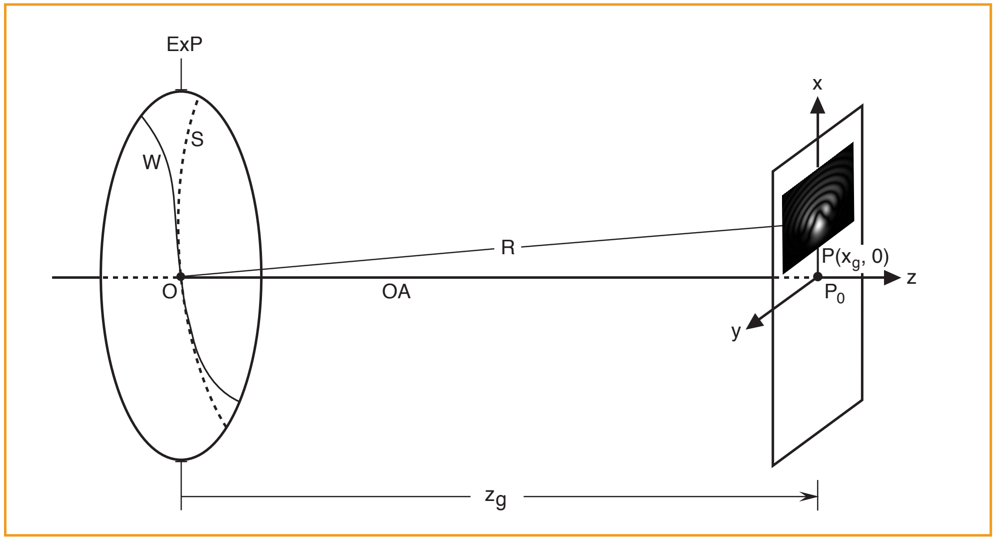
Zernike Polynomials and Beyond

"Introduction to Aberrations"



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Zernike Lecture

12 April 12

References

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2. M. Born and E. Wolf, *Principles of Optics*, 7th edition (Cambridge, New York, 1999), Chapter IX, "The Diffraction Theory of Aberrations," pp. 517-553.

3. V. N. Mahajan, “Zernike polynomials and aberration balancing,” SPIE Proc. Vol 5173, 1-17 (2003).

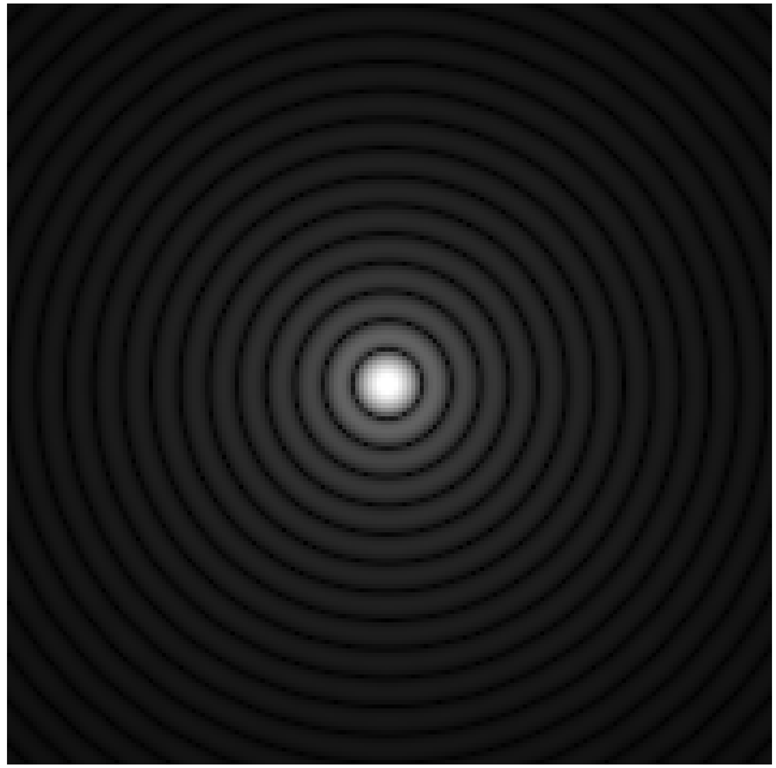
4. V. N. Mahajan, “Zernike polynomials and wavefront fitting,” in *Optical Shop Testing*, D. Malacara ed., 3rd edition (Wiley , 2006).

5. V. N. Mahajan and W. H. Swantner, “Seidel coefficients in optical testing,” Asian J. Phys. **15**, 203–209 (2006).

Summary

- Airy Pattern
- Aberrated PSF
- Strehl Ratio
- Balanced Aberrations
- Zernike Circle Polynomials
- Seidel Coefficients from Zernike Coefficients
- Orthonormal Polynomials for Noncircular Pupils
- Annular Pupils and Zernike Annular Polynomials
- Use of Circle Polynomials for Noncircular Pupils

Airy Pattern



- Central bright spot of radius 1.22, called the Airy disc, containing 83.8% of the total power, surrounded by diffraction rings

$$I(r) = \left[\frac{2J_1(\pi r)}{\pi r}\right]^2 \quad , \qquad P(r_c) = 1 - J_0^2(\pi r_c) - J_1^2(\pi r_c)$$

How is the Airy pattern affected when a system is aberrated?

- Aberration-free central value for total power P: $PS/\lambda^2 R^2$

- Aberration decreases the central value.

- Energy flows from the Airy disc into the rings.

- Peak does not necessarily lie at the center.

- For a small aberration, the relative central value, called the **Strehl ratio**, can be estimated from its **standard deviation**.*

We illustrate reduction in central irradiance of defocused images, and shift in peak of coma-aberrated images.

*** Many optics people (incorrectly) use the term rms wavefront error when they actually mean the standard deviation.**

Defocused PSFs

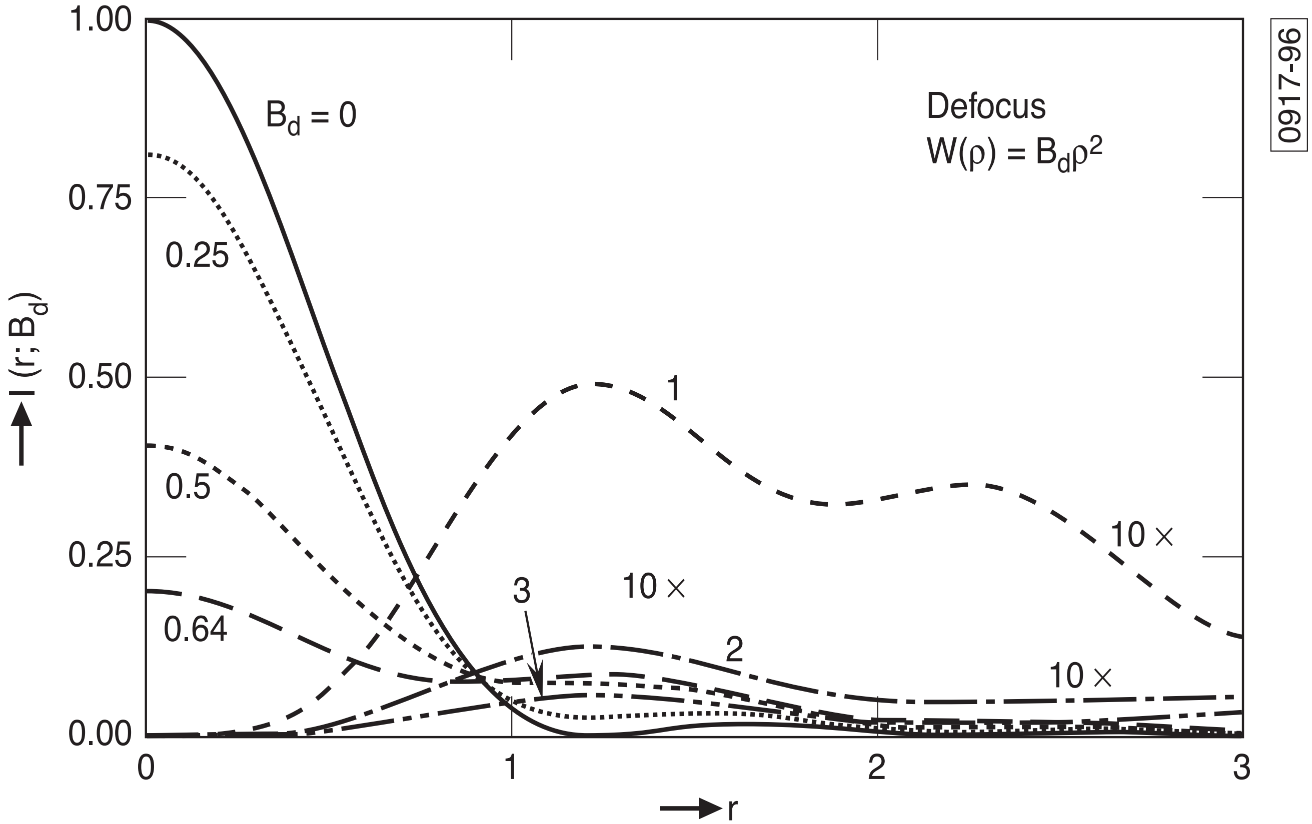
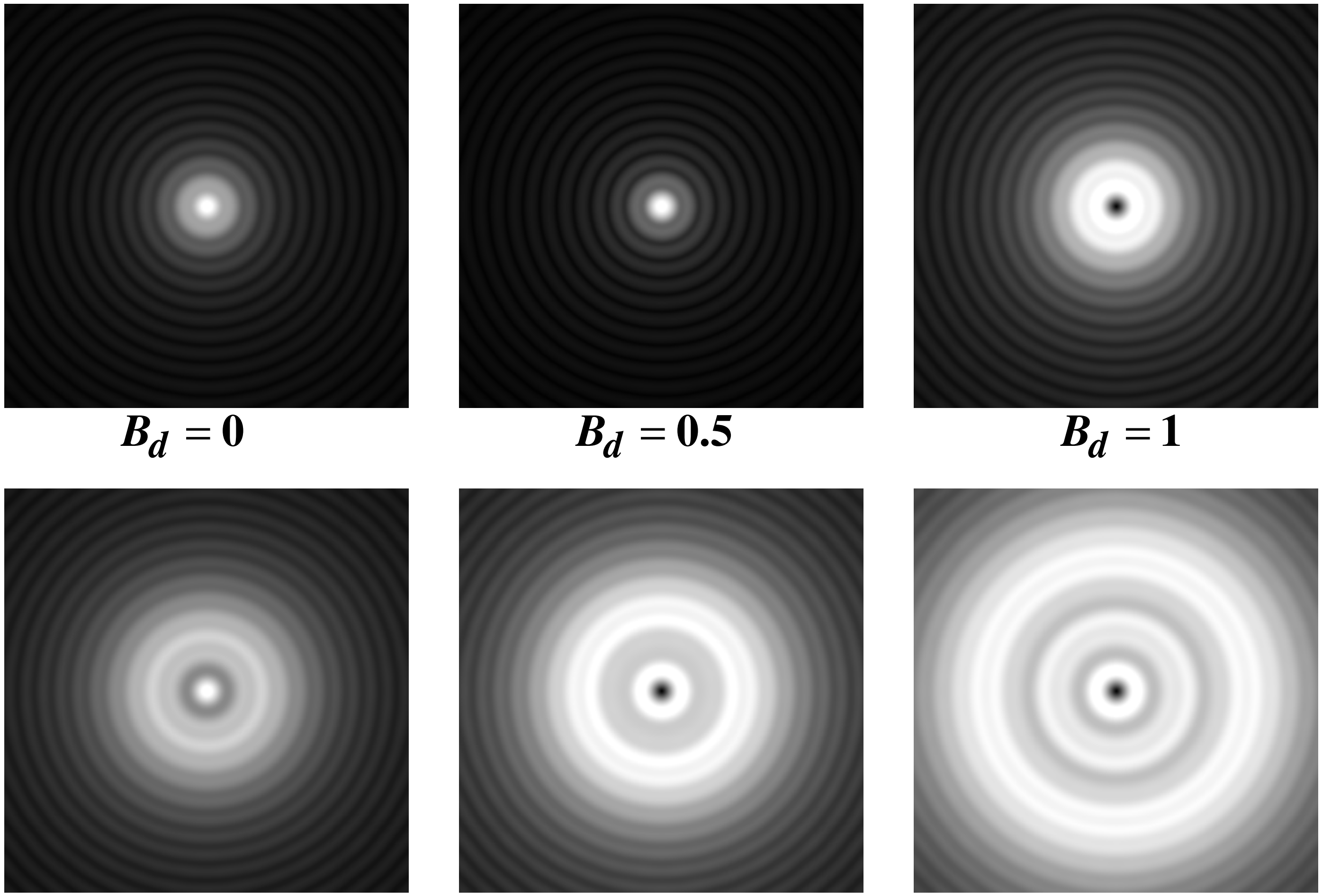


Figure 2-24. Defocused PSF curves for $B_d = 1, 2$, and 3 (in units of λ) have a central value of zero, and have been multiplied by ten for clarity.

- **Central value of PSF decreases as aberration increases.**

Defocused PSFs



$B_d = 0$ $B_d = 0.5$ $B_d = 1$

$B_d = 1.5$ $B_d = 2$ $B_d = 3$

- **Dark spot** at the center obtained for integral number of waves of defocus.

Coma PSFs

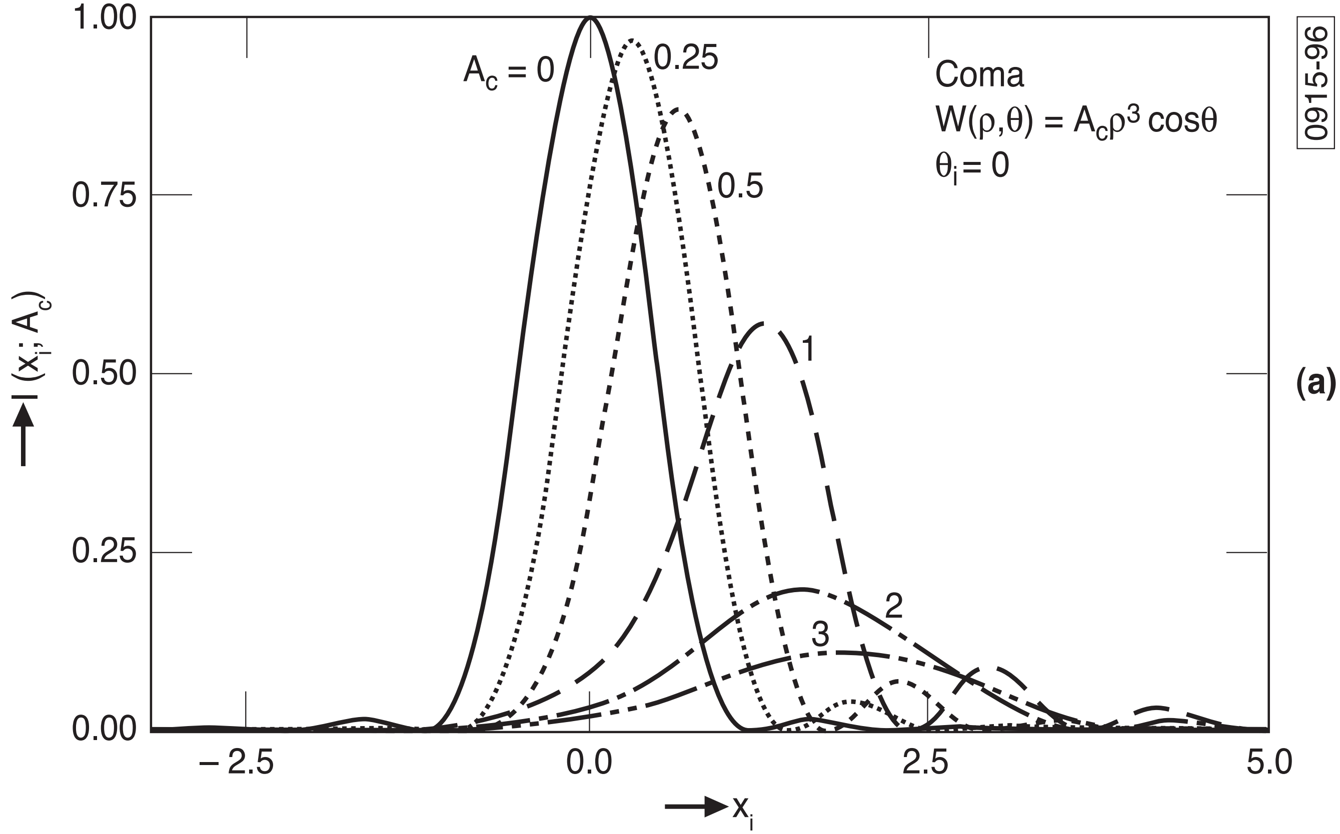


Figure 2-31. Central profiles of PSFs for coma $A_c\rho^3\cos\theta$ along the x_i axis, where the peak value A_c of the aberration is in units of λ.

- **Central value of PSF decreases and its peak shifts.**

Coma PSFs

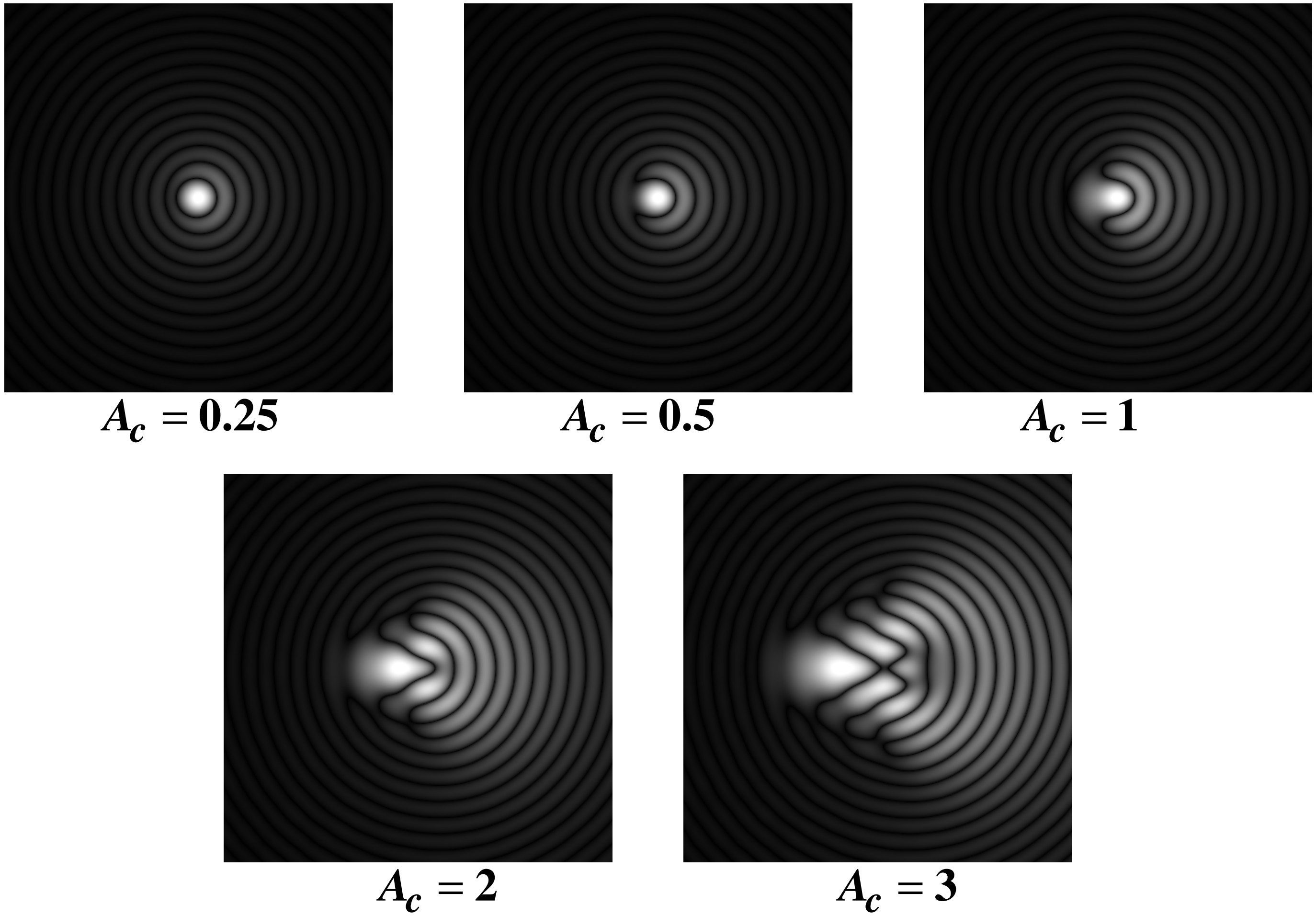


Figure 2-37. PSFs aberrated by **coma** $A_c \rho^3 \cos\theta$ are symmetric about the x_i (horizontal) axis, where A_c is in units of λ.

Strehl Ratio*

• For a given total power, the central value of the image of a point object is maximum for uniform amplitude and phase of the wave at the exit pupil.

• Strehl ratio is the ratio of central irradiances with and without phase/amplitude variations.

$$S = \frac{\text{Aberrated central irraidance}}{\text{Aberration - free central irraidance}} = \frac{I(0)_{\Phi}}{I(0)_{\Phi=0}} \leq 1$$

• Used for aberration tolerancing.

• Apodization or amplitude variations also reduce the central value.**

*K. Strehl, “Ueber Luftschlieren und Zonenfehler,” Zeitschrift fur Instrumentenkunde **22**, 213–217 (1902).

V. N. Mahajan, "Luneburg apodization problem I," Opt. Lett. **5, 267-269 (1980).

How do we determine the Strehl ratio?

- Pupil function: $P\left(\vec{r}_p\right) = A_0 \exp\left[i\Phi\left(\vec{r}_p\right)\right]$ (Uniform amplitude A_0)

- Aberrated PSF: $I(\vec{r}_i) = \dfrac{1}{\lambda^2 R^2}\left|\int P\left(\vec{r}_p\right)\exp\left(-\dfrac{2\pi i}{\lambda R}\vec{r}_p\cdot\vec{r}_i\right)d\vec{r}_p\right|^2$

- Aberration-free central value: $I(0)_{\Phi=0} = \dfrac{PS}{\lambda^2 R^2}$

$$S = \frac{I(0)_\Phi}{I(0)_{\Phi=0}} = \frac{\left|\int \exp\left[i\Phi\left(\vec{r}_p\right)\right]d\vec{r}_p\right|^2}{\left(\int d\vec{r}_p\right)^2}$$

V. N. Mahajan, “Strehl ratio for primary aberrations: some analytical results for circular and annular pupils,” J. Opt. Soc. Am. **7**, 1258–1266 (1983), **10**, 2092 (1993).

V. N. Mahajan, “Strehl ratio for primary aberrations in terms of their aberration variance,” J. Opt. Soc. Am. **73**, 240–241 (1983).

Approximate Expression for Strehl Ratio

$$S = \frac{\text{Aberrated central irraidance}}{\text{Aberration} - \text{free central irraidance}} = \frac{I(0)_{\Phi}}{I(0)_{\Phi=0}}$$

$$= \frac{\left| \int \exp\left[i\Phi\left(\vec{r}_p \right) \right] d\,\vec{r}_p \right|^2}{\left(\int d\,\vec{r}_p \right)^2}$$

$$= \frac{\left| \int_0^1 \int_0^{2\pi} \exp\left[i\Phi(\rho, \theta) \right] \rho\, d\rho\, d\theta \right|^2}{\left(\int_0^1 \int_0^{2\pi} \rho\, d\rho\, d\theta \right)^2}$$

$$= \frac{1}{\pi^2} \left| \int_0^1 \int_0^{2\pi} \exp\left[i\Phi(\rho, \theta) \right] \rho\, d\rho\, d\theta \right|^2$$

$$= \left| \langle \exp(i\Phi) \rangle \right|^2 = \left| \langle \exp\left[i(\Phi - \langle \Phi \rangle) \right] \rangle \right|^2$$

• By expanding the exponent and neglecting terms higher than the quadratic, we obtain **approximate expressions for a small aberration:**

$$S = \left|\left\langle \exp\left[i\left(\Phi - \langle\Phi\rangle\right)\right]\right\rangle\right|^2$$

$$= \left|\left\langle 1 + i\left(\Phi - \langle\Phi\rangle\right) - \frac{1}{2}\left(\Phi - \langle\Phi\rangle\right)^2 + \ldots. \right\rangle\right|^2$$

$$\simeq \left|\left\langle 1 - \frac{1}{2}\left(\Phi - \langle\Phi\rangle\right)^2 \right\rangle\right|^2$$

$$= \left(1 - \frac{1}{2}\sigma_\Phi^2\right)^2$$ (independent of the shape of the pupil)

$$\sigma_\Phi^2 = \left\langle \left(\Phi - \langle\Phi\rangle\right)^2 \right\rangle = \left\langle \Phi^2 + \langle\Phi\rangle^2 - 2\Phi\langle\Phi\rangle \right\rangle = \left\langle \Phi^2 \right\rangle - \langle\Phi\rangle^2$$

Standard deviation: σ_Φ , Variance: σ_Φ^2 , $\mathrm{rms}_\Phi \equiv \sqrt{\left\langle \Phi^2 \right\rangle}$

Various approximate expressions

$$S_1 \simeq \left(1-\sigma_\Phi^2/2\right)^2 = 1-\sigma_\Phi^2+\frac{1}{4}\sigma_\Phi^4 = S_2+\frac{1}{4}\sigma_\Phi^4 \quad \text{(Maréchal)}$$

$$S_2 \simeq 1-\sigma_\Phi^2 \qquad \text{(Nijboer)}$$

$$S_3 \simeq \exp\left(-\sigma_\Phi^2\right) \simeq 1-\sigma_\Phi^2+\frac{1}{2}\sigma_\Phi^4 = S_1+\frac{1}{4}\sigma_\Phi^4 \quad \text{(Mahajan)}$$

$$\sigma_\Phi^2 = \left\langle \Phi^2 \right\rangle - \langle \Phi \rangle^2 \quad \text{(Phase aberration variance)}$$

$$<\Phi^n> = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \Phi^n(\rho,\theta)\,\rho\, d\rho\, d\theta$$

$$S_2 \simeq 0.8 \Rightarrow \sigma_\Phi = \sqrt{0.2} \quad \text{or} \quad \sigma_w = \frac{\lambda}{2\pi}\sqrt{0.2} = \frac{\lambda}{14.05}$$

- **Thus,** $S = 0.8$ **for** $\sigma_w = \lambda/14$**, regardless of the type of the aberration.**

Table 2-4. Standard deviation of and aberration tolerance for Seidel or primary aberrations.

Aberration	$\boldsymbol{\Phi(\rho,\theta)}$	$\boldsymbol{\sigma_\Phi}$	$\boldsymbol{A_i}$ **for** $\boldsymbol{S = 0.8}$
Spherical	$A_s\rho^4$	$\dfrac{2A_s}{3\sqrt{5}} = \dfrac{A_s}{3.35}$	$\lambda/4.19$
Coma	$A_c\rho^3\cos\theta$	$\dfrac{A_c}{2\sqrt{2}} = \dfrac{A_c}{2.83}$	$\lambda/4.96$
Astigmatism	$A_a\rho^2\cos^2\theta$	$\dfrac{A_a}{4}$	$\lambda/3.51$
Defocus	$B_d\rho^2$	$\dfrac{B_d}{2\sqrt{3}} = \dfrac{B_d}{3.46}$	$\lambda/4.06$
Tilt	$B_t\rho\cos\theta$	$\dfrac{B_t}{2}$	$\lambda/7.03$

• Peak aberration A_i for a given Strehl ratio and, therefore, the same standard deviation, is different for a different aberration.

Balanced aberrations:

• Since the Strehl ratio is higher for a smaller aberration variance, we mix a given aberration with one or more lower-order aberrations to reduce its variance.

• The process of reducing the variance of an aberration by mixing it with an other is called *aberration balancing*.

• Consider, for example, balancing of spherical aberration with defocus:

$$\Phi(\rho) = A_s \rho^4 + B_d \rho^2$$

• We calculate its variance as follows:

$$\langle \Phi \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left(A_s \rho^4 + B_d \rho^2 \right) \rho \, d\rho \, d\theta = \frac{A_s}{3} + \frac{B_d}{2}$$

$$\left\langle \Phi^2 \right\rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left(A_s \rho^4 + B_d \rho^2 \right)^2 \rho \, d\rho \, d\theta$$

$$= 2\int_0^1 \left(A_s^2\rho^8 + B_d^2\rho^4 + 2A_sB_d\rho^6\right)\rho\, d\rho$$

$$= \frac{A_s^2}{5} + \frac{B_d^2}{3} + \frac{A_sB_d}{2}$$ **[$\sqrt{\left\langle\Phi^2\right\rangle}$ is the rms value]**

$$\therefore \quad \sigma_\Phi^2 = \left\langle\Phi^2\right\rangle - \langle\Phi\rangle^2 = \frac{4A_s^2}{45} + \frac{B_d^2}{12} + \frac{A_sB_d}{6}$$

- Value of B_d for minimum variance:

$$0 = \frac{\partial\sigma_\Phi^2}{\partial B_d} = \frac{1}{6}\left(B_d + A_s\right) \Rightarrow B_d = -A_s$$

- Balanced spherical aberration:

$$\Phi_{bs}(\rho) = A_s\left(\rho^4 - \rho^2\right) \quad , \quad \sigma_{\Phi_{bs}}^2 = \frac{A_s^2}{180}$$

- Without balancing with defocus:

$$\Phi_s(\rho) = A_s\rho^4 \quad , \quad \sigma_{\Phi_s}^2 = \frac{4A_s^2}{45}$$

• Thus, balancing of spherical aberration with defocus reduces its variance by a factor of 16, or the standard deviation by a factor of 4. Hence, the aberration tolerance for a given Strehl ratio increases by a factor of 4.

• For example, $S = 0.8$ is obtained in the Gaussian image plane for $A_s = \lambda/4$. However, the same Strehl ratio is obtained for $A_s = 1\,\lambda$ in a slightly defocused image plane at a distance z such that $B_d = -\lambda/4$.

• Since the longitudinal defocus $\Delta = z - R = -8F^2 B_d$, the defocused point of maximum irradiance is located at $\left(0, 0, 8F^2 A_s\right)$ and it is called the *diffraction focus*.

• Similarly, we can balance coma with tilt, and astigmatism with defocus and determine the diffraction focus. A wavefront tilt implies that the diffraction focus lies in the Gaussian image plane.

Balancing of Seidel or Primary Aberrations

Spherical: ρ^4 , coma: $\rho^3 \cos\theta$, astigmatism: $\rho^2 \cos^2\theta$

Balanced Aberration	$\boldsymbol{\Phi(\rho,\theta)}$	**Diffraction Focus**	$\boldsymbol{\sigma_\Phi}$	$\boldsymbol{A_i}$ **for** $\boldsymbol{S = 0.8}$
Spherical	$A_s\left(\rho^4 - \rho^2\right)$	$\left(0, 0, 8F^2 A_s\right)$	$\dfrac{A_s}{6\sqrt{5}} = \dfrac{A_s}{13.2}$	0.955 λ
Coma	$A_c\left(\rho^3 - 2\rho/3\right)\cos\theta$	$\left(4FA_c/3, 0, 0\right)$	$\dfrac{A_c}{6\sqrt{2}} = \dfrac{A_c}{8.49}$	0.604 λ
Astigmatism	$A_a\rho^2\left(\cos^2\theta - 1/2\right)$ $= (A_a/2)\rho^2\cos 2\theta$	$\left(0, 0, 4F^2 A_a\right)$	$\dfrac{A_a}{2\sqrt{6}} = \dfrac{A_a}{4.90}$	0.349 λ

• σ_w is reduced by a factor of 4, 3, and 1.225 for spherical, coma, and astigmatism, respectively.

• Aberration tolerance for a given Strehl ratio increases by this factor.

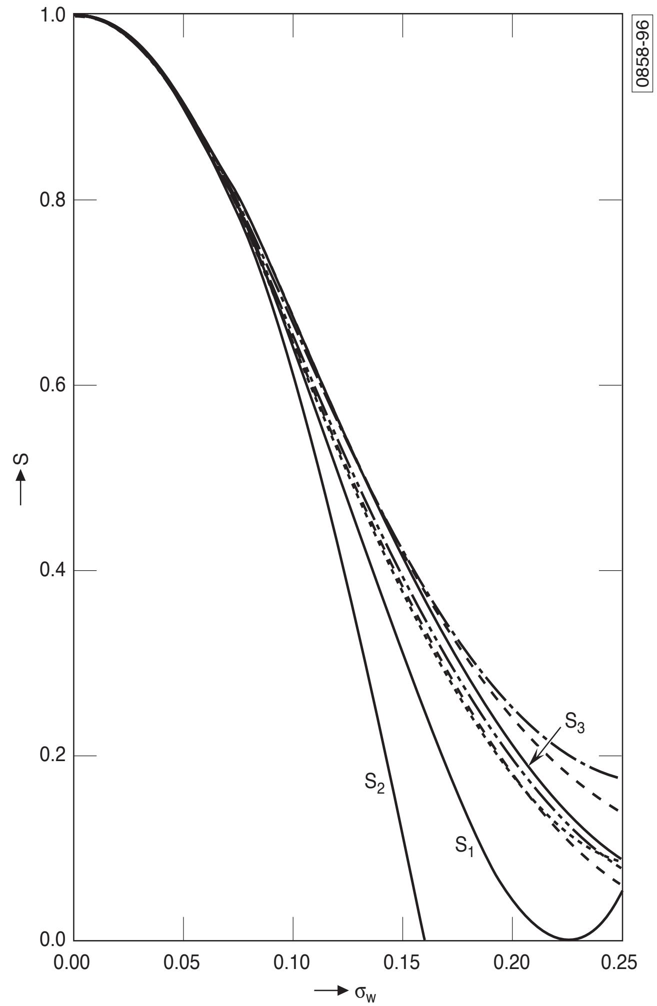


Figure 2-8. Strehl ratio as a function of standard deviation σ_w in units of λ.

- S_3 gives a much better fit than S_1 or S_2 over a larger range of σ_w

Strehl ratio for small and large aberrations

• For a **small** aberration, an optimally balanced aberration with minimum variance yields the maximum Strehl ratio.

$$A_s\left(\rho^4-\rho^2\right) \qquad A_c\left(\rho^3-2\rho/3\right)\cos\theta \qquad A_a\rho^2\left(\cos^2\theta-1/2\right)$$

balanced spherical   balanced coma   balanced astigmatism

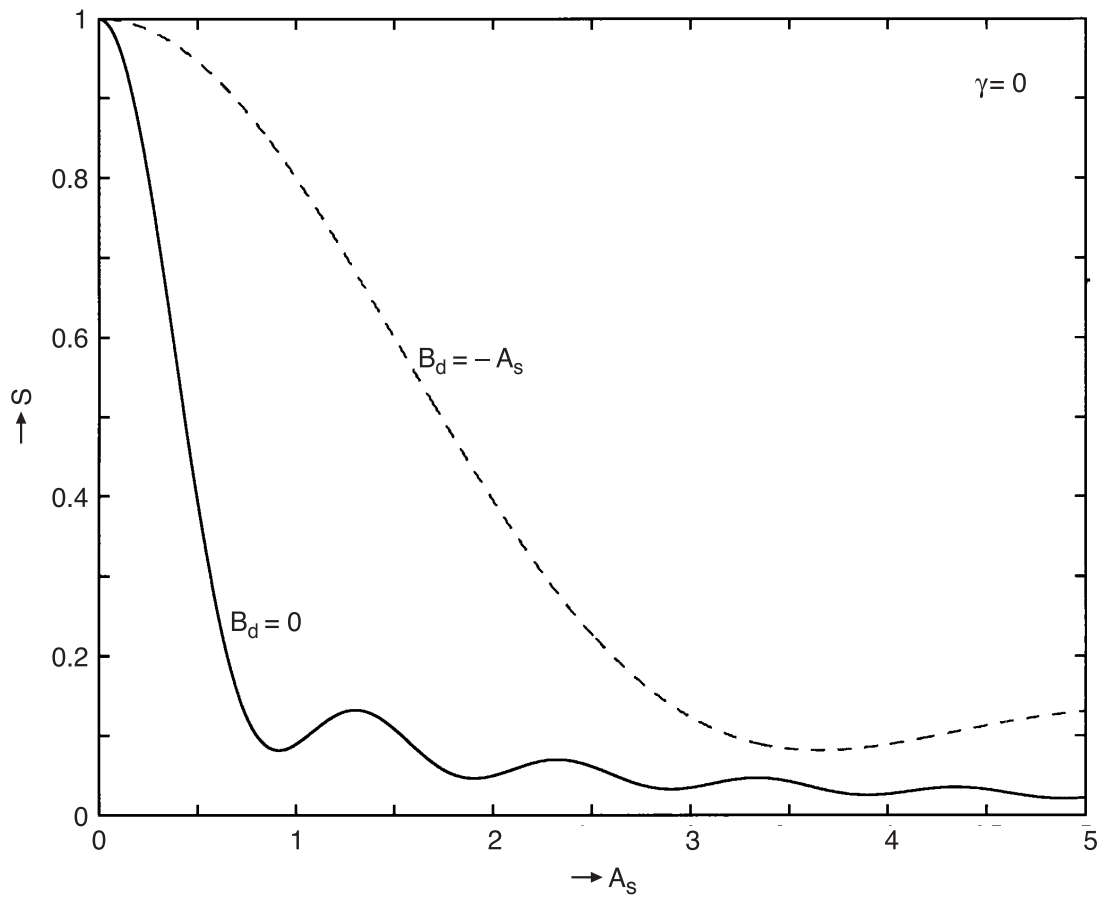
• Maximum Strehl ratio is obtained for

$$A_s \lesssim 2.3\lambda \quad , \qquad A_c \lesssim 0.7\lambda \quad , \qquad A_a < 1\,\lambda$$

• For **large** aberrations, minimum variance does not yield maximum Strehl ratio. Non optimally balanced aberration or one without balancing may yield a higher Strehl ratio.

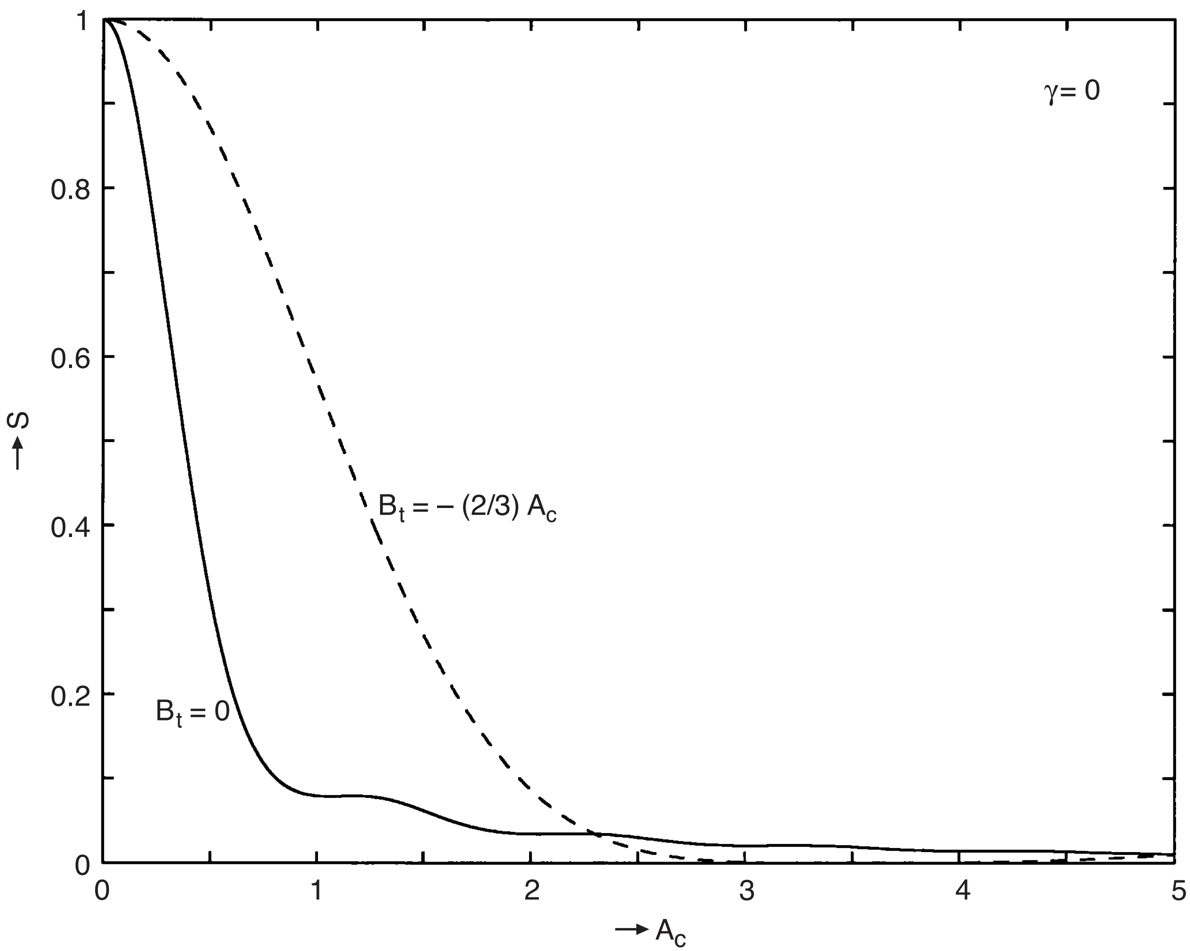
V. N. Mahajan, "Strehl ratio of a Gaussian beam," J. Opt. Soc. Am. **22**, 1824-1833 (2005).

Strehl Ratio for Spherical Aberration



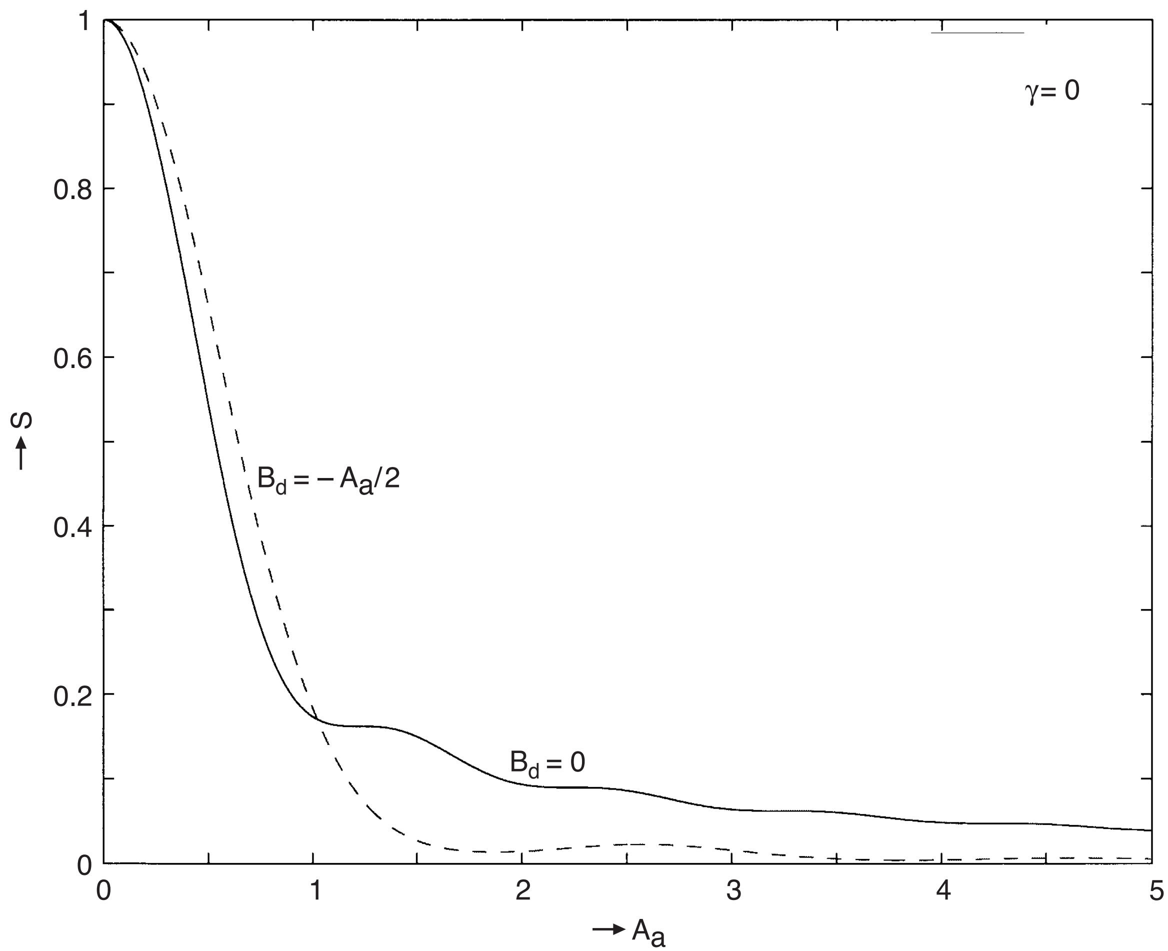
- Balanced spherical aberration yields a much higher Strehl ratio
  - $\gamma = 0$ implies a uniformly illuminated pupil.

Strehl Ratio for Coma



- Seidel coma yields a higher Strehl ratio when $A_c \geq 2.3\,\lambda$

Strehl Ratio for Astigmatism



Balancing of Schwarzschild or Secondary Aberrations

Spherical: ρ^6 , coma: $\rho^5 \cos\theta$, astigmatism: $\rho^4 \cos^2\theta$

Balanced Secondary Aberrations:

• Secondary **spherical** aberration balanced with primary spherical aberration and defocus:

$$\rho^6 - 1.5\rho^4 + 0.6\rho^2$$

• Secondary **coma** balanced with primary coma and tilt:

$$\left(\rho^5 - 1.2\rho^3 + 0.3\rho\right)\cos\theta$$

• Secondary **astigmatism** balanced with primary spherical aberration, primary astigmatism, and defocus:

$$\rho^4 \cos^2\theta - \frac{1}{2}\rho^4 - \frac{3}{4}\rho^2 \cos^2\theta - \frac{3}{8}\rho^2 = \frac{1}{2}\left(\rho^4 - \frac{3}{4}\rho^2\right)\cos 2\theta$$

• **Balanced aberrations lead to Zernike circle polynomials.**

Why Zernike Circle Polynomials in Optics

• Zernike introduced circle polynomials $R_n^m(\rho)\cos m\theta$ in his discussion of the phase-contrast method for testing of mirrors.

• n and m are positive integers including zero, and $n - m \geq 0$ and even.

• $R_n^m(\rho)$: polynomial of degree n in ρ containing terms in ρ^n, $\rho^{n-2}, \cdots$, and ρ^m.

• Nijboer used them to study the diffraction effects of aberrations in systems with circular pupils.

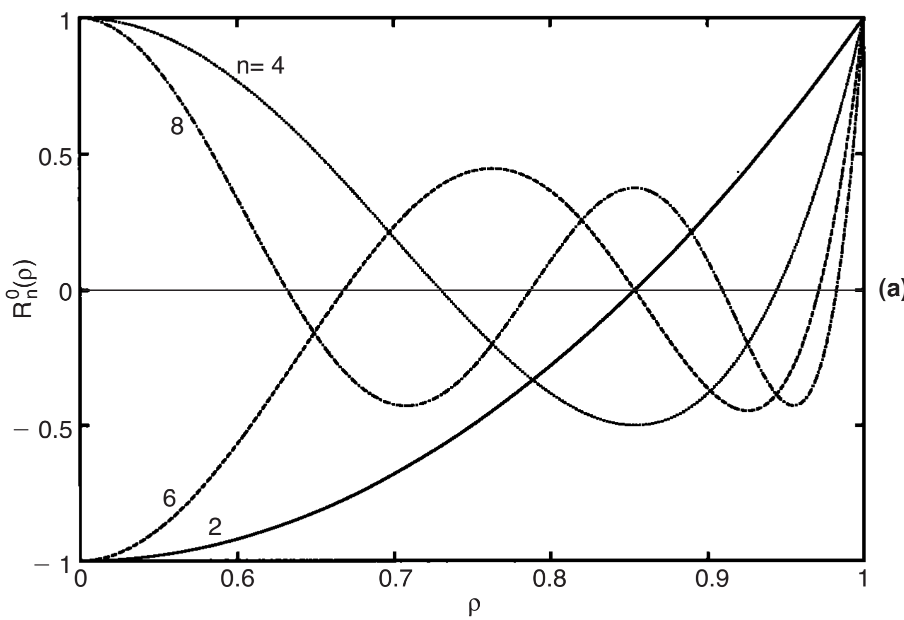
• They are popular in optical design and testing because they are orthogonal over a circular pupil ***and*** represent balanced aberrations.

F. Zernike, "Diffraction theory of the knife-edge test and its improved form, the phase contrast method," *Mon. Notices R. Astron. Soc.* **94**, 377-384 (1934).

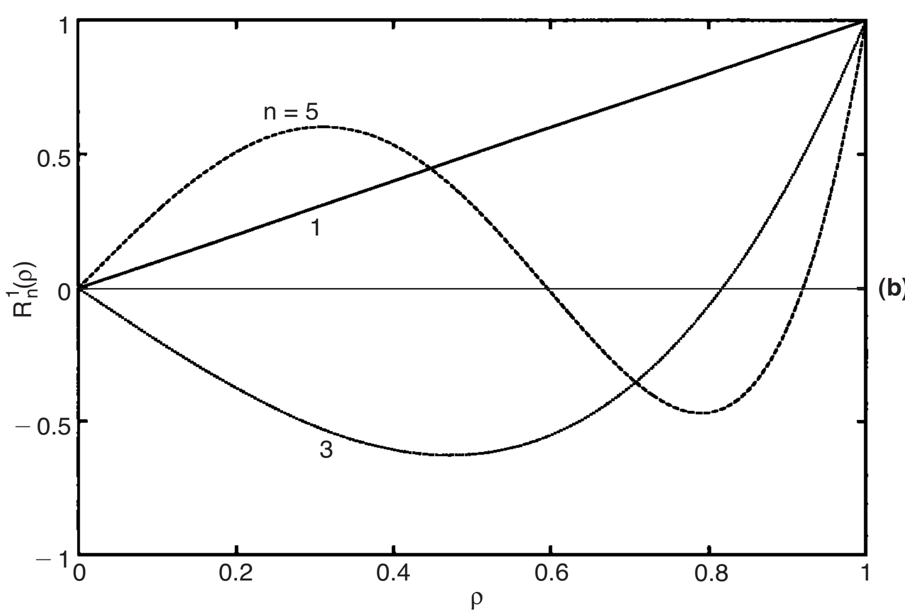
B. R. A. Nijboer, “The diffraction theory of optical aberrations. Part I: General discussion of the geometrical aberrations,” *Physica* **10**, 679–692 (1943); “The diffraction theory of optical aberrations. Part II: Diffraction pattern in the presence of small aberrations,” *Physica* **13**, 605–620 (1947); and K. Nienhuis and B. R. A. Nijboer, “The diffraction theory of optical aberrations. Part III: General formulae for small aberrations: experimental verification of the theoretical results,” *Physica* **14**, 590–608 (1949).

Orthonormal Zernike circle polynomials and balanced aberrations

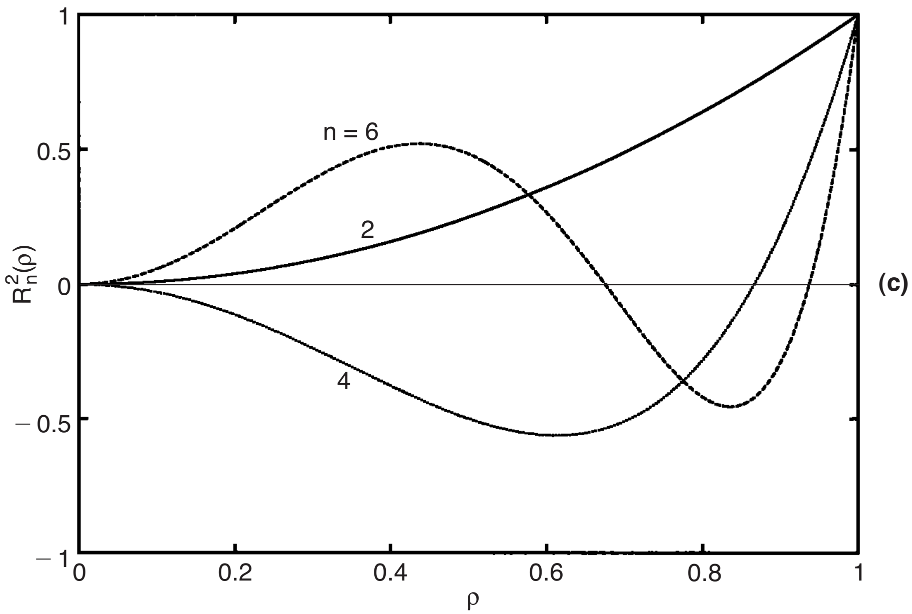
n	m	Orthonormal Zernike Polynomial $Z_n^m(\rho, \theta) = \left[\frac{2(n+1)}{1+\delta_{m0}}\right]^{1/2} R_n^m(\rho)\cos m\theta$	Zernike Aberration Name
2	2	$\sqrt{6}\,\rho^2\cos 2\theta$	Primary astigmatism
3	1	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\theta$	Primary coma
4	0	$\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$	Primary spherical
4	2	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\cos 2\theta$	Secondary astigmatism
5	1	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\cos\theta$	Secondary coma
6	0	$\sqrt{7}\left(20\rho^6 - 30\rho^4 + 12\rho^2 - 1\right)$	Secondary spherical



n= 4
8
6
2
$R_n^0(\rho)$
ρ
(a)
1
0.5
0
− 0.5
− 1
0
0.6
0.7
0.8
0.9
1



n = 5
1
3
$R_n^1(\rho)$
ρ
(b)
1
0.5
0
− 0.5
− 1
0
0.2
0.4
0.6
0.8
1



Variation of a radial circle polynomial $\boldsymbol{R_n^m}(\rho)$ with ρ. (a) Defocus and spherical aberrations. (b) Tilt and coma. (c) Astigmatism.

- Polynomial with even value of n has a value of zero at $n/2$ values of ρ.

- Polynomial with odd value of n has a value of zero at $(n+1)/2$ values of ρ.

- Larger the value of n of a polynomial, the more oscillatory the polynomial.

Zernike Circle Polynomials

• Expand the aberration function in terms of a complete set of Zernike circle polynomials $R_n^m(\rho)\cos m\theta$ that are orthogonal over a unit circle:

$$\Phi(\rho,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{nm}\left[\frac{2(n+1)}{1+\delta_{m0}}\right]^{1/2} R_n^m(\rho)\cos m\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{nm} Z_n^m(\rho,\theta)$$

• c_{nm} are the expansion coefficients

• n and m are positive integers including zero

• $n - m \geq 0$ and even

$$R_n^m(\rho) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!}{s!\left(\frac{n+m}{2}-s\right)!\left(\frac{n-m}{2}-s\right)!}\rho^{n-2s}$$

is a polynomial of degree n in ρ containing terms in ρ^n, ρ^{n-2}, …, and ρ^m.

• $R_n^m(\rho)$ is even or odd in ρ depending on whether n (or m) is even or odd.

• $R_{2n}^0(\rho) = P_n\left(2\rho^2 - 1\right)$, where $P_n(\cdot)$ is a Legendre polynomial

$$R_n^n(\rho) = \rho^n \quad , \quad R_n^m(1) = 1$$

$$R_n^m(0) = \delta_{m0} \quad \text{for even } n/2$$

$$= -\delta_{m0} \quad \text{for odd } n/2$$

- $\delta_{jj'}$ is a Kronecker delta

Orthogonality relations:

$$\int_0^1 R_n^m(\rho)\, R_{n'}^m(\rho)\rho\, d\rho = \frac{1}{2(n+1)}\delta_{nn'}$$

$$\int_0^{2\pi} \cos m\theta \cos m'\theta\, d\theta = \pi(1+\delta_{m0})\delta_{mm'}$$

$$\frac{1}{\pi}\int_0^1 \int_0^{2\pi} Z_n^m(\rho,\theta) Z_{n'}^{m'}(\rho,\theta)\rho\, d\rho\, d\theta = \delta_{nn'}\delta_{mm'}$$ (Orthonormality relation)

Expansion coefficients:

$$c_{nm} = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \Phi(\rho,\theta) Z_n^m(\rho,\theta)\rho\, d\rho\, d\theta$$

Table 3-7 lists **orthonormal** Zernike polynomials and the names associated with some of them when identified with aberrations.

• Polynomials independent of θ are called **spherical** aberrations

• Polynomials varying as $\cos\theta$ are called **coma** aberrations

• Polynomials varying as $\cos 2\theta$ are called **astigmatism** aberrations.

• Number of Zernike (or orthogonal) polynomials in the expansion of an aberration function through a certain order n is given by

$$N_n = \left(\frac{n}{2}+1\right)^2 \quad \text{for even } n$$

$$= (n+1)(n+3)/4 \quad \text{for odd } n$$

n	N_n
1	2
2	4
3	6
4	9
5	12
6	16

Table 3-7. Orthonormal Zernike circle polynomials and balanced aberrations.

n	m	Orthonormal Zernike Polynomial $Z_n^m(\rho,\theta) = \left[\frac{2(n+1)}{1+\delta_{m0}}\right]^{1/2} R_n^m(\rho)\cos m\theta$	Aberration Name*
0	0	1	Piston
1	**1**	$2\rho\cos\theta$	**Distortion (tilt)**
2	**0**	$\sqrt{3}\left(2\rho^2 - 1\right)$	**Field curvature (defocus)**
2	**2**	$\sqrt{6}\,\rho^2\cos 2\theta$	**Primary astigmatism**
3	**1**	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\theta$	**Primary coma**
3	3	$\sqrt{8}\,\rho^3\cos 3\theta$	
4	**0**	$\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$	**Primary spherical**
4	**2**	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\cos 2\theta$	**Secondary astigmatism**
4	4	$\sqrt{10}\,\rho^4\cos 4\theta$	
5	**1**	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\cos\theta$	**Secondary coma**
5	3	$\sqrt{12}\left(5\rho^5 - 4\rho^3\right)\cos 3\theta$	
5	5	$\sqrt{12}\,\rho^5\cos 5\theta$	
6	**0**	$\sqrt{7}\left(20\rho^6 - 30\rho^4 + 12\rho^2 - 1\right)$	**Secondary spherical**

Aberration function in optical design (only $\cos m\theta$ terms):

$$\Phi(\rho,\theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm} \left[\frac{2(n+1)}{1+\delta_{m0}} \right]^{1/2} R_n^m(\rho) \cos m\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm} Z_n^m(\rho,\theta)$$

Orthonormality: $\displaystyle \frac{1}{\pi} \int_0^1 \int_0^{2\pi} Z_n^m(\rho,\theta) Z_{n'}^{m'}(\rho,\theta) \rho \, d\rho \, d\theta = \delta_{nn'} \delta_{mm'}$

Mean value:

$$\langle \Phi \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \Phi(\rho,\theta) \rho \, d\rho \, d\theta = c_{00}$$

Mean square value:

$$\langle \Phi^2 \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \Phi^2(\rho,\theta) \rho \, d\rho \, d\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm}^2$$

Aberration variance:

$$\sigma_\Phi^2 = \langle \Phi^2 \rangle - \langle \Phi \rangle^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm}^2 - c_{00}^2 = \sum_{n=1}^{\infty} \sum_{m=0}^{n} c_{nm}^2$$

Advantages of Zernike Circle Polynomials

- Contain defocus and tilt wave aberration terms.
- Orthonormal and represent balanced wave aberrations.
- Mean value of an aberration function is the piston coefficient c_{00}.
- Coefficient of an orthonormal polynomial term represents its standard deviation.
- Sum of the squares of the coefficients (excluding c_{00}) yields the aberration variance.
- Value of a Zernike coefficient is independent of the number of polynomials used to represent an aberration function:

$$c_{nm} = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \Phi(\rho,\theta) Z_n^m(\rho,\theta)\, \rho\, d\rho\, d\theta$$

- Number of polynomials through order n (cos and sin terms):

$$N_n = (n+1)(n+2)/2$$

Seidel Coefficients from Zernike Coefficients

• Each primary Seidel aberration can be written in terms of orthonormal Zernike polynomials $Z_n^m(\rho, \theta)$.

• See the Homework problem on spherical aberration.

• Cannot determine distortion from the tilt coefficient c_{11}, or field curvature from the defocus coefficient c_{20}, or lateral spherical aberration $h'^2\rho^4$ from the spherical aberration coefficient c_{40} (where h' is the image height).

• Similarly, Seidel aberrations can be obtained from the Zernike aberrations.

• Since a Seidel term is contained in several Zernike terms, the value of a Seidel coefficient as obtained from the Zernike coefficients depends on the number of Zernike terms used to represent an aberration function.

• For example, Seidel spherical aberration, which varies as ρ^4, is contained in Zernike polynomials $Z_4^0(\rho)$, $Z_4^2(\rho, \theta)$, $Z_4^4(\rho, \theta)$, $Z_6^0(\rho)$, etc.

Zernike Circle Polynomials in Optical Testing

- Because of fabrication and assembly errors, the aberrations in optical testing consist of both cosine and sine polynomials.
- Hence, we consider Zernike polynomials with cos and sin dependence.
- **Three indices** describe Zernike circle polynomials $Z_j(\rho,\theta)$:
  - Polynomial number j
  - Radial degree n
  - Azimuthal frequency m
- **Polynomial ordering:**
  - Even j for a symmetric polynomial varying as $\cos m\theta$
  - Odd j for an antisymmetric polynomial varying as $\sin m\theta$
  - Polynomial with a lower value of n is ordered first
  - For a given value of n, a polynomial with a lower value of m is ordered first

Orthonormal Zernike Circle Polynomials

j	n	m	$Z_j(\rho,\theta)$	Aberration Name
1	0	0	1	**Piston**
2	1	1	$2\rho\cos\theta$	***x* tilt**
3	1	1	$2\rho\sin\theta$	***y* tilt**
4	2	0	$\sqrt{3}\left(2\rho^2 - 1\right)$	**Defocus**
5	2	2	$\sqrt{6}\,\rho^2 \sin 2\theta$	**Primary astigmatism at** $45°$
6	2	2	$\sqrt{6}\,\rho^2 \cos 2\theta$	**Primary astigmatism at** $0°$
7	3	1	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\sin\theta$	**Primary *y* coma**
8	3	1	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\theta$	**Primary *x* coma**
9	3	3	$\sqrt{8}\,\rho^3 \sin 3\theta$	
10	3	3	$\sqrt{8}\,\rho^3 \cos 3\theta$	
11	4	0	$\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$	**Primary spherical**
12	4	2	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\cos 2\theta$	**Secondary astigmatism at** $0°$
13	4	2	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\sin 2\theta$	**Secondary astigmatism at** $45°$
14	4	4	$\sqrt{10}\,\rho^4 \cos 4\theta$	
15	4	4	$\sqrt{10}\,\rho^4 \sin 4\theta$	
16	5	1	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\cos\theta$	**Secondary *x* coma**
17	5	1	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\sin\theta$	**Secondary *y* coma**
18	5	3	$\sqrt{12}\left(5\rho^5 - 4\rho^3\right)\cos 3\theta$	
19	5	3	$\sqrt{12}\left(5\rho^5 - 4\rho^3\right)\sin 3\theta$	
20	5	5	$\sqrt{12}\,\rho^5 \cos 5\theta$	
21	5	5	$\sqrt{12}\,\rho^5 \sin 5\theta$	
22	6	0	$\sqrt{7}\left(20\rho^6 - 30\rho^4 + 12\rho^2 - 1\right)$	**Secondary spherical**

Combination of Zernike Tilts

$$W(\rho,\theta) = a_2 Z_2(\rho,\theta) + a_3 Z_3(\rho,\theta)$$

$$= a_2(2\rho\cos\theta) + a_3(2\rho\sin\theta)$$

$$= \left(a_2^2 + a_3^2\right)^{1/2}\left\{2\rho\cos\left[\theta - \tan^{-1}(a_3/a_2)\right]\right\}$$

- Represents a Zernike tilt of an angle $2\left(a_2^2 + a_3^2\right)^{1/2}$ about an axis that is orthogonal to a line making an angle of $\tan^{-1}(a_3/a_2)$ with the *x* axis.

- Does not give the full tilt, since there are tilt terms in Zernike comas also.

V. N. Mahajan and W. H. Swantner, "Seidel coefficients in optical testing," Asian J. Phys. **15**, 203–209 (2006).

J. C. Wyant and K. Creath, "Basic wavefront aberration theory for optical metrology," *Appl. Opt. and Optical Eng.* **XI**, 1–53 (1992).

Combination of Zernike Comas

$$
\begin{aligned}
W(\rho,\theta) &= a_8 Z_8(\rho,\theta) + a_7 Z_7(\rho,\theta) \\
&= a_8\left[\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\theta\right] + a_7\left[\sqrt{8}\left(3\rho^3 - 2\rho\right)\sin\theta\right] \\
&= \left(a_7^2 + a_8^2\right)^{1/2}\left\{\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\left[\theta - \tan^{-1}(a_7/a_8)\right]\right\}
\end{aligned}
$$

• Zernike coma of magnitude $\left(a_7^2 + a_8^2\right)^{1/2}$ inclined at an angle of $\tan^{-1}(a_7/a_8)$ with x axis.

• Does not give full coma, since there are coma terms in the secondary Zernike comas also.

Combination of Zernike Astigmatisms

$$W(\rho,\theta) = a_6 Z_6(\rho,\theta) + a_5 Z_5(\rho,\theta)$$

$$= a_6\left(\sqrt{6}\rho^2 \cos 2\theta\right) + a_5\left(\sqrt{6}\rho^2 \sin 2\theta\right)$$

$$= \left(a_5^2 + a_6^2\right)^{1/2} \sqrt{6}\rho^2 \cos\left\{2\left[\theta - \frac{1}{2}\tan^{-1}(a_5/a_6)\right]\right\}$$

- Represents Zernike astigmatism of magnitude $\left(a_5^2 + a_6^2\right)^{1/2}$ at an angle of $(1/2)\tan^{-1}(a_5/a_6)$ with the *x* axis.

- Does not give full astigmatism, since there are astigmatism terms in the secondary Zernike astigmatism also.

Spherical Aberration

• Similarly, Seidel spherical aberration cannot be obtained from the primary Zernike spherical aberration, since the former is contained in the secondary and tertiary Zernike spherical aberrations.

$$Z_{11} = \frac{1}{4}\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$$

$$Z_{22} = \sqrt{7}\left(20\rho^6 - 30\rho^4 + 12\rho^2 - 1\right)$$

• All significant Zernike aberration terms must be considered (and not just the primary Zernike terms) to infer the Seidel coefficients.

Dall-Kirkham Telescope

- Primary or Seidel spherical aberration is zero
- Consider axial image with one wave of secondary spherical aberration ρ^6.
- In terms of Zernike circle polynomials:

$$W(\rho) = \rho^6$$

$$= a_{22} Z_{22}(\rho) + a_{11} Z_{11}(\rho) + a_4 Z_4(\rho) + a_1 Z_1(\rho)$$

$$a_{22} = 1/20\sqrt{7},\ a_{11} = 1/4\sqrt{5},\ a_4 = 9/20\sqrt{3},\ \text{and}\ a_1 = 1/4$$

• Seidel spherical aberration inferred from only the primary Zernike aberration

$$a_{11}Z_{11}(\rho) = \frac{1}{4}\left(6\rho^4 - 6\rho^2 + 1\right)$$

is 1.5 waves, which is an erroneous conclusion.

• Expanding the aberration function up to, say, as many as the first 21 terms, will in fact yield an incorrect result that the amount of Seidel spherical aberration is 1.5 waves.

• However, the Seidel spherical aberration will correctly reduce to zero when at least the first 22 terms are included in the expansion.

• Hence, an expansion must be up to as many terms as are necessary so that any additional terms do not change significantly the mean square difference between the function and its estimate.

• Otherwise, the inferred Seidel aberrations will be erroneous.

Systems With Noncircular Pupils

• Centered mirror telescopes, e. g., Hubble, have an **annular** pupil.

• Primary mirrors of large telescopes, such as the Keck, consist of **hexagonal segments**.

• Off-axis pupil of a rotationally symmetric system is **elliptical** as are the pupils of a shearing interferograms.

• High-power laser beams have **square cross-sections**.

• For systems with noncircular pupils, we need polynomials that are orthogonal and represent balanced aberrations yielding minimum variance over the pupil under consideration.

• Such polynomials can be obtained from the circle polynomials by the recursive Gram-Schmidt orthogonalization or the nonrecursive matrix approach.

V. N. Mahajan and G.-m. Dai, "Orthonormal polynomials in wavefront analysis: Analytical solution," J. Opt. Soc. Am. **A24**, 2994-31016 (2007).

Basis Functions

• For orthogonalization, we need a complete set of basis functions to obtain the desired orthogonal polynomials.

• Use of Zernike circle polynomials as the basis functions has the advantage that the orthonormal polynomials thus obtained are easily related to them.

• Moreover, the error encountered in using circle polynomials for the noncircular pupil can be determined.

• Orthognalization process yields different orthogonal polynomials depending on the order in which they are used.

• Hence, we need a logical **ordering of the circle polynomials**.

• We use the same ordering as by Noll.

R. J. Noll, “Zernike polynomials and atmospheric turbulence,” J. Opt. Soc. Am. **66**, 207–211 (1976).

Circle Polynomial Ordering

• **Fabrication errors** consist of both $\cos m\theta$ and $\sin m\theta$ Zernike polynomials:

$$Z_j(\rho,\theta) \equiv Z_n^m(\rho,\theta) = \left[\frac{2(n+1)}{1+\delta_{m0}}\right]^{1/2} R_n^m(\rho)\begin{cases}\cos m\theta \\ \sin m\theta\end{cases}$$

• Index j is a polynomial-ordering number and is a function of radial degree n and azimuthal frequency m.

• Polynomials are ordered such that an even j corresponds to a symmetric polynomial varying as $\cos m\theta$, while an odd j corresponds to an antisymmetric polynomial varying as $\sin m\theta$.

• A polynomial with a lower value of n is ordered first, while for a given value of n, a polynomial with a lower value of m is ordered first.

Orthonormal Zernike *circle* polynomials $Z_j(\rho, \theta)$

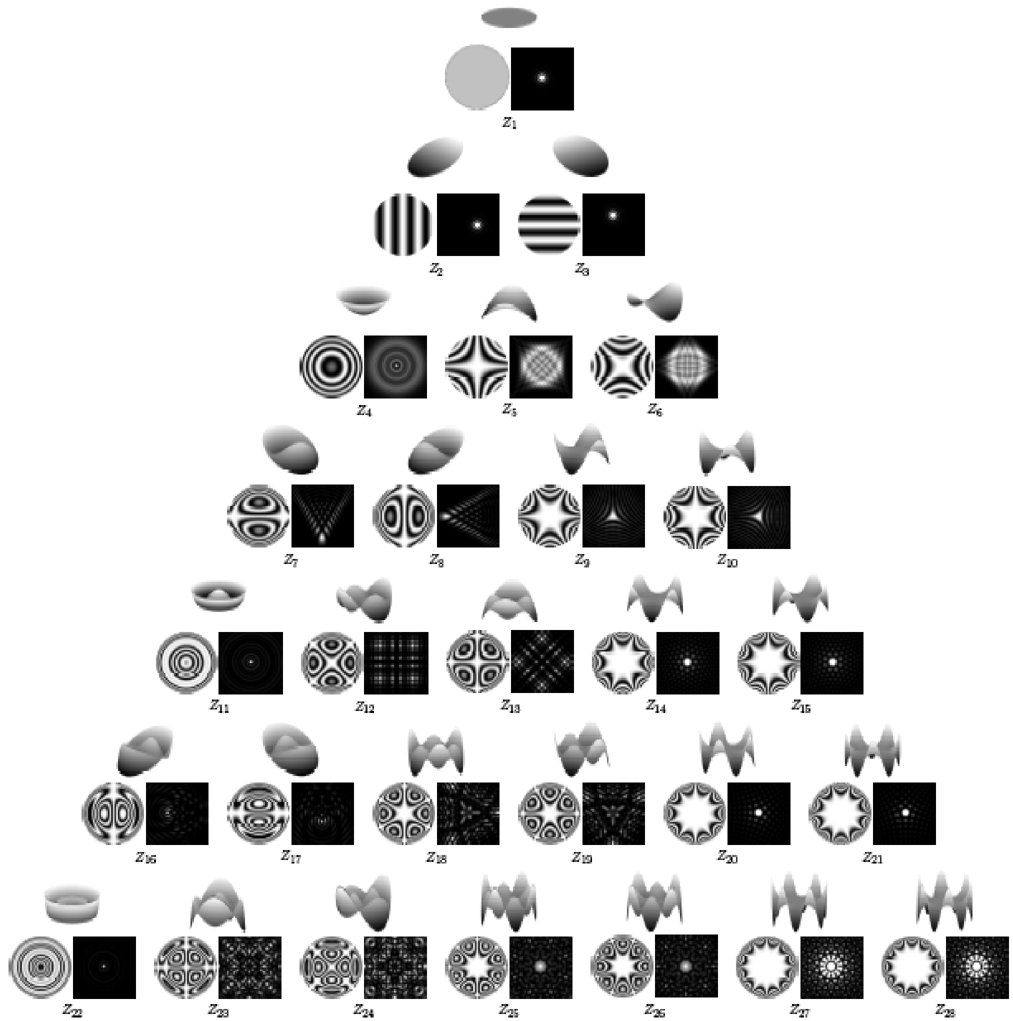
j	*n*	*m*	$Z_j(\rho, \theta)$	**Aberration Name**
1	0	0	1	Piston
2	1	1	$2\rho\cos\theta$	*x* tilt
3	1	1	$2\rho\sin\theta$	*y* tilt
4	2	0	$\sqrt{3}\left(2\rho^2 - 1\right)$	**Defocus**
5	2	2	$\sqrt{6}\,\rho^2 \sin 2\theta$	**Primary astigmatism at** $45°$
6	2	2	$\sqrt{6}\,\rho^2 \cos 2\theta$	**Primary astigmatism at** $0°$
7	3	1	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\sin\theta$	**Primary *y* coma**
8	3	1	$\sqrt{8}\left(3\rho^3 - 2\rho\right)\cos\theta$	**Primary *x* coma**
9	3	3	$\sqrt{8}\,\rho^3 \sin 3\theta$	
10	3	3	$\sqrt{8}\,\rho^3 \cos 3\theta$	
11	4	0	$\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$	**Primary spherical**
12	4	2	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\cos 2\theta$	Secondary astigmatism at $0°$
13	4	2	$\sqrt{10}\left(4\rho^4 - 3\rho^2\right)\sin 2\theta$	Secondary astigmatism at $45°$
14	4	4	$\sqrt{10}\,\rho^4 \cos 4\theta$	
15	4	4	$\sqrt{10}\,\rho^4 \sin 4\theta$	
16	5	1	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\cos\theta$	Secondary *x* coma
17	5	1	$\sqrt{12}\left(10\rho^5 - 12\rho^3 + 3\rho\right)\sin\theta$	Secondary *y* coma
18	5	3	$\sqrt{12}\left(5\rho^5 - 4\rho^3\right)\cos 3\theta$	
19	5	3	$\sqrt{12}\left(5\rho^5 - 4\rho^3\right)\sin 3\theta$	
20	5	5	$\sqrt{12}\,\rho^5 \cos 5\theta$	
21	5	5	$\sqrt{12}\,\rho^5 \sin 5\theta$	
22	6	0	$\sqrt{7}\left(20\rho^6 - 30\rho^4 + 12\rho^2 - 1\right)$	**Secondary spherical**

23	6	2	$\sqrt{14}\left(15\rho^6 - 20\rho^4 + 6\rho^2\right)\sin 2\theta$	Tertiary astigmatism at $45°$
24	6	2	$\sqrt{14}\left(15\rho^6 - 20\rho^4 + 6\rho^2\right)\cos 2\theta$	Tertiary astigmatism at $0°$
25	6	4	$\sqrt{14}\left(6\rho^6 - 5\rho^4\right)\sin 4\theta$	
26	6	4	$\sqrt{14}\left(6\rho^6 - 5\rho^4\right)\cos 4\theta$	
27	6	6	$\sqrt{14}\,\rho^6 \sin 6\theta$	
28	6	6	$\sqrt{14}\,\rho^6 \cos 6\theta$	
29	7	1	$4\left(35\rho^7 - 60\rho^5 + 30\rho^3 - 4\rho\right)\sin\theta$	Tertiary *y* coma
30	7	1	$4\left(35\rho^7 - 60\rho^5 + 30\rho^3 - 4\rho\right)\cos\theta$	Tertiary *x* coma
31	7	3	$4\left(21\rho^7 - 30\rho^5 + 10\rho^3\right)\sin 3\theta$	
32	7	3	$4\left(21\rho^7 - 30\rho^5 + 10\rho^3\right)\cos 3\theta$	
33	7	5	$4\left(7\rho^7 - 6\rho^5\right)\sin 5\theta$	
34	7	5	$4\left(7\rho^7 - 6\rho^5\right)\cos 5\theta$	
35	7	7	$4\rho^7 \sin 7\theta$	
36	7	7	$4\rho^7 \cos 7\theta$	
37	8	0	$3\left(70\rho^8 - 140\rho^6 + 90\rho^4 - 20\rho^2 + 1\right)$	**Tertiary spherical**

Defocus: $j = 4$, Astigmatism: $j = 5, 6$; coma: $j = 7, 8$; and spherical: $j = 11$

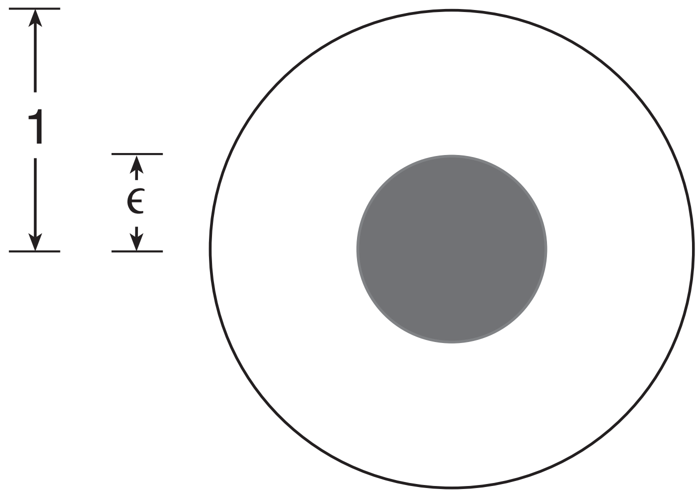
V. N. Mahajan, “Zernike polynomials and aberration balancing,” in Current Developments in Lens Design and Optical Engineering, 1-16, SPIE Proc. 5173 (2003).

V. N. Mahajan, “Zernike polynomials and wavefront fitting,” in *Optical Shop Testing*, D. Malacara, ed., 3rd edition (Wiley, 2007) pp. 498-546.



Zernike circle polynomials pyramid showing isometric plot on the top, interferogram on the left, and PSF on the right for each polynomial. The standard deviation of each polynomial aberration is one wave.

Annular Pupil: Obscuration ratio ϵ



Unit annulus

Obscuration ratio ϵ of some well-known telescopes:

0.36 for the 200-inch telescope at Mount Palomar

0.37 for the 84-inch telescope at the Kitt-Peak observatory

0.5 for the telescope at the McDonald Observatory

0.33 for the 2.4 m Hubble Space Telescope.

Balanced Primary Aberrations and Diffraction Focus

Balanced Aberration	$\Phi(\rho,\theta;\varepsilon)$	Diffraction Focus
Spherical	$A_s\left[\rho^4-\left(1+\epsilon^2\right)\rho^2\right]$	$\left[0,0,8\left(1+\epsilon^2\right)F^2A_s\right]$
Coma	$A_c\left(\rho^3-\dfrac{2}{3}\dfrac{1+\epsilon^2+\epsilon^4}{1+\epsilon^2}\rho\right)\cos\theta$	$\left[\dfrac{4\left(1+\epsilon^2+\epsilon^4\right)}{3\left(1+\epsilon^2\right)}FA_c,0,0\right]$
Astigmatism	$A_a\rho^2\left(\cos^2\theta-1/2\right)$ $=(A_a/2)\rho^2\cos 2\theta$	$\left(0,0,4F^2A_a\right)$

• **Balancing defocus for astigmatism $A_a\rho^2\cos^2\theta$ is independent of ϵ.**

V. N. Mahajan, "Zernike annular polynomials for imaging systems with annular pupils," J. Opt. Soc. Am. **71**, 75–85 (1981), **71**, 1408 (1981), **A1**, 685 (1984).

V. N. Mahajan, "Zernike annular polynomials and optical aberrations of systems with annular pupils," Appl. Opt. **33**, 8125–8127 (1994).

Orthonormal Zernike Annular Polynomials

n	m	$Z_n^m(\rho,\theta;\epsilon)$
2	2	$\sqrt{6}\left[\rho^2\Big/\left(1+\epsilon^2+\epsilon^4\right)^{1/2}\right]\cos 2\theta$
3	1	$\sqrt{8}\,\dfrac{3\left(1+\epsilon^2\right)\rho^3-2\left(1+\epsilon^2+\epsilon^4\right)\rho}{\left(1-\epsilon^2\right)\left[\left(1+\epsilon^2\right)\left(1+4\epsilon^2+\epsilon^4\right)\right]^{1/2}}\cos\theta$
4	0	$\sqrt{5}\left[6\rho^4-6\left(1+\epsilon^2\right)\rho^2+1+4\epsilon^2+\epsilon^4\right]\Big/\left(1-\epsilon^2\right)^2$

Orthonormality: $$\frac{1}{\pi\left(1-\epsilon^2\right)}\int_{\epsilon}^{1}\int_{0}^{2\pi} Z_n^m(\rho,\theta;\epsilon)Z_{n'}^{m'}(\rho,\theta;\epsilon)\rho\,d\rho\,d\theta = \delta_{nn'}\,\delta_{mm'}$$

Table 3-7. Radial Zernike *annular* polynomials.

n	m	$R_n^m(\rho;\epsilon)$
0	0	1
1	1	$\rho \Big/ \left(1+\epsilon^2\right)^{1/2}$
2	0	$\left(2\rho^2 - 1 - \epsilon^2\right) \Big/ \left(1-\epsilon^2\right)$
2	2	$\rho^2 \Big/ \left(1+\epsilon^2+\epsilon^4\right)^{1/2}$
3	1	$\dfrac{3\left(1+\epsilon^2\right)\rho^3 - 2\left(1+\epsilon^2+\epsilon^4\right)\rho}{\left(1-\epsilon^2\right)\left[\left(1+\epsilon^2\right)\left(1+4\epsilon^2+\epsilon^4\right)\right]^{1/2}}$
3	3	$\rho^3 \Big/ \left(1+\epsilon^2+\epsilon^4+\epsilon^6\right)^{1/2}$
4	0	$\left[6\rho^4 - 6\left(1+\epsilon^2\right)\rho^2 + 1 + 4\epsilon^2 + \epsilon^4\right] \Big/ \left(1-\epsilon^2\right)^2$

4	2	$\dfrac{4\rho^4 - 3\left[\left(1-\epsilon^8\right)\big/\left(1-\epsilon^6\right)\right]\rho^2}{\left\{\left(1-\epsilon^2\right)^{-1}\left[16\left(1-\epsilon^{10}\right)-15\left(1-\epsilon^8\right)^2\big/\left(1-\epsilon^6\right)\right]^{1/2}\right\}}$
4	4	$\rho^4\big/\left(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8\right)^{1/2}$
5	1	$\dfrac{10\left(1+4\epsilon^2+\epsilon^4\right)\rho^5-12\left(1+4\epsilon^2+4\epsilon^4+\epsilon^6\right)\rho^3+3\left(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8\right)\rho}{\left(1-\epsilon^2\right)^2\left[\left(1+4\epsilon^2+\epsilon^4\right)\left(1+9\epsilon^2+9\epsilon^4+\epsilon^6\right)\right]^{1/2}}$
5	3	$\dfrac{5\rho^5 - 4\left[\left(1-\epsilon^{10}\right)\big/\left(1-\epsilon^8\right)\right]\rho^3}{\left\{\left(1-\epsilon^2\right)^{-1}\left[25\left(1-\epsilon^{12}\right)-24\left(1-\epsilon^{10}\right)^2\big/\left(1-\epsilon^8\right)\right]\right\}^{1/2}}$
5	5	$\rho^5\big/\left(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}\right)^{1/2}$
6	0	$\left[20\rho^6-30\left(1+\epsilon^2\right)\rho^4+12\left(1+3\epsilon^2+\epsilon^4\right)\rho^2-\left(1+9\epsilon^2+9\epsilon^4+\epsilon^6\right)\right]\big/\left(1-\epsilon^2\right)^3$

Using Circle Polynomials for Noncircular Pupils

• For systems with noncircular pupils, Zernike circle polynomials are neither orthogonal over such pupils nor do they represent balanced aberrations.

• Hence their special utility is lost.

• However, since they form a complete set, an aberration function over a noncircular wavefront can be expanded in terms of them.

• Since an orthonormal polynomial is a linear combination of the Zernike polynomials, each Zernike coefficient is also a linear combination of the orthonormal coefficients.

• Hence, wavefront fitting with a certain number of orthonormal polynomials is identical to fitting with the corresponding Zernike polynomials.

So What is Wrong With Using Circle Polynomials for Noncircular Pupils

• Zernike coefficients are **not** independent of each other, and their values change as the number of polynomials used in the expansion changes.

• Piston term does **not** represent the mean value of the aberration function.

• Sum of the squares of the other Zernike coefficients does **not** yield the aberration variance.

• Zernike coefficients do **not** represent coefficients of balanced aberrations.

• Interferometer setting errors of tip, tilt, and defocus are in error, except when obtained from a 4-polynomial expansion.

• Zernike coefficients are **incorrect** when calculated as for a circular pupil:

$$c_{nm} \neq \frac{1}{\text{pupil area}} \int_{\text{pupil}} \int \Phi(\rho,\theta) Z_n^m(\rho,\theta)\, \rho\, d\rho\, d\theta$$

G.-m. Dai and V. N. Mahajan, "Orthonormal polynomials in wavefront analysis: Error analysis," Appl. Opt. **47**, 3433-3445 (1 July 2008).

Circle and Annular Polynomial Analysis of an Annular Seidel Aberration Function

$$\hat{W}(\rho,\theta;\epsilon) = \sum_{j=1}^{J} a_j A_j(\rho,\theta;\epsilon) = \sum_{j=1}^{J} \hat{b}_j Z_j(\rho,\theta)$$

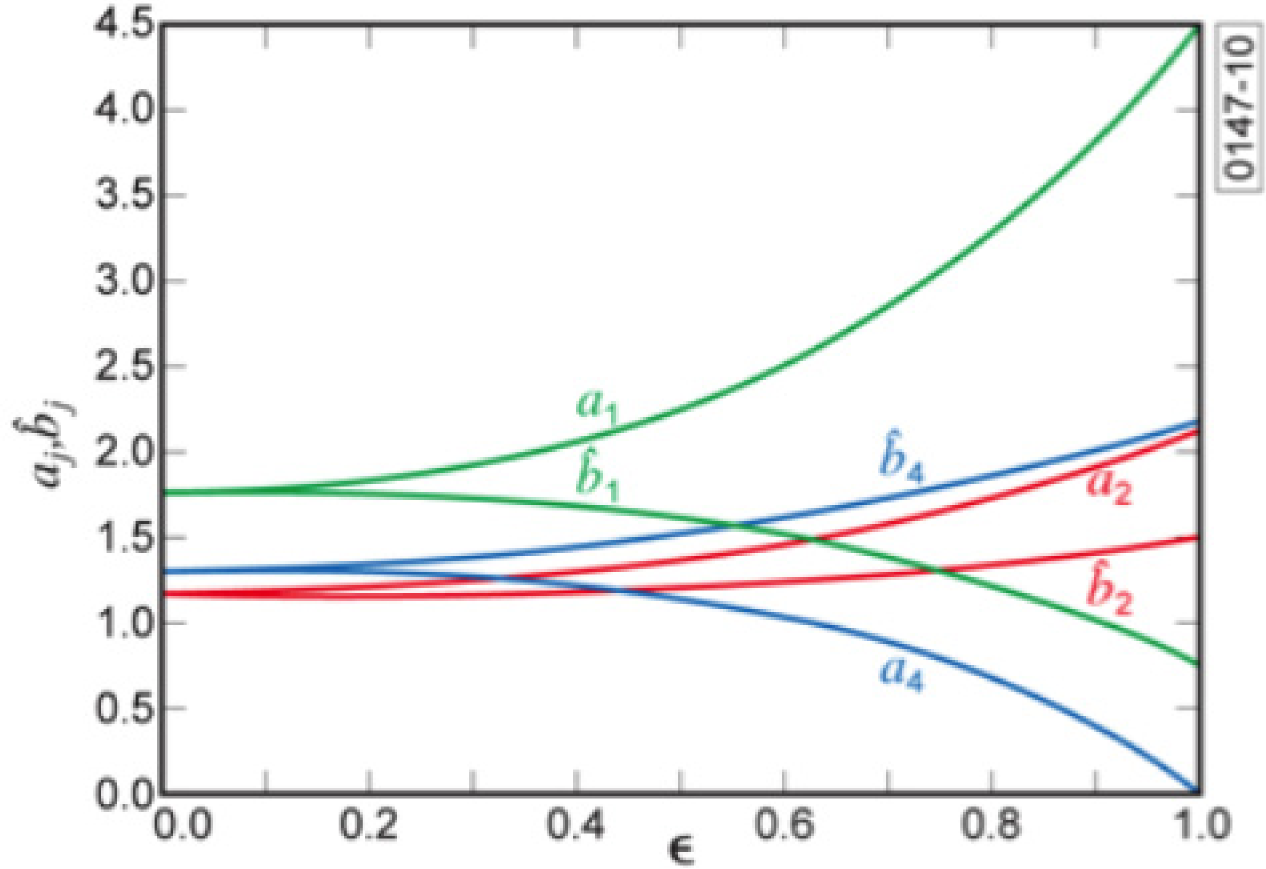
- Wavefront fit with a certain number J of circle polynomials is identically the same as that with the corresponding annular polynomials.

- But the piston circle coefficient does **not** represent the mean value of the aberration function, and the sum of the squares of the other coefficients does **not** yield its variance.

- Interferometer setting errors of tip, tilt, and defocus from a 4-circle-polynomial expansion are the same as those from the annular-polynomial expansion.

- However, if these errors are obtained from, say, an 11-circle-polynomial expansion, and are removed from the aberration function, **wrong polishing** will result by zeroing out the residual aberration function.

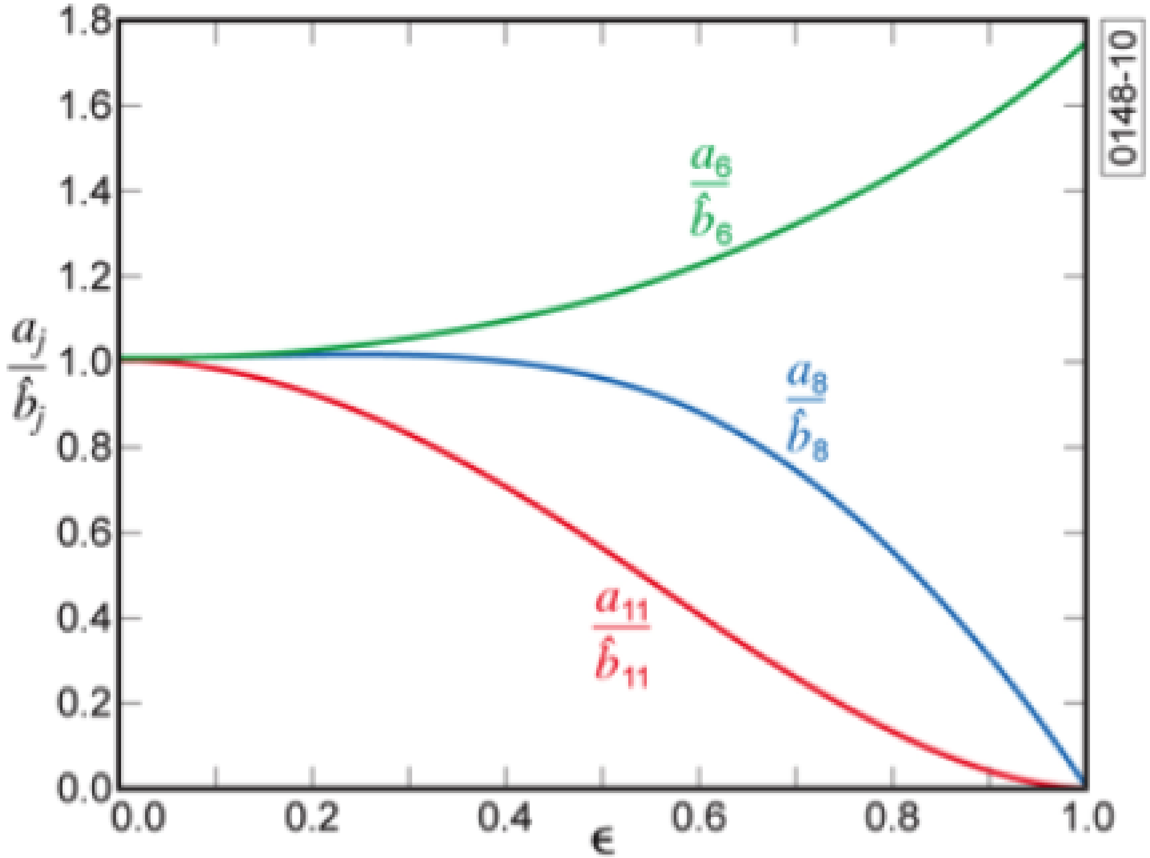
4-Polynomial Expansion Coefficients



- **Reconstructed wavefront is the same with either set of coefficients**

V. N. Mahajan and M. Aftab, "Systematic comparison of the use of annular and Zernike circle polynomials for annular wavefronts," Appl. Opt. **49**, 6489–6501 (2010).

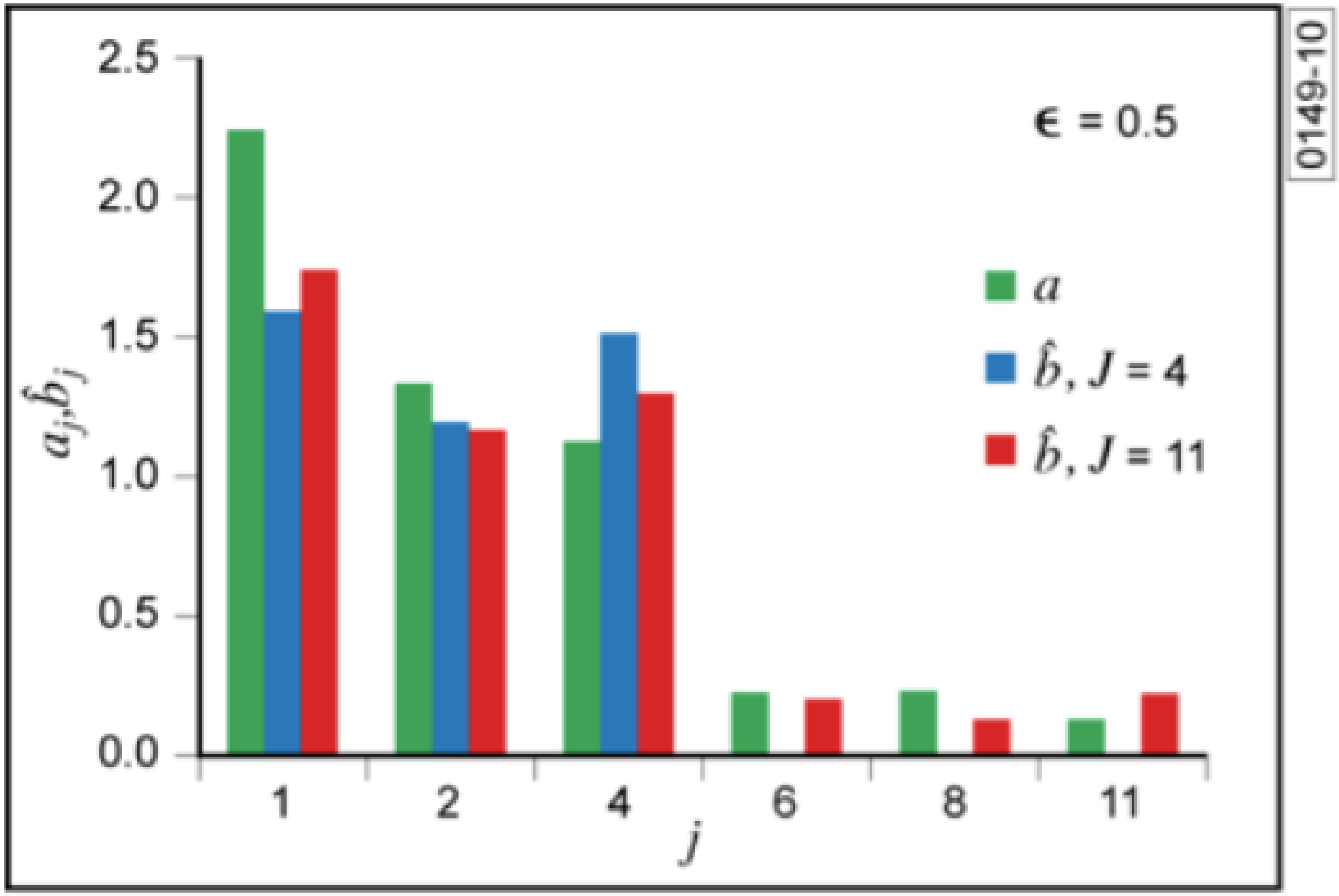
Coefficient Ratio With 11-Polynomial Expansion



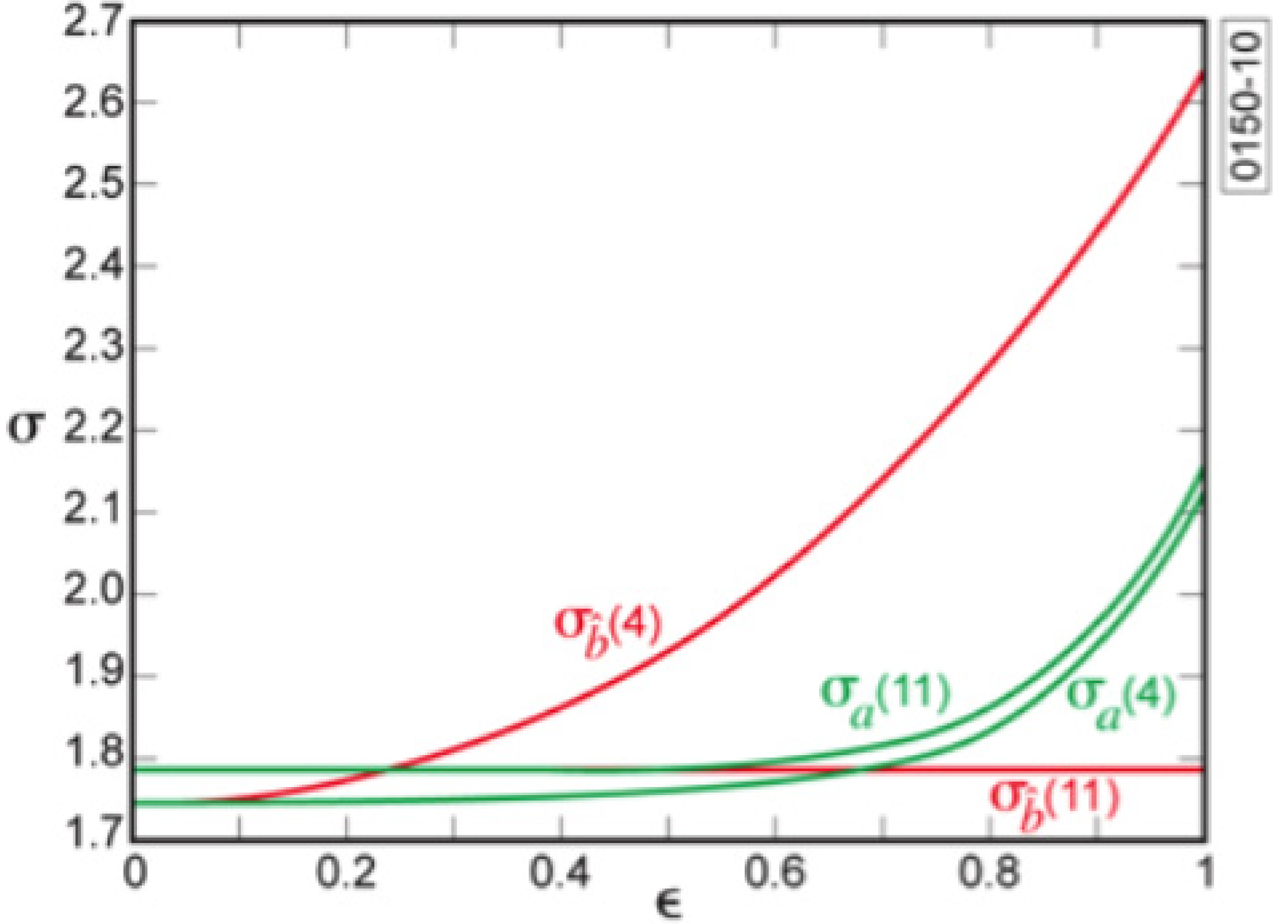
- $a_6/\hat{b}_6$ (astigmatism), $a_8/\hat{b}_8$ (coma) and $a_{11}/\hat{b}_{11}$ (spherical)

- Circle coefficients $\hat{b}_6 = 1/2\sqrt{6}$, $\hat{b}_8 = 1/6\sqrt{2}$, $\hat{b}_{11} = 1/6\sqrt{5}$

Change of Circle Coefficients With a Change of the Number of Polynomials From 4 to 11



Standard Deviation Obtained From a 4- and 11-Polynomial Expansion Coefficients



Home Work

1. Consider a Seidel spherical aberration $A_s\rho^4$ in a system with circular pupil.

(a) Write it in terms of orthonormal Zernike polynomials.

(b) Determine its mean value.

(c) Determine its rms value and standard deviation.

(d) Repeat the problem for a system with an annular pupil of obscuration ratio ϵ by writing the aberration in terms of the orthonormal annular polynomials.

2. Consider a defocused system with aberration $B_d\rho^2$.

(a) Determine the standard deviation of the aberration.

(b) Balance defocus with spherical aberration $A_s\rho^4$ and determine the value of A_s that minimizes aberration variance. Give the value of the standard deviation of the balanced aberration.