M_i}{\lambda_i^2 \cdot \frac{1}{20} \cdot \text{arc } 1'' (\tan 3^\circ)^2} = \frac{C}{\lambda_i^2} \cdot 1,75323 = 56,654 \cdot \frac{M_i}{\lambda_i}
\]

\[
D' = \frac{2 D M_i}{\lambda_i^2 \cdot \frac{1}{20} \cdot \text{arc } 1'' (\tan 3^\circ)^2} = \frac{D}{\lambda_i} \cdot 1,75323 = 56,654 \cdot \frac{M_i^2}{\lambda_i^2}
\]

\[
E' = \frac{E(\tan 3^\circ)^2}{\text{arc } 1''} = \frac{E}{1,47263} = 29,692 \cdot E
\]

\[
F' = \frac{3 E M_i^2}{\lambda_i^2 \cdot \frac{1}{20^2 \cdot \text{arc } 1''}} = \frac{E}{\lambda_i^2} \cdot 1,90889 = 81,076 \cdot \frac{M_i^2}{\lambda_i^2}
\]
and terms the variables $B'$ … $F'$ as the numerical aberrations of the system.

Then follows in arc seconds:

- $B' \cdot v^3$ diameter of the blur circle of the spherical aberration
- $E' \cdot g^3$ distortion
- $F' \cdot gv^2$ the radial extent of the coma (the tangential extent is $2/3$ of the radial)
- $(2C' + D') \cdot g^2v$ the radial axis of the ellipse produced by astigmatism and image curvature.
- $D' \cdot g^2v$ the tangential axis of the ellipse produced by astigmatism and image curvature.

Later I will name these variables (apart from the distortion) as individual aberrations.

In the case of the object moving into infinity, the factor $\frac{M_I}{\lambda_i}$ obtains a particularly simple meaning. In general it is:

$$Y_1 = \frac{Y_0}{M_0} \cdot \frac{Y_1}{M_1}$$

or according to (8) for $n_0 = n_1 = \frac{\lambda_0}{\lambda_1} = 1$:

$$Y_1 = \frac{Y_0}{M_0} \cdot \frac{M_1}{\lambda_1}.$$

now $\frac{Y_0}{M_I}$ is as mentioned the apparent size of the object, $Y_I$ is the size of the image. For an infinitely distant object the proportion factor gives therefore the focal length $f$ of an optical instrument, in which however the negative sign has to be introduced because image inversion is linked to a positive focal length. Hence the following relationship:

$$\frac{M_I}{\lambda_1} = -f.$$

Further in relation to the signs you can read the following from the formulae given above:

Positive $B'$ means rays from the marginal zone of the aperture come to a focus closer to the exit pupil than rays from the central zone (so called “under correction” of the spherical aberration).

Positive $E'$ means a compression of the outer parts of the object (so called barrel distortion, because a square depicts with barrel-like outwardly-curving sides), negative $E'$ delivers the so called pin cushion distortion (the corners of a square become pointed, distanced further from the centre).

Positive $F'$ means the coma is orientated towards the axis.
Positive $2C' + D'$ means the position of the tangential image surface lies behind the Gaussian image plane.

Positive $D'$ means the position of the sagittal image surface lies behind the Gaussian image plane.

11. **Influence of the aperture stop position.** Using the Eikonal the question of the influence of the aperture stop position can be simply managed. We think of the limitation of the effective aperture of the system to be caused by the entrance pupil (or any reproduced image of it through part of the optical system). The entrance pupil was defined by its distance $M_0$ from the object plane. The entrance pupil shall now be displaced and assigned a new distance $\overline{M}_0$ from the object plane. In this way all variables depending on the position of the entrance pupil, such as the magnification ratio $\frac{\lambda_1}{\lambda_0}$ between exit and entrance pupil and the entire set of Siedel variables, take on new values marked by an overscore.

Equations (7) (I leave out the analogous relationship for the $z$ coordinates)

\[
y_0 = \frac{n_0 \lambda_0}{\overline{M}_0} \cdot Y_0, \quad \eta_0 = \frac{Y_0}{\lambda_0} + \frac{\overline{M}_0}{\lambda_0} \overline{p}_0,
\]

\[
y_1 = \frac{n_1 \lambda_1}{\overline{M}_1} \cdot Y_1, \quad \eta_1 = \frac{Y_1}{\lambda_1} + \frac{\overline{M}_1}{\lambda_1} \overline{p}_1.
\]

are replaced by the equations

\[
y_{\overline{0}} = \frac{n_0 \lambda_0}{\overline{M}_0} \cdot \overline{Y}_0, \quad \eta_{\overline{0}} = \frac{\overline{Y}_0}{\lambda_0} + \frac{\overline{M}_0}{\lambda_0} \overline{p}_0,
\]

\[
y_{\overline{1}} = \frac{n_1 \lambda_1}{\overline{M}_1} \cdot \overline{Y}_1, \quad \eta_{\overline{1}} = \frac{\overline{Y}_1}{\lambda_1} + \frac{\overline{M}_1}{\lambda_1} \overline{p}_1.
\]

from which the relations between the old and the new Seidel variables result:

\[
y_{\overline{0}} = \frac{\overline{y}_0}{\lambda_0} \cdot \frac{\overline{M}_1}{\overline{M}_0}, \quad \eta_{\overline{0}} = \frac{\overline{\eta}_0}{\lambda_0} \cdot \frac{\overline{M}_1}{\overline{M}_0} + \overline{y}_{\overline{1}} \left( \frac{\overline{M}_1 - \overline{M}_0}{n_0 \lambda_0 \lambda_0} \right),
\]

\[
y_{\overline{1}} = \frac{\overline{y}_1}{\lambda_1} \cdot \frac{\overline{M}_1}{\overline{M}_1}, \quad \eta_{\overline{1}} = \frac{\overline{\eta}_1}{\lambda_1} \cdot \frac{\overline{M}_1}{\overline{M}_1} + \overline{y}_{\overline{1}} \left( \frac{\overline{M}_1 - \overline{M}_0}{n_0 \lambda_0 \lambda_0} \right),
\]

making the following substitutions

\[
\frac{\overline{\lambda}_0 \overline{M}_0}{\lambda_0 M_0} = \beta, \quad \frac{\overline{M}_0 - M_0}{n_0 \lambda_0 \lambda_0} = \gamma,
\]

gives:

\[
y_{\overline{0}} = \beta y_0, \quad \eta_{\overline{0}} = \beta \eta_0 + \gamma y_0,
\]

\[
y_{\overline{1}} = \beta y_1, \quad \eta_{\overline{1}} = \beta \eta_1 + \gamma y_1,
\]

in which consideration is taken for the invariant (8) valid for each pair of conjugated planes.

If you replace the old variables by the new ones in the Eikonal derivation (18), then one obtains the derivation of the exact same form, in which the coefficients, that is the
aberrations corresponding to the altered aperture stop adjustment, have the following values:

\[
\begin{align*}
\bar{B} &= B \beta^4, \\
\bar{F} &= F \beta^2 - B \beta^2 \gamma, \\
\bar{C} &= C - 2F \beta \gamma + B \beta^4 \gamma, \\
\bar{D} &= D - 2F \beta \gamma + B \beta \gamma^2, \\
\bar{E} &= \frac{E}{\beta} - (D + 2C) \frac{\gamma}{\beta} + 3F \gamma^2 - B \beta \gamma^2.
\end{align*}
\]

These are already noted in such a sequence that each aberration becomes independent from the aperture stop position adjustment (except from a factor \( \beta^4 \) or \( 1/\beta^2 \)) as soon as all preceding aberrations disappear.

It shall be noted that with the elimination of spherical aberration and coma \((B = F = 0)\) the image curvatures \(C\) and \(D\) become independent of the aperture adjustment.

This property is easy to understand in that you have influence on the shape of the image with stopping down/lowering aperture of certain rays only as long as not all rays unite in one point.

12. The 9 aberrations of the 5th order of an optical system. The derivation of 5th order aberrations is completely analogous to that of the 3rd order aberrations. The general expression of \(S^6\) is:

\[
S^6 = S_1 R_0^0 + S_2 R_0^2 \eta_1 + S_3 R_0^2 \eta_0 + S_4 R_0^2 \eta_0 + S_5 R_0^2 \eta_0 + S_6 R_0^2 \eta_0 + S_7 R_0^2 \eta_0 + S_8 R_0^2 \eta_0 + S_9 R_0^2 \eta_0 + S_{10} R_0^2 \eta_0 + \frac{\delta}{\delta \eta_1} + \frac{\delta}{\delta \eta_0}
\]

in which again \(S_1\) to \(S_{10}\) represent arbitrary constants. The differentiation according to equations (12) delivers, under the condition that \(z_\theta\) is set to zero as above,

\[
y_0 - y_1 = \frac{\partial S^4}{\partial \eta_1} + 2\eta_1 S_2 y_0^2 \eta_1 + 2S_2 y_0^2 \eta_1 + 3S_1 y_0^2 \eta_1 + 2S_6 y_0^2 \eta_1 + \frac{\delta}{\delta \eta_1}
\]

\[
\frac{\delta}{\delta \eta_1} + \frac{\delta}{\delta \eta_0}
\]

Since \(S_1\) is eliminated, there are all together 9 independent aberrations of the 5th order. I isolate these by considering the aberration curves of each individual aberration, for which we again set \(\eta_1 = \sigma \cos \phi\) , \(\zeta_1 = \sigma \sin \phi\) and the corresponding displacements of \(y_1\) and \(z_1\) are named \(\Delta y_1\) and \(\Delta z_1\). At the same time I allow myself to suggest names for these aberrations. The aberrations are arranged according to their dimensions in relation to \(y_\theta\), the distance of the object from the axis.

a) \(S_7 < 0\). Secondary spherical aberration.
The aberration curves are circles whose radii are independent of the height above the axis of the object increase with the 5th power of the instrument's aperture.

b) $S_8 \gg 0$. Secondary coma.

\[
\begin{align*}
\Delta y_i &= -6s_i\sigma_i^5 \cos \varphi \\
\Delta z_i &= -6s_i\sigma_i^5 \sin \varphi.
\end{align*}
\]

The aberration curves are circles with a radius of $2s_iy_i\sigma_i^4$ which touch straight lines inclined at an angle of $41.8^\circ$ to the $y$-axis ($\sin 41.8^\circ = 2/3$).

c) $S_4 \gg 0$. Lateral spherical aberration.

\[
\begin{align*}
\Delta y_i &= -4s_iy_i\sigma_i^4 \cos \varphi \\
\Delta z_i &= -4s_iy_i\sigma_i^4 \sin \varphi.
\end{align*}
\]

The aberration curves are circles whose radii increase as the square of the axis distance and the third power of the aperture.

d) $S_9 \gg 0$. ...Propellor aberration.

\[
\begin{align*}
\Delta y_i &= -2s_iy_i\sigma_i^4 \cos \varphi (1 + \cos^2 \varphi) \\
\Delta z_i &= -2s_iy_i\sigma_i^4 \cos^2 \varphi \sin \varphi.
\end{align*}
\]

The aberration curves will become curves of the 6th order in the wing shape shown alongside. The curves that belong to the same object have a common intersection point and differ only in magnitude.

e) $S_{10} \gg 0$. Arrow aberration.

\[
\begin{align*}
\Delta y_i &= -3s_iy_i^2\sigma_i^4 \cos^3 \varphi \\
\Delta z_i &= 0.
\end{align*}
\]

The aberration curve consists of a straight line, which extends from the Gaussian image point to one side.

f) $S_5 \gg 0$. Lateral coma.

\[
\begin{align*}
\Delta y_i &= -s_iy_i^2\sigma_i^4 (1 + 2 \cos^2 \varphi) \\
\Delta z_i &= -s_iy_i^2\sigma_i^4 2 \sin \varphi \cos \varphi.
\end{align*}
\]
The aberration curves possess the same shape as with the “usual” coma (\(F\) in no. 9); but its dimensions grow with the third power of the axis distances.

\(g)\) \(S_2 > 0\) and \(S_6 > 0\). Lateral image curvature/astigmatism.

\[\Delta y_1 = -2(S_2 + S_4) y'_0 \cos \varphi \]
\[\Delta z_1 = -2 S_2 y'_0 \sin \varphi.\]

We take these two aberrations together, as the image curvature and the astigmatism above. The aberration curves are ellipses. The half difference of the two axis \(S_6\) will be suitably named as lateral astigmatism, half of the sum \(2S_2 + S_6\) as lateral image curvature.

\(g)\) \(S_3 > 0\). Lateral mapping aberration/distortion.

\[\Delta y_1 = - S_3 y'_0 \]
\[\Delta z_1 = 0.\]

This aberration does not disturb the point shaped depiction/image, it only changes the distortion.

13. Note on the aberrations in aplanatic point pairs. If the point \(y_0 = z_0 = 0\) is depicted stigmatically in the point \(y_1 = z_1 = 0\), then this says that all aberrations independent of the perpendicular distance of the object from the axis disappear, that is the spherical aberration of the first and second level \(B\) and \(S_1\). If additionally the sine condition is fulfilled, then the aberrations \(F\) and \(S_3\) proportional to the first power of \(y_0\) also disappear. The existence of an aplanatic point pair necessitates the freedom of the system from primary and secondary spherical aberration and the coma.

§ 5. The composition of several optical systems.

14. If the Eikonal theorem gives an overview by implication on the number and type of possible aberrations, then there remains now the far more difficult task to calculate accurately the Eikonal for an optical system and derive from this the values of the aberrations themselves. The first part of the task will be to calculate the Eikonal of a single reflecting or refracting surface, the second part consists of assembling any number of such single systems to make up multi-element systems. We deal first with the second task, by restricting ourselves to a two element system. The systems are always assumed to be coaxial.

One first uses Gaussian optics to image the planes \(x = c_1\) and \(x = c_1 + M_1\) through the second element into two planes which might be given by \(x = c_2\) and \(x = c_2 + M_2\) and which obviously represent the image plane and the exit pupil of the entire system. The angle Eikonal of the first system shall be:

\[W_1 = \widehat{W}_1(p_0, q_0; p_1, q_1),\]

the one of the second in analogous terms:

\[W_2 = \widehat{W}_2(p_1; q_1, p_2, q_2).\]
The Angle Eikonal of the entire system consists of the sum of these two variables according to the geometric interpretation of $W$:

$$W = W_1 + W_2,$$

in which the challenge lies in expressing $p_1, q_1$ through $p_0, q_0, p_2, q_2$ and therefore illustrating $W$ as a function of the last 4 variables.

This elimination of the intermediate variables shall meanwhile not be carried out on $W$ but on the Seidel Eikonal where it can be performed exceptionally more simply.

One obtains, according to (10):

$$S_1 = W_1 + \frac{M_1 y_1^2 + z_1^2}{2\lambda_0} - \frac{M_1 y_1^2 + z_1^2}{2\lambda_1} + y_1(\eta_1 - \eta_0) + s_1(\xi_1 - \xi_0).$$

Accordingly:

$$S_2 = W_2 + \frac{M_1 y_2^2 + z_2^2}{2\lambda_0} - \frac{M_1 y_2^2 + z_2^2}{2\lambda_2} + y_2(\eta_2 - \eta_0) + s_2(\xi_2 - \xi_0)$$

And for the entire system:

$$S = W + \frac{M_1 y_0^2 + z_0^2}{2\lambda_0} - \frac{M_1 y_2^2 + z_2^2}{2\lambda_2} + y_0(\eta_2 - \eta_0) + s_1(\xi_2 - \xi_0).$$

According to (36):

$$S = S_1 + S_2 + (y_0 - y_1) (\eta_2 - \eta_1) + (s_0 - s_1) (\xi_2 - \xi_1)$$

and according to (12):

$$S = S_1 + S_2 + S_0 + S_1^4 + S_2^4 + \frac{\partial S_1^4}{\partial \eta_1} \frac{\partial S_2^4}{\partial \xi_1} \frac{\partial S_1^4}{\partial \eta_1} \frac{\partial S_2^4}{\partial \xi_1}.$$

It is unnecessary here to express $y_1, z_1, \eta_1, \xi_1$ in terms of $y_0, z_0, \eta_2, \xi_2$ and to find $S$ as a function of the last four variables. We introduce now a series expansion and restrict ourselves to include terms of the 6th order in $S$. If you separate the terms of various orders from $S_1$ and $S_2$, then the terms for $S$ up to the 6th order inclusive become:

$$S = S_1 + S_2 + S_0 + S_1^4 + S_2^4 + \frac{\partial S_1^4}{\partial \eta_1} \frac{\partial S_2^4}{\partial \xi_1} \frac{\partial S_1^4}{\partial \eta_1} \frac{\partial S_2^4}{\partial \xi_1}.$$

In the last four terms of this expression one can easily set $y_1 = y_0$, $z_1 = z_0$, $\eta_1 = \eta_2$, $\xi_1 = \xi_2$, because here the consideration of the differences of these terms would only generate terms of the 8th order. This equality does not apply to $S_1^4$ and $S_2^4$. They are replaced by:

$$S_1^4(y_0, \eta_0, \xi_0) = S_1^4(y_0, \eta_0, \eta_0, \xi_0 - \xi_0)$$

or expanded accurately up to the 6th order:

$$S_1^4(y_0, \eta_0, \xi_0) = S_1^4(y_0, \eta_0, \eta_0, \xi_0) - \frac{\partial S_2^4}{\partial \eta_1} \frac{\partial S_1^4}{\partial \xi_1} \frac{\partial S_2^4}{\partial \xi_1} \frac{\partial S_1^4}{\partial \xi_1}.$$
and correspondingly:

$$S_1^4(y_1, z_1, \eta_1, \xi_1) = S_1^4(y_0, z_0, \eta_2, \xi_2) - \frac{\partial S_1^4}{\partial y_0} \frac{\partial S_2^4}{\partial y_1} - \frac{\partial S_3^4}{\partial \xi_1} \frac{\partial S_2^4}{\partial \xi_1}.$$

If we now mark with an overscore terms in which we have replaced \( y_1, z_1 \) by \( y_0, z_0 \), and \( \eta_1, \xi_1 \) by \( \eta_2, \xi_2 \) in a function, then we obtain for the 4\(^{th}\) and 6\(^{th}\) order terms of \( S \):

\[ S^4 = \overline{S}_1^4 + \overline{S}_2^4 \]

\[ S^6 = \overline{S}_1^6 + \overline{S}_2^6 - \frac{\partial \overline{S}_1^4}{\partial \eta_0} \frac{\partial \overline{S}_2^4}{\partial \eta_0} - \frac{\partial \overline{S}_1^4}{\partial \xi_0} \frac{\partial \overline{S}_2^4}{\partial \xi_0}.

With that we have carried out the elimination of the intermediate variables with the intended accuracy and found the expression of the entire Eikonal. In order to visualise the significance of formula (38) completely, we want to derive explicitly the rules contained within it. By adding the index 1 everywhere, according to (18) we obtain:

Analogously we have:

$$S_1^4 = -\frac{A_2}{4} R_1^2 - \frac{B_2}{4} q_1^2 - C_2 z_1^2 - D_2 R_1 q_2 + E_2 R_1 \eta_2 + F_2 q_2 \zeta_2,$$

$$R_1 = y_1^2 + z_1^2, \quad q_2 = \eta_1^2 + \xi_1^2, \quad z_1 = y_1 \eta_1 + z_1 \xi_1.$$

Replacing here \( y_1, z_1 \) by \( y_0, z_0 \) and \( \eta_1, \xi_1 \) by \( \eta_2, \xi_2 \) and introducing the following notation:

$$z_{02} = y_0 \eta_2 + z_0 \xi_2,$$

we obtain:

$$S^4 = -\frac{A_1 + A_2}{4} R_0^2 - \frac{B_1 + B_2}{4} q_0^2 - (C_1 + C_2) z_{02}^2 - \frac{D_1 + D_2}{2} R_0 q_2 + (E_1 + E_2) R_0 \zeta_{02} + (F_1 + F_2) q_0 \zeta_{02}.$$

This equation shows that the aberrations of the 3\(^{rd}\) order of an entire system are made up from the accumulated aberrations of the individual systems. If this result seems simple the credit for this is solely due to the use of the Seidel variables and the definition of the individual image aberrations by the coefficients of the Eikonal expansion especially with these variables. As soon as you turn to other linear combinations of the Seidel variables, which change from system to system, one obtains for each expansion coefficient of the combined Eikonal a complicated linear equation of all the aberrations of the individual Eikonals. So this is the point, where the advantage of the Seidel variables is clearly evident.

The formula (39) shows that the 5\(^{th}\) order aberrations are not directly subjected to the accumulative rule, but their composition is readily apparent.
The transition from the composition of two surfaces to an arbitrary number is so obvious, that writing of equations is unnecessary, I suppose.

§ 5. The third order aberrations of a centred lens system.
The Seidels formulae.

15. At this point we carry out the complete calculation of the 4th order Eikonal $S^4$ of a centred lens system, in which we can also consider surface deviations from spherical without effort.

We consider first the calculation for a single surface. The refractive indices of both sides shall be $n_0$ and $n_1$. The x-axis shall be counted positive in the direction of the light propagation. If the surface is spherical, then the equation reads:

$$X - a = r - \sqrt{r^2 - Y^2 - Z^2},$$

in which $a$ is the sagitta of the spherical section, $r$ is the radius and a positive $r$ corresponds to a convex surface with respect to the incoming light. Up to the terms of the fourth order we have:

$$X = a + \frac{Y^2 + Z^2}{2r} + \frac{(Y^2 + Z^2)^2}{8r},$$

By ascribing an arbitrary non-spherical rotational shape to the surface, up to the 4th order terms we set exactly:

$$X = a + \frac{Y^2 + Z^2}{2r} + \frac{(Y^2 + Z^2)^2}{8r} + (1 + b),$$

in which $b$ can be termed as "deformation" of the surface.

For the distances of the four planes $X = c_0$, $c_1$, $c_0 + M_0$ and $c_1 + M_1$ from the vertex of the surface, one introduces the abbreviations:

$$s = a - c_0, \quad s' = a - c_1, \quad t = a - c_0 - M_0, \quad t' = a - c_1 - M_1,$$

then applies according to the known formulae of Gaussian optics based on the conjugated position of object plane and image plane, entry and exit pupils:

$$n_0 \left( \frac{1}{s} + \frac{1}{r} \right) = n_1 \left( \frac{1}{s'} + \frac{1}{r} \right) = K,$$

$$n_0 \left( \frac{1}{t} + \frac{1}{r} \right) = n_1 \left( \frac{1}{t'} + \frac{1}{r} \right) = L.$$
In this calculation $K$ and $L$ represent two invariant variables which are named Abbe’s Invariants. The magnification between the two plane pairs becomes, as the images lie perspectively as seen from the curvature centre point of the refracting surface:

\[
\frac{t_1}{t_0} = \frac{s'+r}{s+r} = \frac{n_0 s'}{n_1 s}, \quad \frac{\lambda_1}{\lambda_0} = \frac{t'+r}{t+r} = \frac{n_0 t'}{n_1 t}.
\]

We now form the Angle Eikonal of this refracting surface by producing perpendicular lines $c_0N_0$ and $c_1N_1$ from the points $x = c_0$ and $x = c_1$ on the axis onto the incoming and refracted ray, and calculate the expression:

\[
W = n_0 N_0 P + n_1 N_1 P,
\]

in which $P$ refers to the intersection point of the ray with the refracting surface, with the coordinates $X, Y, Z$. If $n_0, p_0, q_0, n_1, p_1, q_1$ are the same as above, the direction cosines of the ray before and after the calculation then follow according to:

\[
W = n_0 [(X - c_0) m_0 + Y p_0 + Z q_0] - n_1 [(X - c_1) m_1 + Y p_1 + Z q_1]. \tag{44}
\]

Replacing $m$ by $\sqrt{1-p^2-q^2}$, replacing $X$ by its expression (40) as a function of $Y$ and $Z$ and expanding up to the terms of the 4th order, one obtains:

\[
\begin{align*}
W &= n_0 s - n_1 s' \\
&+ n_0 \left[ Y^2 + Z^2 + \frac{s}{2(p_1^2 + q_1^2)} + \frac{Y^2 + Z^2}{8r^2} \left( 1 + \frac{1}{8} \right) \left( \frac{Y^2 + Z^2}{p_1^2 + q_1^2} \right) \right] \\
&- n_1 \left[ Y^2 + Z^2 + \frac{s'}{2(p_1^2 + q_1^2)} + \frac{Y^2 + Z^2}{8r^2} \left( 1 + \frac{1}{8} \right) \left( \frac{Y^2 + Z^2}{p_1^2 + q_1^2} \right) \right]. \tag{45}
\end{align*}
\]

The next step is now to eliminate $Y$ and $Z$ from this expression, to obtain $W$ as a function of $p_0, q_0, p_1, q_1$ only. Within the accuracy of the Gaussian optics one obtains from Snell’s law immediately:

\[
Y \quad \frac{n_0 - n_0}{r} = n_0 p_0 - n_1 p_1, \tag{46}
\]

\[
Z \quad \frac{n_0 - n_0}{r} = n_0 q_0 - n_1 q_1.
\]

With strict calculation 3rd order terms would be added, instead one finds it unnecessary to derive these, if one remembers the minimal property of $W$. The latter has the effect that small variations of $Y$ and $Z$ effect $W$ only as their squared values so that 3rd order corrections of $Y$ and $Z$ deliver only 6th order contributions to $W$. If one keeps therefore
within the 4th order, then follows that we can insert the expressions (46) for $Y$ and $Z$ into $W$.

We must now convert from the angle Eikonal to the Seidel Eikonal. For this purpose we have to replace $p$ and $q$ according to (7a) by the variables of the Seidel Eikonal and to add according to (10) a quadratic expression in these variables to $W$. However, on the one hand all the relations between old and new variables including the equations (46) are linear and therefore the individual terms of the expansion (45) of $W$ do not change their order at the transition to the new variables. On the other hand we know that the expansion of $S$ begins with terms of the 4th order. Hence, $S^4$ can only consist of 4th order terms of $W$. Thus it is valid:

$$S^4 = n_0 \left\{ \frac{(Y^2 + Z^2) (1 + b)}{8r^2} - \frac{(Y^2 + Z^2) (p_i^4 + q_i^4)}{4r} - \frac{s (p_i^4 + q_i^4)}{8} \right\}$$

which can also be converted due to equation (42) into the form:

$$4T S^4 = \frac{1}{8 n_n s^3} \left[ n_i \frac{Y^2 + Z^2}{r} + n_i s (p_i^4 + q_i^4) \right]^3 - \frac{1}{8 n_n s^3} \left[ n_i \frac{Y^2 + Z^2}{r} + n_i s (p_i^4 + q_i^4) \right]^3 + b (n_n - n_i)(Y^2 + Z^2)^3.$$

Here we can insert the values from Gaussian optics for all variables within the intended accuracy, we can therefore exchange in particular the Seidel variables before and after the calculation as required. To obtain $S^4$ directly as a function of $y_0, z_0, \eta_1$ and $\zeta_1$ we will use instead of the equation (7a) the following:

$$p_0 = \frac{\eta_1}{\lambda - y_0} n_n \lambda_0, \quad q_0 = \frac{\eta_1}{\lambda - z_0} n_n \lambda_0,$$

$$p_1 = \frac{\eta_1}{\lambda - y_0} n_n \lambda_0, \quad q_1 = \frac{\eta_1}{\lambda - z_0} n_n \lambda_0,$$

Before inserting these values in $S^4$ small conversions are recommended. Here we set for simplicity:

$$H = \frac{t}{\lambda - n_n \lambda_0}, \quad \delta = \frac{\eta_1}{\lambda - n_n \lambda_0}.$$

Then one obtains:

$$p_0 = \frac{\eta_1}{\lambda - y_0 H} \frac{t}{s}, \quad q_0 = \frac{\eta_1}{\lambda - z_0 H} \frac{t}{s},$$

$$p_1 = \frac{\eta_1}{\lambda - y_0 H} \frac{t}{s}, \quad q_1 = \frac{\eta_1}{\lambda - z_0 H} \frac{t}{s},$$

And from (46) with consideration of (41) – (43):

$$Y = \eta_1 \delta - y_0 H, \quad Z = \eta_1 \delta - z_0 H.$$
Using the former relationships:

\[ y_0^2 + z_0^2 = R_0, \quad \eta_1^2 + z_1^2 = q_1, \quad y_0 \eta_1 + z_0 \eta_1 = \kappa_0, \]

then follows with consideration of (42),

\[ Y^2 + Z^2 = H^2 R_s - 2Hb\kappa_s + h' \eta_1 \]

\[ n_s \left( \frac{Y^2 + Z^2}{r} \right) + n_s (y_1^2 + z_1^2) = H^2 R_s \left( L - (K - L)^{s'} \right) + h' \eta_1, K - 2\frac{Hb\kappa_s}{L}, \]

\[ n_i \left( \frac{Y^2 + Z^2}{r} \right) + n_i (y_0^2 + z_0^2) = H^i R_i \left( L - (K - L)^{s'} \right) + h' \eta_i, K - 2\frac{Hb\kappa_s}{L}. \]

If we insert these expressions in \( S^d \), then we obtain the finished representation of the desired Eikonal:

\[ 8S^d = + R_s H^2 \left( \frac{b}{r^2} (n_s - n_i) + L \left( \frac{1}{n_s} - \frac{1}{n_i} \right) - 2L(K - L) \left( \frac{1}{n_s} - \frac{1}{n_i} \right) \left( s' - \frac{s'}{n_s} \right) \right), \]

\[ + q_{0 i} H^2 \left( \frac{b}{r^2} (n_0 - n_i) + K \left( \frac{1}{n_0} - \frac{1}{n_i} \right) \right) \]

52) \[ + 4\kappa_0 H^2 \left( \frac{b}{r^2} (n_0 - n_i) + L \left( \frac{1}{n_0} - \frac{1}{n_i} \right) \right) \]

\[ + 2R_s \eta_0 H^2 \left( \frac{b}{r^2} (n_s - n_i) + KL \left( \frac{1}{n_s} - \frac{1}{n_i} \right) - K(K - L) \left( \frac{1}{n_s} - \frac{1}{n_i} \right) \right) \]

\[ - 4R_s \eta_0 H^2 \left( \frac{b}{r^2} (n_s - n_i) + L \left( \frac{1}{n_s} - \frac{1}{n_i} \right) - L(K - L) \left( \frac{1}{n_s} - \frac{1}{n_i} \right) \right) \]

\[ - 4\kappa_0 H^3 \left( \frac{b}{r^2} (n_0 - n_i) + KL \left( \frac{1}{n_0} - \frac{1}{n_i} \right) \right). \]

The factors of the 5 last lines give the 3rd order aberrations introduced by a single surface.

16. We proceed at once to the consideration of any number of refracting surfaces. All variables belonging to the \( i \)-th surface shall obtain the subscript \( i \). The value of the refractive index after the \( i \)-th surface shall be \( n_i \).

According to the theorem from § 5, that the individual 3rd order aberrations are cumulative and by comparison with the emerging Eikonal expansion one obtains with the former general approach (18):

\[ B = \frac{1}{2} \sum_i b_i \left( \frac{1}{r^2} (n_i - n_{i-1}) + K \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \right) \]

\[ C = \frac{1}{2} \sum_i H_i^2 \left( \frac{b_i}{r^2} (n_i - n_{i-1}) + L \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \right) \]

54) \[ D = \frac{1}{2} \sum_i H_i^2 \left( \frac{b_i}{r^2} (n_i - n_{i-1}) + K_i L_i \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) - K_i (K_i - L_i) \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \right) \]

\[ E = \frac{1}{2} \sum_i H_i^2 \left( \frac{b_i}{r^2} (n_i - n_{i-1}) + L_i \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) - L_i (K_i - L_i) \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \right) \]

\[ F = \frac{1}{2} \sum_i H_i^2 \left( \frac{b_i}{r^2} (n_i - n_{i-1}) + K_i L_i \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \right). \]
These are the Seidel formulae for 3rd order aberrations of any centred lens system. They allow one, in a very simple way, to calculate these aberrations once all variables are known that come into question with the Gaussian optics of a lens system. The previous equations regarding the latter are composed here again with a small rearrangement and in more generalised terms, in which the so far arbitrary variable shall be set to $\lambda_0 = 1$.

\[
\begin{align*}
55) \quad n_{i-1}\left(\frac{1}{s_i} + \frac{1}{r_j}\right) &= n_i\left(\frac{1}{s_i} + \frac{1}{r_j}\right) = K_i, \\
56) \quad H_i &= \frac{t_i}{n_i}, \quad h_i = \frac{s_i}{s_i - t_i}, \quad \frac{H_{i+1}}{H_i} = \frac{t_{i+1}}{t_i}, \quad \frac{h_{i+1}}{h_i} = \frac{s_{i+1}}{s_i}, \quad H_i h_i = \frac{s_i t_i}{n_{i+1}(s_{i+1} - t_{i+1})}.
\end{align*}
\]

Naming $d_i$ as the distance of the vertex of the $(i + 1)$th surface from the vertex of the $i$th surface, one obtains:

\[
57) \quad d_i = s_{i+1} - s_i' = t_{i+1} - t_i'.
\]

If a lens system is given by the refractive index $n_i$, the radii $r_i$, the distances $d_i$ and the deformations $b_i$ and if the object plane and entrance pupil are fixed by their distances $s_1$ and $t_1$ from the first vertex, then one can calculate in turn all variables found in the Seidel formulae. Within the accuracy of Gaussian optics, the variables $h_i$ are obviously proportional to the distances perpendicular to the axis in which the individual refracting surfaces are intersected by rays that originate from axial object points. $s_i$ and $s_i'$ are the distances of the points of intersection of the same ray with the optical axis from the pole of the surface before and after refraction. The variables $H_i$, $t_i$ and $t_i'$ have an analogous meaning for a ray originating from the centre of the entrance pupil. For building the aberration expressions there are therefore two rays in total to track through the system according to the Gaussian optics.

If you want to change to the numerical aberrations, then you have to multiply the aberration given in 21a) with $\frac{M_{M}}{\lambda_0} = \frac{s_i'}{h_x}$ (x is the order number of the last refracting surface) and substituting the numerical factors stated there.

### 17. The Petzval condition.

The Seidel formulae contain a special theorem, that refers to astigmatism and image curvature. The subtraction of the aberration $D_i$ from $C_i$ results in:

\[
C - D = \frac{1}{2} \sum H_i h_i (L_i - K_i) \left\{ L_i \left(\frac{1}{n_{i-1} s_i} - \frac{1}{n_i s_i'}\right) - K_i \left(\frac{1}{n_{i-1} t_i} - \frac{1}{n_i t_i'}\right)\right\}.
\]

From (55) follows:

\[
\begin{align*}
\frac{1}{n_{i-1} s_i} - \frac{1}{n_i s_i'} &= K_i \left(\frac{1}{n_{i-1}^2} - \frac{1}{n_i^2}\right) - \frac{1}{r_i} \left(\frac{1}{n_{i-1}} - \frac{1}{n_i}\right), \\
\frac{1}{n_{i-1} t_i} - \frac{1}{n_i t_i'} &= L_i \left(\frac{1}{n_{i-1}^2} - \frac{1}{n_i^2}\right) - \frac{1}{r_i} \left(\frac{1}{n_{i-1}} - \frac{1}{n_i}\right).
\end{align*}
\]
And therefore:

\[ L \left( \frac{1}{n_{i-1} s_i} - \frac{1}{n_i} \right) - K_1 \left( \frac{1}{n_{i-1} t_i} - \frac{1}{n_i} \right) = \frac{K_i}{r_i} \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right). \]

Also:

\[ C - D = \frac{i}{4} \sum H_i H_i' (L_i - K_i)^2 \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right) \frac{1}{r_i}. \]

Also it follows from equation (55) that:

\[ K_i - L_i = n_i - s_i \frac{t_i - s_i}{t_i s_i}, \]

which with consideration of the last equation (56) gives:

\[ (L_i - K_i) H_i = 1 \]

and therefore:

\[ C - D = \frac{i}{4} \sum \frac{1}{r_i} \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right). \]

If we express \( C \) and \( D \) by the curvature radii \( \rho_s \) and \( \rho_t \) of the sagittal and tangential image surface and set the refractive index of the last medium (\( n_1 \) in the formulae (25) and (26)) equal to 1, then one obtains:

\[ \frac{1}{\rho_i} - \frac{3}{\rho_i} = 2 \sum \frac{1}{r_i} \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right). \]

We find therefore a relation between the radii of curvature of the two image surfaces which depend only on the radii, but not on the distances of the refracting surfaces. If the system is free from spherical aberration, coma and astigmatism, so that a sharp image occurs on a surface with a curvature of \( \rho_s = \rho_t = \rho \), then the expression states:

\[ \frac{1}{\rho} = \sum \frac{1}{r_i} \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right), \]

directly the radius of curvature of the spherical surface on which the focused image lies. This law is named after its discoverer as “Petzval’s theorem”. Under the Petzval condition we understand the requirement:

\[ 0 = \sum \frac{1}{r_i} \left( \frac{1}{n_i} - \frac{1}{n_{i-1}} \right), \]

which needs to be fulfilled for an optical system to be completely free of 3rd order aberrations.

18. Final remark.

From the Seidel formulae we can tackle the practical task of designing optical systems, for which one or more aberrations of the 3rd order are eliminated. It is apparent see that we have to deal with nothing but algebraic problems. In a later report older and newer problems of this kind shall be considered.