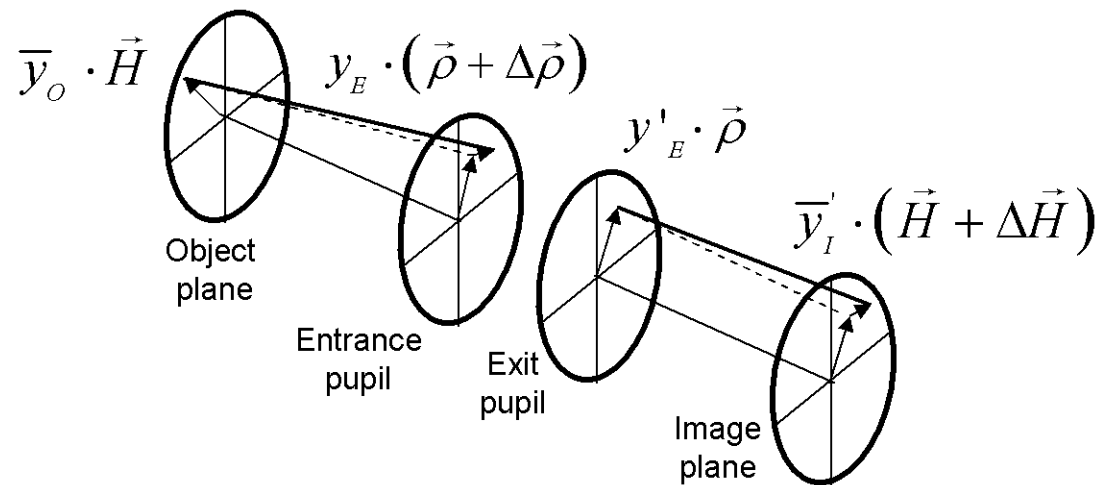


Introduction to aberrations

OPTI 518

Lecture 8

Image and aberration evaluation



Aberrations and image:

- Aberration theory
- Aberration evaluation
- Image formation
- Image evaluation
- Geometrical and physical theories

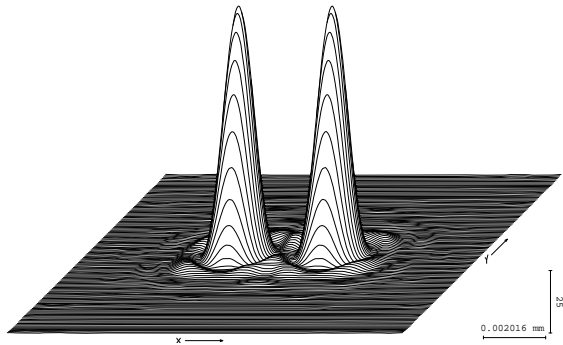
- The image of a “point” object

Criteria

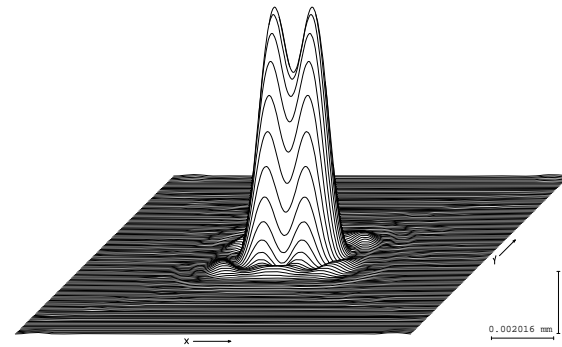
- The application determines the aberration tolerances
- Most usually the detector pixel size or the specification on resolution
- The case of visual instruments

Resolution

- Resolution is an important image quality metric
 - It determines whether two closely spaced, and equal irradiance, objects can be identified as two



Well resolved



Rayleigh
resolution
peak of 2nd at
1st zero of first

Resolution



Rules that rule optics

$$RES = k_1 \frac{\lambda}{NA}; \quad DOF = k_2 \frac{n\lambda}{NA^2}$$

$k_1 \sim 0.4-0.8$

Raleigh resolution $k_1 = 0.61$

$k_2 \sim 0.5$

Wavelength = 193 nm

R(193 dry) = 147 nm
(NA = 0.8; $k_1 = 0.61$)

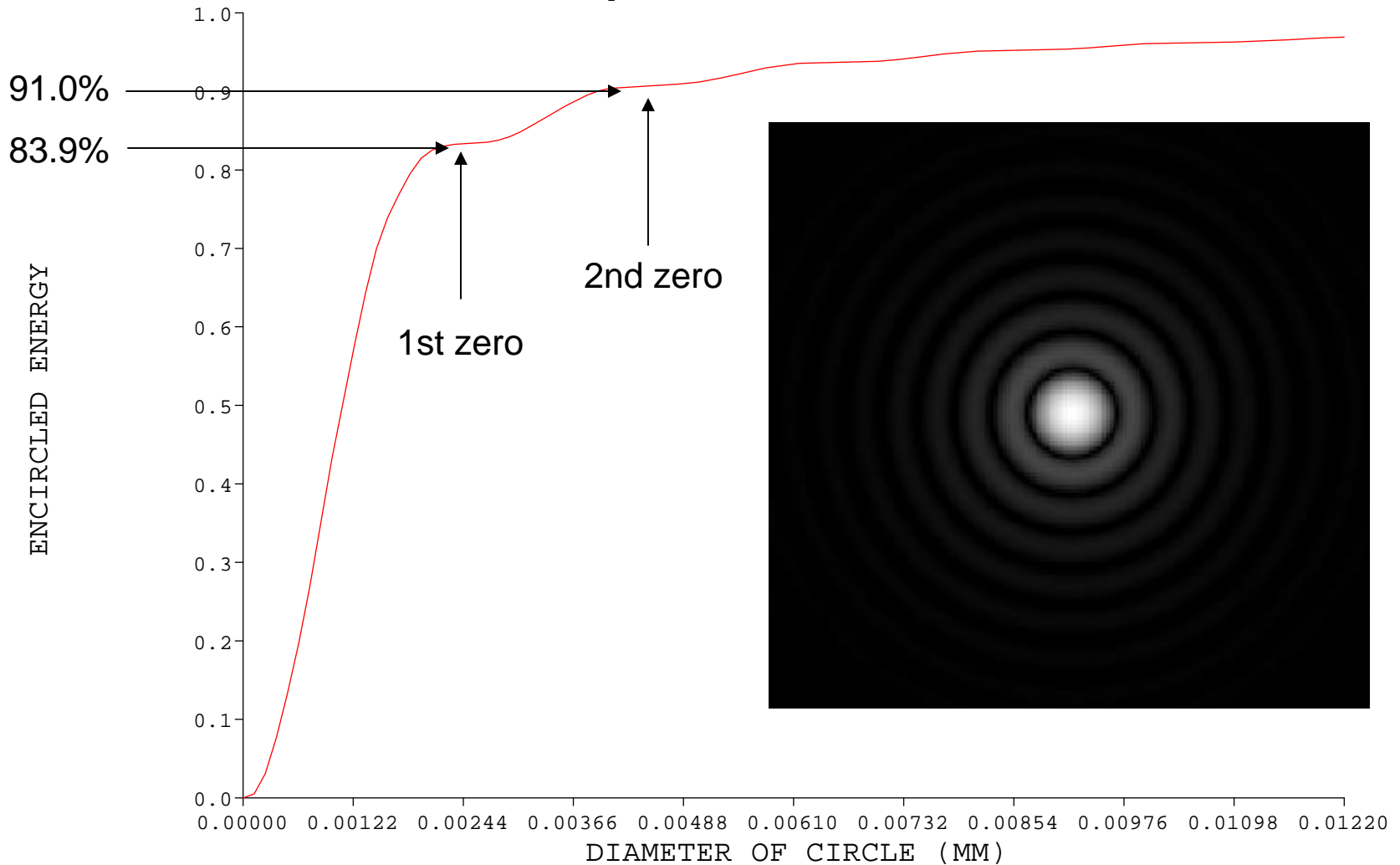
DOF = 151 nm

R(193 water) = 110 nm

Must preserve imaging volume!

Resolution and depth of focus are critical.

Diffraction Encircled Energy in the Airy pattern



Wavefront variance

One metric to quantify aberrations is the use of their amplitude; this is given by the aberration coefficients. Another important metric is the variance σ_W^2 of the wavefront deformation,

$$\sigma_W^2 = \overline{W^2} - \overline{W}^2,$$

where the mean square deformation is,

$$\overline{W^2} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W^2 \rho d\rho d\phi$$

and the mean deformation is,

$$\overline{W} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W \rho d\rho d\phi.$$

Let us consider the case of spherical aberration and change of focus. We have that the aberration function is,

$$W(\bar{H}, \bar{\rho}) = W_{040} (\bar{\rho} \cdot \bar{\rho})^2 + W_{020} (\bar{\rho} \cdot \bar{\rho}).$$

The mean square deformation and the square of the mean deformation are respectively,

$$\overline{W^2} = \frac{1}{3} W_{020}^2 + \frac{1}{2} W_{020} W_{040} + \frac{1}{5} W_{040}^2,$$

$$\overline{W}^2 = \frac{1}{2} W_{020} + \frac{1}{3} W_{040}.$$

and the variance becomes,

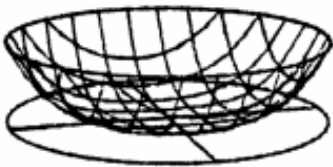


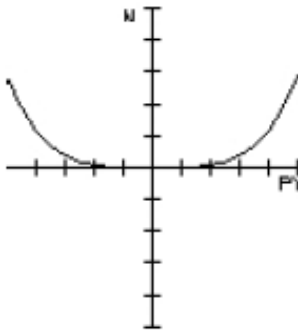
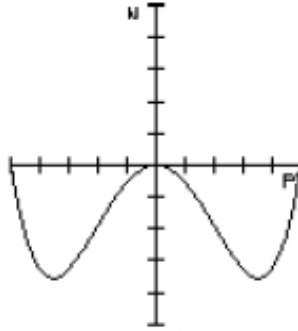
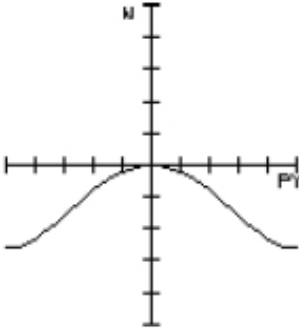
$$\sigma_W^2 = \frac{1}{12} (W_{020} + W_{040})^2 + \frac{1}{180} W_{040}^2.$$

If we use a minimum wavefront variance as a criteria for best image, then we find that the best image occurs when the amount of change of focus is,

$$W_{020} = -W_{040} .$$

We are using two different terms in the aberration function to minimize the wavefront variance. This process is known as aberration balancing.

Aberration balancing

	Paraxial focus	Minimum OPD variance	Marginal ray focus
S H A P E S			
W A V E F A N S			

Representation of spherical aberration and focus balancing.

Wavefront variance in the presence of primary aberrations

$$\sigma_w^2 = \frac{1}{12} \left(W_{020} + W_{040} + \left(W_{220} + \frac{1}{2} W_{222} \right) H^2 \right)^2 + \frac{1}{4} \left(W_{111} + \frac{2}{3} W_{131} H + W_{311} H^3 \right)^2 + \frac{1}{180} (W_{040})^2 + \frac{1}{72} (W_{131} H)^2 + \frac{1}{24} (W_{222} H^2)^2$$

For a given field point \bar{H} the best focus under minimum wavefront variance takes place when the change of focus W_{020} and magnification W_{111} satisfy,

$$W_{020} + W_{040} + \left(W_{220} + \frac{1}{2} W_{222} \right) H^2 = 0$$

and,

$$W_{111} + \frac{2}{3} W_{131} H + W_{311} H^3 = 0.$$

Clearly the even aberrations, spherical aberration and astigmatism, can be balanced with a change of focus W_{020} ; the odd aberrations, coma and distortion, can be balanced with a change of magnification W_{111} for a given field point.

Rayleigh-Strehl ratio

Imaging with light waves is obtained as a convolution of the geometrical image with the point spread function of the imaging system. For the case of a system with a circular aperture the irradiance of the point spread function is the Airy pattern. In the presence of aberrations the central peak of the Airy pattern decreases in peak value.

A first estimate of the decreases of image quality in a system that has small amounts of aberrations is the ratio of the peak of the point spread function in the presence of aberrations to the peak in the absence of aberrations. The concept of using the decrease in the peak of the point spread function as a metric for image quality has been pioneered by Lord Raleigh. In his investigations Lord Raleigh had, in the absence of aberrations, normalized the peak irradiance to unity. However, the ratio is commonly referred as the Strehl ratio.

Rayleigh-Strehl ratio

Let us assume an imaging system that may have small amounts of wavefront $W(x,y)$ aberrations. The point spread function for incoherent illumination is,

$$psf(x, y) = \left(\frac{A_p}{f\lambda} \right)^2 \left| FT \left\{ t(x, y) e^{ikW(x,y)} \right\} \right|_{\xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}}^2,$$

where A_p is the amplitude of an incoming plane wave, $t(x,y)$ is the transmittance function, and f is the focal length. Using the central limit theorem we find that the peak irradiance of the point spread function is,

Use of the central-ordinate theorem

$$\begin{aligned}
 I_{peak} &= \left(\frac{A_p}{f\lambda} \right)^2 \left| \iint_{\infty} t(x, y) e^{ikW(x, y)} dx dy \right|^2 \\
 &\cong \left(\frac{A_p}{f\lambda} \right)^2 \left| \iint_{Aperture} \left(1 + ikW(x, y) - k^2 W^2(x, y) / 2 \right) dx dy \right|^2 \\
 &\cong \left(\frac{A_p}{f\lambda} \right)^2 \left| \iint_{Aperture} dx dy + ik \iint_{Aperture} W(x, y) dx dy - \frac{k^2}{2} \iint_{Aperture} W^2(x, y) dx dy \right|^2. \\
 &\cong \left(\frac{A_p}{f\lambda} \right)^2 \left(\left(\iint_{Aperture} dx dy \right)^2 + k^2 \left(\iint_{Aperture} W(x, y) dx dy \right)^2 \right. \\
 &\quad \left. - k^2 \left(\iint_{Aperture} dx dy \right) \left(\iint_{Aperture} W^2(x, y) dx dy \right) \right).
 \end{aligned}$$

In the absence of aberrations the peak of the point spread function is,

$$I_0 = \left(\frac{A_p}{f\lambda} \right)^2 \left(\iint_{Aperture} dx dy \right)^2.$$

Then the Raleigh-Strehl ratio becomes,

$$\frac{I_{peak}}{I_0} \cong 1 - k^2 \frac{\left(\iint_{Aperture} dx dy \right) \left(\iint_{Aperture} W^2(x, y) dx dy \right) - \left(\iint_{Aperture} W(x, y) dx dy \right)^2}{\left(\iint_{Aperture} dx dy \right)^2}.$$
$$= 1 - \left(\frac{2\pi}{\lambda} \right)^2 \sigma_W^2$$

This simple expression that relates the variance of the wavefront σ_w^2 to the drop in the peak of the un-aberrated point spread function is insightful. First, for systems with small amounts of aberration, $\sim \lambda / 2$, it makes the variance of the wavefront an important image quality metric. Second, the term $(2\pi / \lambda)^2 \sigma_w^2$ represents the energy that is removed from the central peak and redistributed elsewhere in the diffraction pattern.

Andre Marechal formula

$$\frac{I_W}{I} = \left(1 - \frac{1}{2} \left(\frac{2\pi}{\lambda} \right)^2 \sigma_W^2 \right)^2$$

Shack's formula

$$\frac{I_w}{I} = \exp \left\{ - \left(\frac{2\pi}{\lambda} \right)^2 \sigma_w^2 \right\}$$

Shack's formula derivation

Sol.

$$RSR = \frac{1}{\pi^2} \left| \int_0^{2\pi} \int_0^1 \exp\left[\frac{i2\pi}{\lambda} W(\rho, \theta)\right] \rho d\rho d\theta \right|^2$$

$$\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$$

$$\therefore RSR = \left| \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1 + \frac{i2\pi}{\lambda} W(\rho, \theta) - \frac{1}{2} \left(\frac{2\pi}{\lambda} \right)^2 W^2(\rho, \theta) + \dots \right) \rho d\rho d\theta \right|^2$$

Recall the moments of the wavefront error:

$$\overline{W^n} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W^n(\rho, \theta) \rho d\rho d\theta$$

Since the aberrations are small, we can neglect the 3rd and higher order terms of the power series.

$$\begin{aligned} RSR &\approx \left| 1 + \frac{i2\pi}{\lambda} \overline{W} - \frac{1}{2} \left(\frac{2\pi}{\lambda} \right)^2 \overline{W^2} \right|^2 = \left[1 - \frac{1}{2} \left(\frac{2\pi}{\lambda} \right)^2 \overline{W^2} \right]^2 + \left[\left(\frac{2\pi}{\lambda} \right) \overline{W} \right]^2 \\ &\approx 1 - \left(\frac{2\pi}{\lambda} \right)^2 \left(\overline{W^2} - \overline{W}^2 \right) = 1 - \left(\frac{2\pi}{\lambda} \right)^2 \sigma_w^2 = 1 - \left(\frac{2\pi}{\lambda} \sigma_w \right)^2 \end{aligned}$$

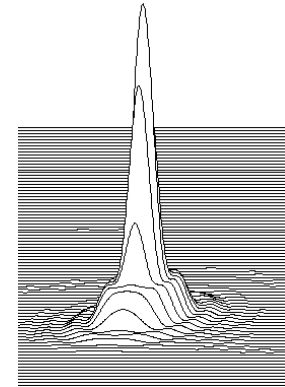
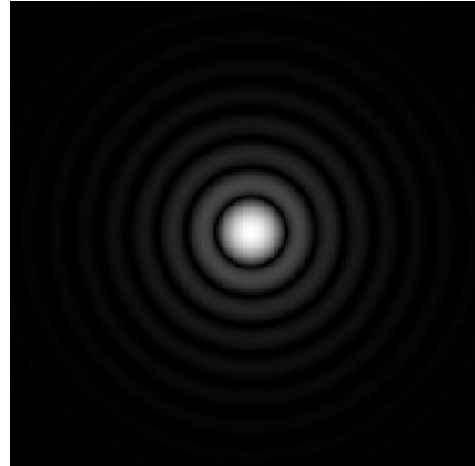
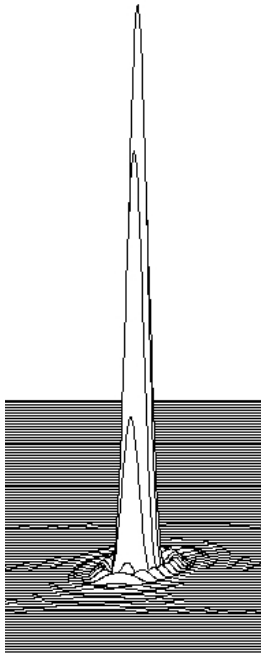
For $RSR > 0.1$, we keep the high order terms of the power series:

Shack's formula derivation cont.

$$\begin{aligned}
 RSR &= \left| \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1 + \frac{i2\pi}{\lambda} W(\rho, \theta) - \frac{1}{2!} \left(\frac{2\pi}{\lambda} \right)^2 W^2(\rho, \theta) - \frac{i}{3!} \left(\frac{2\pi}{\lambda} \right)^3 W^3(\rho, \theta) + \frac{1}{4!} \left(\frac{2\pi}{\lambda} \right)^4 W^4(\rho, \theta) - \dots \right) \rho d\rho d\theta \right|^2 \\
 &= \left| 1 + \frac{i2\pi}{\lambda} \overline{W} - \frac{1}{2!} \left(\frac{2\pi}{\lambda} \right)^2 \overline{W}^2 - \frac{i}{3!} \left(\frac{2\pi}{\lambda} \right)^3 \overline{W}^3 + \frac{1}{4!} \left(\frac{2\pi}{\lambda} \right)^4 \overline{W}^4 - \dots \right|^2 \\
 &= \left[1 - \frac{1}{2!} \left(\frac{2\pi}{\lambda} \right)^2 \overline{W}^2 + \frac{1}{4!} \left(\frac{2\pi}{\lambda} \right)^4 \overline{W}^4 - \dots \right]^2 + \left[\left(\frac{2\pi}{\lambda} \right) \overline{W} - \frac{1}{3!} \left(\frac{2\pi}{\lambda} \right)^3 \overline{W}^3 + \dots \right]^2 \\
 &\approx 1 - \left(\frac{2\pi}{\lambda} \right)^2 (\overline{W}^2 - \overline{W}^2) + \frac{1}{2!} \left(\frac{2\pi}{\lambda} \right)^4 (\overline{W}^2{}^2 - 2\overline{W}^2 \overline{W}^2 + \overline{W}^4) + \dots \\
 &= 1 - \left(\frac{2\pi}{\lambda} \sigma_w \right)^2 + \frac{1}{2!} \left(\frac{2\pi}{\lambda} \sigma_w \right)^4 + \dots \\
 &= e^{-\left(\frac{2\pi}{\lambda} \sigma_w \right)^2}
 \end{aligned}$$

Given by an OPTI518 student

Rayleigh-Strehl ratio



$$\text{Rayleigh - Strehl} = \frac{\text{Aberrated PSF peak}}{\text{Unaberrated PSF peak}}$$

- See Lord Rayleigh, "Investigations in optics, with special reference to the spectroscope," *Phil. Mag.* Vol. 5(8), 403-411, 1879.
- Strehl, K. 1895, *Aplanatische und fehlerhafte Abbildung im Fernrohr*, *Zeitschrift für Instrumentenkunde* 15 (Oct.), 362-370.
- Strehl, K. 1902, *Über Luftschlieren und Zonenfehler*, *Zeitschrift für Instrumentenkunde*, 22 (July), 213-217.
- V. N. Mahajan, "Strehl ratio for primary aberrations in terms of their aberration variance," *J. Opt. Soc. Am.* Vol. 73(6) 860-861, 1983.

Rayleigh-Strehl ratio

$$S \sim 1 - \left(\frac{2\pi}{\lambda} \right)^2 \sigma_w^2$$

Approximation
Good for $S \sim > 0.3$

$$S \sim e^{-\left(\frac{2\pi}{\lambda} \right)^2 \sigma_w^2}$$

Semi-Empirical !?
Good for $S \sim > 0.1$

**It is the variance of the
wavefront deformation**

Some Rayleigh-Strehl ratio insights

- Ratio of aberrated to unaberrated PSF peaks
- Scattered energy
- Contributions from refracting and reflecting surfaces
- Contributions from rough surfaces
- DOE efficiency
- Keep in mind formula limits

$$S \sim 1 - \left(\frac{2\pi}{\lambda} \right)^2 \sigma_w^2$$

The case of refraction vs. reflection

$$S \sim 1 - \left(\frac{2\pi}{\lambda}\right)^2 \sigma_w^2 = 1 - \left(\frac{2\pi}{\lambda}\right)^2 (n' - n)^2 \sigma_{surface}^2$$

$$(n' - n)_{refraction}^2 = 0.25$$

$$(n' - n)_{reflection}^2 = 4$$

- Ratio of scattered energy is 16!
- Thus for a given surface roughness, reflection scatters 16 times more than refraction.
- Note dependence with wavelength

Marechal Criterion ~1947

For SR=0.8, RMS $\sim 0.07 \lambda$

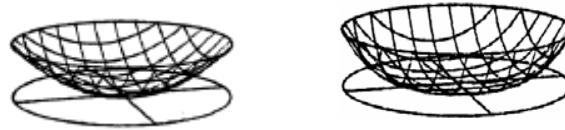
Marechal Criteria

	P-V	PV/RMS	
W_{020}	$\sim 0.247 \lambda$	~ 3.5	
W_{040}	$\sim 0.239 \lambda$	~ 3.5	
W_{040}	$\sim 0.955 \lambda$	~ 13.4	Best focus
W_{131}	$\sim 0.604 \lambda$	~ 8.6	
W_{222}	$\sim 0.349 \lambda$	~ 5	Medial image

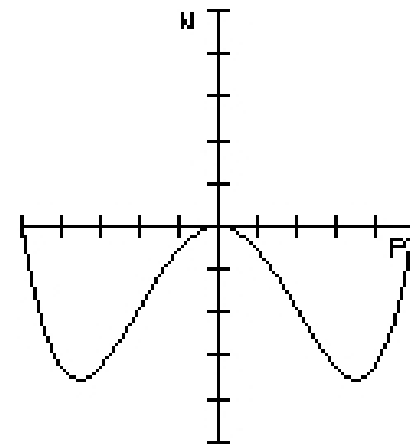
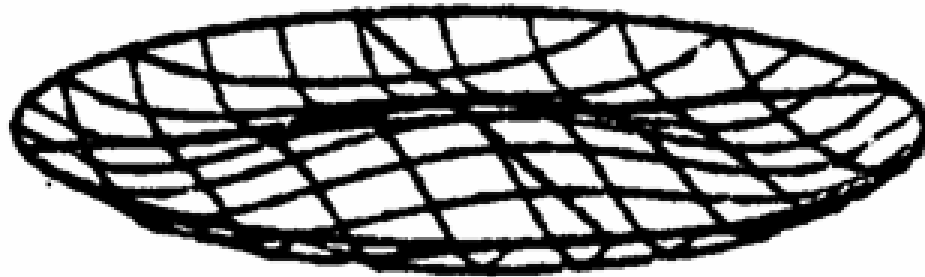
Keep in mind that is not only the resolution but also
the depth of focus.

MUST PRESERVE ALL DOF!!!

Best focus according to variance

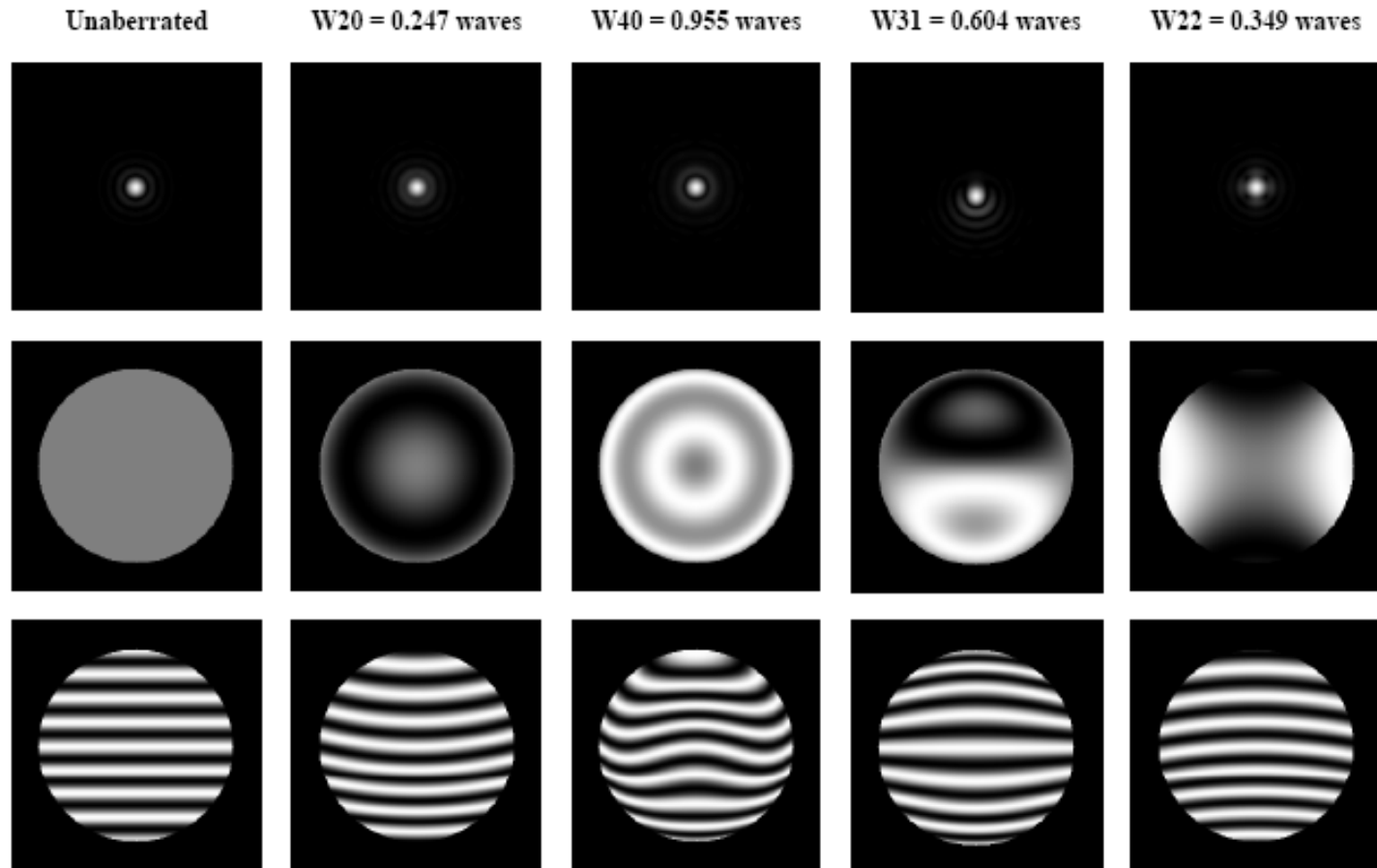


$$W_{020} = -W_{040}$$



The concept of “Diffraction limited”

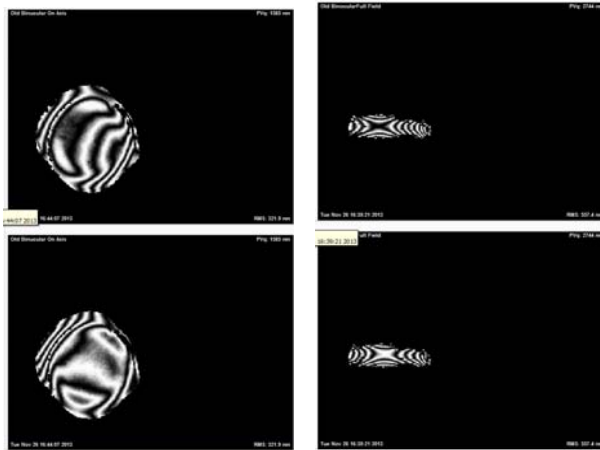
- Not limited by geometrical aberrations
- Depends on the particular field
- Usually means $RSR > 0.8$
- The human eye is diffraction limited at a pupil diameter of ~ 1.5 mm



Tolerably Aberrated Images
Strehl Ratio = 0.80

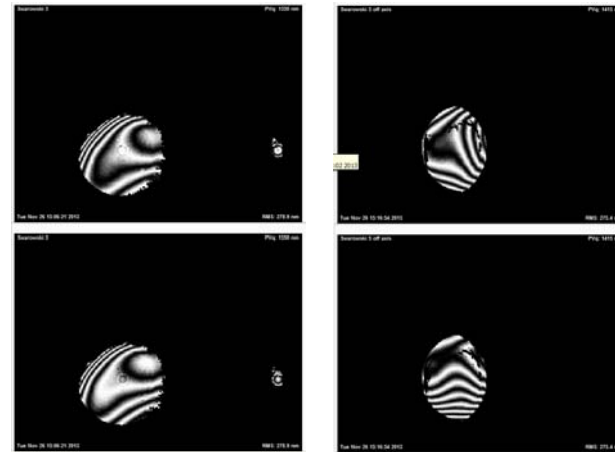
Binocular quality

Inexpensive binos



Double pass

Expensive binos 42X10



Double pass

Buchroeder rule for visual instruments: $\sim 1/4$ wave of monochromatic aberration per $1/2$ mm of pupil

For a 4 mm in diameter eye pupil

$$\sqrt{\sigma_{eye}^2} \sim 2\lambda$$

@ $\lambda = 0.5 \mu m$

$$\sigma_w^2 = \sigma_{eye}^2 + \sigma_{bino}^2$$

Topics

- Aberration/image evaluation
- Image of a point object
- Rules that rule the optics world
- Wavefront variance
- Aberration balancing
- Raleigh-Strehl ratio