Carl Friedrich Gauss Werke

Band V

Herausgegeben von der Königlichen Gesselschaft der Wissenschaften zu Göttingen - 1867

DIOPTRISCHE UNTERSUCHUNGEN

von

Carl Friedrich Gauss

Der Königl. Societät übergeben MDCCCXL Dec. X

Abhandlungen der königlichen Gesellschaft der Wissenschaften zu Göttingen, Band I. Göttingen, MDCCCXLIII

- - - - -

Works of Carl Friedrich Gauss

Volume V

Published by the Royal Society of Sciences in Göttingen-1867

OPTICAL INVESTIGATIONS

by

Carl Friedrich Gauss

Delivered to the Royal Society - Dec. 10th, 1840

Transactions of the Royal Society of Sciences at Göttingen.

Volume I, Göttingen, 1843

Translated by:
Bernard Rosett

Appendix by:
Bernard Rosett

July, 1941

Translator's Note

Despite the fact that Gauss' celebrated "Dioptrische Untersuchungen" was written more than one hundred years ago, the English-speaking world has never enjoyed its complete translation. This essay forms the basis of all modern paraxial or "Gaussian" optics and is a masterpiece of analytical reasoning. It is hoped that this translation will be useful to physicists and optical engineers and, likewise, will serve to grant students an insight into the brilliance that was Gauss'.

In the original German, the language, by modern standards, is archaic, clumsy, and involved. Although the translation has not been free, I have endeavored to remove all ambiguities by the simple expedients of breaking up long sentences and avoiding pronouns of obscure antecedence. I have not hesitated to insert editorial notes wherever they might aid in overcoming any obscurity. In his article, Gauss included no diagrams whatsoever, incredible though that may seem. Such diagrams have been included wherever necessary in the trust that they will clarify the text.

The treatment offered by Gauss is very general, dealing analytically with a coaxial system of refracting surfaces. However, the special case of thick lenses, such as telescope objectives, is given chief attention. The mathematical theory is greatly simplified by special substitutions, to whose choice he was probably led by previous knowledge of the analogies existing between a single lens and a system of lenses, but which otherwise would appear rather arbitrary to the reader. The constructions and formulae based upon Gauss' Cardinal Points are very practical, and appear in most texts on Geometrical Optics. Although the symbolism and conventions employed by Gauss are now obsolete, they have been for the most part retained, for historical and sentimental reasons. The few Greek symbols appearing in the original have been replaced by Latin ones in order to simplify the typography. The reader who is already conversant with modern optical terminology will readily perceive the analogies between Gauss' and the present forms of optical formulae.

Bernard Rosett,

New York, July, 1941

I N D E X

OPTICAL INVESTIGATIONS

In the consideration of the paths taken through lenses by those light rays which are very slightly inclined toward the common axis of those lenses, and of associated phenomena, many very elegant results have been presented. It may seem that the work of Cotes, Euler, Lagrange, and Möbius has left the subject exhausted; still, more remains to be desired. An outstanding fault of the equations formulated by these mathematicians is that the thicknesses of the lenses are therein neglected. Because of this, considerable limitations, as regards accuracy and naturalness, are imposed upon these equations. It is not to be disputed that in many other optical considerations, (namely in those wherein the so-called Aberration due to the spherical form of the lens surfaces is taken into account) the initial neglect of the lens thicknesses becomes useful, and indeed necessary, in order to obtain simpler and more flexible expressions for computations and first approximations. Nevertheless, it would still be desirable to avoid such sacrifices, wherever possible without any important loss in the simplicity of the results.

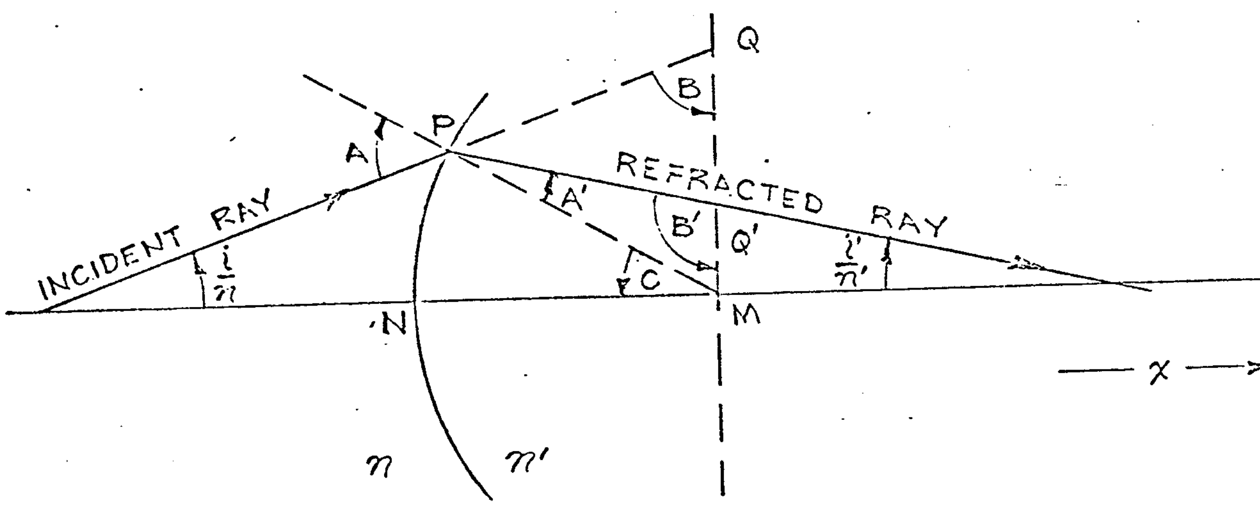
In the primary definitions of optics, we encounter a pernicious deficiency, of mathematical nature. The concepts of Axis and Focal Point of a lens are precisely defined, but this is not true of the Focal Length. Most authors define this as the distance of the focal point of a lens from its center, thus either tacitly assuming or actually advocating that the thickness of the lens be considered infinitely small. Hence, according to these writers, the focal length of an actual lens retains an uncertainty of the order of the thickness of the lens. In more accurate determinations of the focal length, several definitions, not consistent with each other, are employed. Thus, the focal length is defined sometimes as the distance from the rear surface to the focal point, again as the distance from the so-called "optical center" of a lens to its focal point, and other times as the distance from the point midway between the front and rear surfaces to the focal point. Different from all these definitions is the one which uses as a basis the ratio of the size of the image of an infinitely distant object to the apparent, or angular size of that object. This last is indeed the only definition with a meaning.

Therefore, I have not considered it superfluous to devote several pages to these very elementary investigations, with the intent of showing that one may consider the thickness of the lens in the aforementioned formulae. The only limitation we shall permit is that edge rays, which suffer from spherical aberration shall be neglected. Only such rays as are slightly inclined to the axis (Ed. note: "paraxial rays" in modern terminology) are to be retained.

1

The location of all the points appearing in this investigation is determined by means of rectangular coordinates, x, y, and z. It is presupposed that the focal points of all the different refracting surfaces lie on the x-axis; only those rays forming small angles with this axis are to be considered. By an arbitrary convention, the x coordinate is considered as growing positively in the direction of propogation of the light rays.

We consider first the effect of a refraction upon the path of a light ray. Let the relative index of refraction from the first medium to the second be in the ratio of $\frac{n'}{n}$. We let M denote the center of curvature of the interface between the two media; N denotes the point of intersection of this surface with the first coordinate axis (x).



These points may also be represented by the corresponding values of x. This, in the future, will also be done with the other points on the first coordinate axis. Furthermore, let the radius of the interface be $r = M - N$. The value of r will be positive or negative, depending on whether the second medium is on the concave or convex side. Let P denote the point of intersection of the light ray and the refracting surface, and C, the acute angle between MP and the x-axis.

The straight line described by the incident ray is expressed by two equations -:

$$y = \frac{i}{n}(x-N) + b$$

$$z = \frac{j}{n}(x-N) + c$$

In like manner, the equations expressing the straight line described by the refracted rays are -:

$$y = \frac{i'}{n'}(x-N) + b'$$

$$z = \frac{j'}{n'}(x-N) + c'$$

We shall endeavor to derive the relationship of the four values i', j', b', and c' to i, j, b, and c. The point P is determined by

$$x = N + r\,(1-\cos C)$$

and, because both equations hold true for the point P, we have

$$\frac{i}{n}\,r\,(1-\cos C) + b = \frac{i'}{n}\,(r)\,(1-\cos C) + b'$$

Hence, since i, i' and C are infinitely small values, we may put

(1) $$\begin{cases} b' = b \\ \text{and} \\ c' = c \end{cases}$$

A plane through M and perpendicular to the x-axis cuts the path (prolonged, if necessary), of the incident ray at Q, and cuts the path of the refracted ray at Q'. Since PQ' lies in the same plane as PQ and PM, M, Q, and Q' lie along the same straight line. Let us denote by B and B' those angles made by the straight line and PQ and PQ' respectively. Then it becomes obvious that $\frac{MQ}{\sin A} = \frac{r}{\sin B}$ and $\frac{MQ'}{\sin A'} = \frac{r}{\sin B'}$

where A and A' are the angles of incidence and refraction.

Consequently, $\frac{MQ'}{MQ} = \frac{\sin A'}{\sin A} . \frac{\sin B'}{\sin B}$

or $MQ' = MQ \frac{n}{n'} \frac{\sin B}{\sin B'}$.

Since for the point Q

$$y = b + \frac{i}{n} r$$

$$z = c + \frac{j}{n'} r$$

and for the point Q'

$$y = b' + \frac{i'}{n} r$$

$$z = c' + \frac{j'}{n'} r$$

and the last two coordinates are related to the two previous ones in the ratio $\frac{MQ'}{MQ}$, we see that

$$\left(\begin{array}{l} b' + \frac{i'r}{n'} = \frac{n}{n'} \frac{\sin B}{\sin B'} (b + \frac{ir}{n}) \\ \\ c' + \frac{j'r}{n'} = \frac{n}{n'} \frac{\sin B}{\sin B'} (c + \frac{jr}{n}) \end{array} \right.$$

$$\text{and}\quad\begin{cases} i' = \dfrac{nb + ir}{r}\left(\dfrac{\sin B}{\sin B'}\right) - \dfrac{n'b'}{r} \\[2ex] j' = \dfrac{nc + jr}{r}\left(\dfrac{\sin B}{\sin B'}\right) - \dfrac{n'c'}{r} \end{cases}$$

These expressions are strictly correct. Since B and B' differ from a right angle by small values only, it becomes sufficiently accurate to write: -

(2)
$$i' = i - \frac{n' - n}{r}(b) = i + \frac{n' - n}{N - M}(b)$$

$$j' = j - \frac{n' - n}{r}(c) = j + \frac{n' - n}{N - M}(c)$$

These equations, (1 and 2) contain the solution to our problem.

The fact deserves mention that these formulae are also immediately applicable to a beam of reflected light if we merely substitute -n for n'. With the aid of such a method, the following investigations are also very easily extended for the case where, instead of refractions, one or more reflections occur.

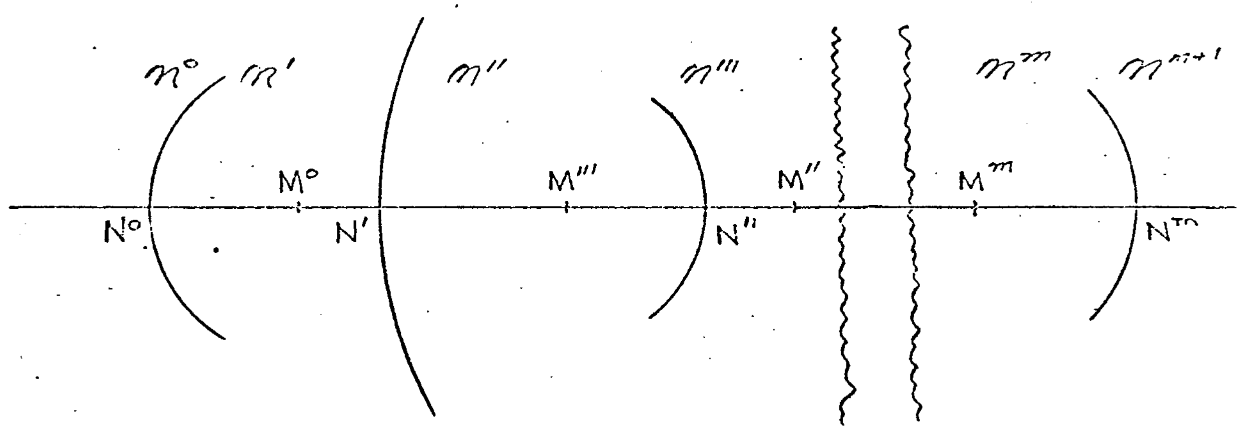
2

For the solution of the general problem of determining the paths of light rays after repeated refractions (m + 1), we shall use the following symbols: -

N^o, N', N'', ... N^m are the points where the x-axis crosses the refracting surfaces.

$M^\circ, N', M'', \ldots M^m$ represents the centers of the refracting surfaces, all lying in the x-axis.

$\frac{n'}{n^\circ}, \frac{n''}{n'}, \frac{n'''}{n''}, \ldots \frac{n^{m+1}}{n^m}$ are the relative indices of refraction upon crossing from the first medium (before N°) into the second (between N° and N'); from the second into the third; and so on.



In the corpuscular theory, the symbols n°, n', n'', etc. are directly proportional # to the velocity of propagation of the light; in the undulatory theory, the symbols n°, n', n'', etc. are inversely proportional to the velocity of propagation of the light waves. If the last medium is the same as the first, n^{m+1} becomes equal to n°.

Let the equations for the paths of the light rays before the first refraction be -:

#(Ed.Note: This is not strictly true. See Newton's "Opticks", Book I, Part I, Exp. 15.)

$$\begin{cases} y = \dfrac{i^\circ}{n^\circ}(x - N^\circ) + b^\circ \\ z = \dfrac{j^\circ}{n^\circ}(x - N^\circ) + c^\circ \end{cases}$$

Similarly the equations for the path of the ray after the first refraction would be :

$$\begin{cases} y = \dfrac{i'}{n'}(x - N^\circ) + b^\circ \\ z = \dfrac{j'}{n'}(x - N^\circ) + c^\circ \end{cases}$$

or, using N' in place of N°, we have

$$\begin{cases} y = \dfrac{i'}{n'}(x - N') + b' \\ z = \dfrac{j'}{n'}(x - N') + c' \end{cases}$$

Then, the equations for the path after the second refraction would be :

$$\begin{cases} y = \dfrac{i''}{n''}(x - N') + b' \\ z = \dfrac{j''}{n''}(x - N') + c' \end{cases}$$

or

$$\begin{cases} y = \dfrac{i''}{n''}(x - N'') + b'' \\ z = \dfrac{j''}{n''}(x - N'') + c'' \end{cases}, \text{ etc.}$$

Therefore, if we take the last terms in the series of i, j, n, N, b, c, namely i^{m+1}, j^{m+1}, n^{m+1}, N^m, b^m, c^m, and denote them by means of an asterisk for a superscript, in order to make them recognizable as such, we have i^*, j^*, N^*, b^*, and c^* symbolizing the last terms of the series. The equations for the last path of the light ray, after the last refraction, then become :

$$\left(\begin{array}{l} y = \frac{i^*}{n^*}(x - N^*) + b^* \\ \\ z = \frac{j^*}{n^*}(x - N^*) + c^* \end{array}\right.$$

Finally, if we abbreviate by setting

$$(3) \quad \left(\begin{array}{lll} \frac{N' - N^\circ}{n'} = t' & \frac{N'' - N'}{n''} = t'' & \frac{N''' - N''}{n'''} = t''' \\ \\ \frac{n' - n^\circ}{N^\circ - M^\circ} = u^\circ & \frac{n'' - n'}{N' - M'} = u' & \frac{n''' - n''}{N'' - M''} = u'' \end{array}\right.$$

.... etc.

then, according to the proposed convention for the superscripts of the last terms in the series of refractions, we may write

$$t^m = t^* \qquad \text{and} \qquad u^m = u^*$$

Therefore, as a result of the previous statements,

$$\left(\begin{array}{l} i' = i^\circ + u^\circ b^\circ \\ \\ b' = b^\circ + t'i' \end{array}\right.$$

$$\left(\begin{array}{l} i'' = i' + u'b' \\ \\ b'' = b' + t''i'' \end{array}\right.$$

$$\left(\begin{array}{l} i''' = i'' + u''b'' \\ \\ b''' = b'' + t'''i''' \end{array}\right.$$

and so forth; whence it follows logically that b^* and i^* are determined linearly by b° and i°. Thus, if we substitute

$$(4)\qquad \begin{cases} b^* = gb^\circ + hi^\circ \\ i^* = kb^\circ + li^\circ \end{cases}$$

in the symbolism introduced by Euler# (Comment. Nov. Acad. Petropol. Vol. IX), we get-:

$$(5)\qquad \begin{cases} g = (u^\circ,\ t',\ u',\ t'',\ u'' \ldots t^*) \\ h = (t',\ u',\ t'',\ u'', \ldots t^*) \\ k = (u^\circ,\ t',\ u',\ t'',\ u'' \ldots u^*) \\ l = (t',\ u',\ t'',\ u'' \ldots u^*) \end{cases}$$

The meaning of this symbolism becomes evident from the fact that if from a given series of values a, a', a", a''', etc., another series A, A', A", A''', etc. is formed after the following algorithm, or mathematical pattern,-: $A = a$, $A' = a' A + 1$, $A'' = a'' A' + A$, $A''' = a''' A'' + A'$, and so on, we may write: $A = a$, $A' = (a, a')$, $A'' = (a, a', a'')$, $A''' = (a, a', a'', a''')$ etc.

Further, it becomes self-evident that in the equations for the third coordinate, z, the constants for the last path are derived from those of the first (incident) path in a manner analogous to that employed in the equation for y. Hence, we will have:

$$(4)\qquad \begin{cases} c^* = gc^\circ + hj^\circ \\ j^* = kc^\circ + lj^\circ \end{cases}$$

In the equations (3) (5) (4) are contained the complete

See translator's appendix at end of article.

solution to our problem.

Euler has elsewhere developed the most elegant relations of the previously mentioned algorithms, of which only two need be recalled to your mind.

Firstly, it is always true that

$$(a, a', a'' \ldots a^{m})\,(a', a'' \ldots a^{m+1}) - (a, a', a'' \ldots a^{m+1})\,(a', a'' \ldots a^{m}) = \pm 1 \text{ (unity)}$$

where the upper or the lower sign is operative, depending upon the number of all the elements $a, a' \ldots a^{m+1}$ (i.e., the number $m + 2$ respectively odd or even.)

Secondly, it is permissible to reverse the order of the elements. Then $(a, a', a'' \ldots a^{m}) = (a^{m} \ldots a'', a', a)$.

From the application of the first of these equations to the values of g, h, k, and l, it follows that

$$gl - hk = 1 \text{ (unity)}$$

The equations (4) can therefore be presented in the following form:

$$b^{\circ} = lb^{*} - hi^{*}$$

$$i^{\circ} = -kb^{*} + gi^{*}$$

$$c^{\circ} = lc^{*} - hj^{*}$$

$$j^{\circ} = -kc^{*} + gj^{*}$$

4.

Let P be a given point on the straight line (prolonged if necessary), representing the first path of the light ray, and let X, Y, and Z be its coordinates. Therefore, we have

$$n^\circ Y = i^\circ \; (X - N^\circ) + n^\circ b^\circ$$

or, if we substitute for i° and b° those expressions at the end of the previous article,

$$n^\circ Y = (gi* - kb*) \; (X - N^\circ) - n^\circ \; (hi* - lb*)$$

Consequently,

$$b* = \frac{n^\circ Y + \left(n^\circ h - g \; (X - N^\circ)\right) \; i*}{n^\circ l - k \; (X - N^\circ)}$$

If we substitute this value in the first equation for the path of the light ray after the last refraction, namely in

$$y = \frac{i*}{n*} \; (x - N*) + b*,$$

and if we abbreviate as follows:

$$N* - \frac{\left(n^\circ h - g \; (X - N^\circ)\right) n*}{n^\circ l - k \; (X - N^\circ)} = X*$$

$$\frac{n^\circ Y}{n^\circ l - k \; (X - N^\circ)} = Y*$$

then this equation becomes

$$y = Y* + \frac{i*}{n*} \; (x - X*)$$

In similar manner, if we also write

$$\frac{n^\circ Z}{n^\circ l - k \; (X - N^\circ)} = Z*$$

for the second equation for the path of the light ray after the last refraction (page 9), this becomes,

$$z = Z^* + \frac{j^*}{n^*} \; (r - Y^*).$$

The point P*, whose coordinates are X*, Y*, and Z*, therefore lies on the straight line (produced backwards if necessary), representing this last path. Also, it is obvious that the coordinates of p* are independent of i°, b°, j°, c°, and that these coordinates, X*, Y* and Z*, are the same for all incident rays which pass through the point P. We can consider the point P as an object and P* as its image; the object can be a real one only if P lies in the first medium, or $(X - N^\circ)$ is negative, and also the image is a real one only if p* lies in the last medium, or $(X^* - N^*)$ is positive. In other cases, the object or image is virtual.

The points P and P* lie in the same plane as the x-axis and at distances therefrom which are in the ratio of unity to $\frac{n^\circ}{n^\circ I - k\,(X - N^\circ)}$. A positive or negative sign of this ratio indicates the position of these points respectively on the same or on opposite sides of the x-axis. We can regard a system of points in one plane normal to the x-axis as a composite object whose composite image also lies in a plane normal to this axis, and is similar to the object so that the linear relationships of the parts (Ed. note: or the lateral magnification) can be expressed by:

$$\frac{n^\circ}{n^\circ I - k\,(X - N^\circ)} = g + \frac{k}{n^*} \; (X^* - N^*).$$

The sign differentiates between erect or inverted image.

5.

The material thus far developed contains the entire theory of the deviations induced in the paths of light rays by refraction, and also permits extension to the case where one or more reflections are associated. The cases of reflection, however, will not be specifically presented here.

It is not superfluous to bring the results into different form by using as points of reference, instead of the vertices of the first and last surfaces (N° and N^*), two other points Q and Q^*.

The equations for the first path of the light ray would then be:

$$\left(\begin{aligned} y &= \frac{i^\circ}{n^\circ}\,(x - Q) + B \\ z &= \frac{j^\circ}{n^\circ}\,(x - Q) + C \end{aligned} \right. ,$$

and the equations for the last path of the light ray would be:

$$\left(\begin{aligned} y &= \frac{i^*}{n^*}\,(x - Q^*) + B^* \\ z &= \frac{j^*}{n^*}\,(x - Q^*) + C^* \end{aligned} \right.$$

If we set $\frac{N^\circ - Q}{n^\circ} = w$ and $\frac{Q^* - N^*}{n^*} = w^*$, we have:

$$b^\circ = B + wi^\circ \qquad c^\circ = C + wj^\circ$$

$$B^* = b^* + w^*i^* \qquad C^* = c^* + w^*j^*$$

Then, if in connection with equations (4) we substitute

$$G = g + w^*k$$

$$H = h + wg + ww^*k + w^*l$$

$$K = k$$

$$L = l + wk \quad ,$$

it follows that

$$B^* = GB + Hi^\circ \qquad c^* = GC + Hj^\circ$$

$$i^* = KB + Li^\circ \qquad j^* = KC + Lj^\circ$$

The coefficients G, H, K, and L, which in this manner replace the former symbols g, h, k, l, also obey the relation

$$GL - HK = 1 \quad \text{(unity)}$$

6.

The introduction of other reference points is made in order to present a simpler dependency of the position of the emergent ray upon that of the incident ray. For this purpose, two pairs of points, which are to be denoted by E, E* (I) and F, F*, (II) are admirably suited.

The values of the quantities associated with these points easily lend themselves to arrangement in tabular form.

	I (E, E*)	II (F, F*)
w	$\frac{l - 1}{k}$	$\frac{-1}{k}$
w*	$\frac{1 - g}{k}$	$\frac{-g}{k}$
Q	$E = N^\circ - \frac{n^\circ(l-1)}{k}$	$F = N^\circ + \frac{n^\circ l}{k} = E + \frac{n^\circ}{k}$
Q*	$E^* = N^* + \frac{n^*(1-g)}{k}$	$F^* = N^* - \frac{n^* g}{k} = E^* - \frac{n^*}{k}$
G	1	0
H	0	$\frac{-1}{k}$
K	k	k
L	1	0

The result is, therefore, that if the equations of the incident ray are put into the form

$$\begin{cases} y = \dfrac{i^\circ}{n^\circ}(x - E) + B \\ z = \dfrac{j^\circ}{n^\circ}(x - E) + C \end{cases},$$

or into the following form, (where we differentiate the constants from those of the previous form, by using accents),

$$\begin{cases} y = \dfrac{i^\circ}{n^\circ}(x - F) + B' \\ z = \dfrac{j^\circ}{n^\circ}(x - F) + C' \end{cases},$$

the equations for the emerging ray become

$$\begin{cases} y = \dfrac{i^\circ + kB}{n^*}(x - E^*) + B \\ z = \dfrac{j^\circ + kC}{n^*}(x - E^*) + C \end{cases}$$

or

$$\begin{cases} y = \dfrac{kB'}{n^*}(x - F^*) - \dfrac{i^\circ}{k} \\ z = \dfrac{kC'}{n^*}(x - F^*) - \dfrac{j^\circ}{k} \end{cases}$$

- 7. -

Thru the use of points E, E*, the dependency of the last path of the light ray on the first permits itself to be expressed simply, as follows:

The last path has the same position with reference to the point E* as would be found for a ray refracted at a single spherical refracting surface of radius $\frac{n^\circ - n^*}{k}$ at E, separating the first medium from the second. This is applicable to the case where the first and last media are dissimilar. However, if they are similar, or $n^* = n^\circ$, as by refraction thru one or several lenses surrounded by air, then the last path has the same position relative to E* as it would have relative to E if the refracted thru an infinitely thin lens at E of focal length $\frac{-n^\circ}{k}$. In other words, it is permissible to replace the transition of light from the first to the last medium, incurring several refractions, either by a single refraction at one refractory surface or by refraction thru a single lens of infinitesimally small thickness, depending on whether the first and last media are dissimilar or identical. In the first case, the refracting surface is given the radius $\frac{n^\circ - n^*}{k}$, and in the second case, the thin lens is given a focal length of $\frac{-n^\circ}{k}$; in either case, the vertex or the thin lens is assumed to be at E. Then the emergent ray is displaced or shifted the same amount as E* is from E. Incidentally, the sign of the radius of the refracting surface is to be taken as in Article I; the sign of the focal length will be considered later, in Article 9.

Because of the significance of the points E and E*, they seem well to merit special nomenclature. I shall call them the PRINCIPAL POINTS ("Hauptpunkte") of the systems of media, or the lens, or the systems of lenses to which they refer. E may be taken as the First Principal Point, and E* as the Second

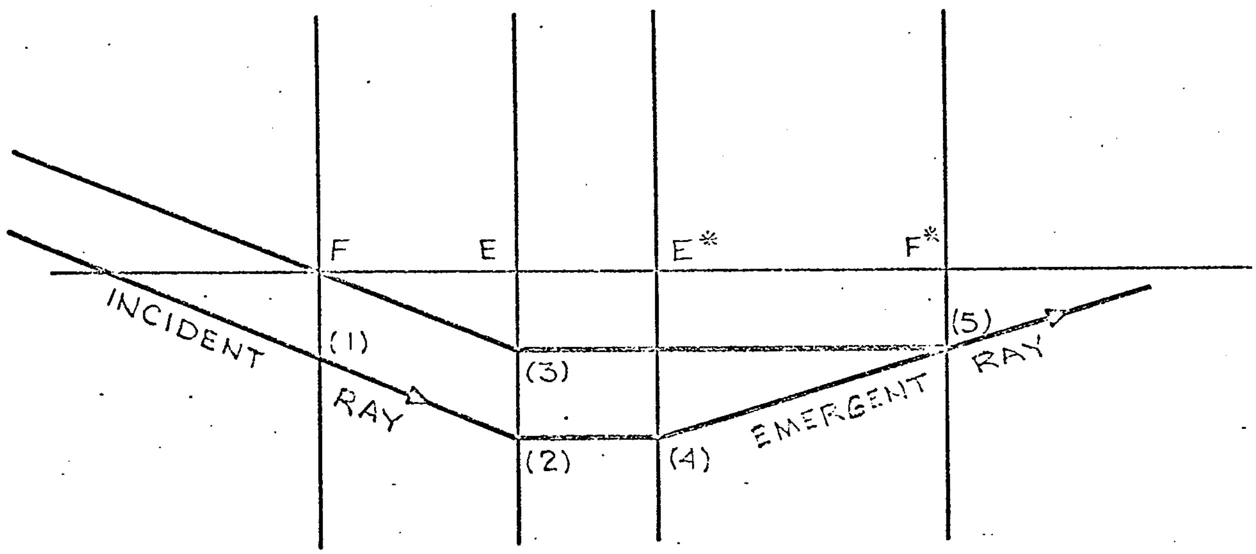
Principal Point. By Principal Planes are to be understood those thru the Principal Points and perpendicular to the x-axis.

8.

Looking back upon the points F and F*, the formulae of Article 6 show that all incident light rays passing thru the point F correspond to emergent ones which are parallel to the optical axis; however, incident rays which are parallel to the axis emerge so as to intersect in the point F*. For rays traveling in the opposite direction, it is merely necessary to interchange the functions of these points. If, therefore, we extend the convention existing for individual lenses, we may call F and F* the Focal Points of the system of media or lenses to which they refer, F representing the First Focal Point, and F* representing the Second Focal Point. The planes, perpendicular to the x-axis and passing thru these points, may be called the Focal Planes. The formulae of Article 6 show that all rays which cross in any extra-axial point of the focal plane correspond to emergent rays which are inclined to the axis, but which are parallel amongst themselves; conversely the same formulae indicate that all rays which are not parallel to the axis correspond to emergent rays which cross the Second Focal Plane in points differing from F*.

9.

With the aid of these four planes, we arrive at a very simple construction for the position of the emergent ray.



The construction is performed as follows:

Let the incident ray cut the First Focal Plane at the point (1), the First Principal Plane at the point (2). Let a line thru F and parallel to (1) (2) meet the First Principal Plane at (3). A line parallel to the axis thru (2) meets the Second Principal Plane at (4). Finally, let a line parallel to the axis, thru (3) meet the Second Focal Plane at (5). Then the position of the emerging ray will be given by (4) (5) or (5) (4).

The values of the coordinates are given in this little table:

Points	Coordinates x	y	z
F	F	o	o
(1)	F	B'	C'
(2)	E	B	C
(3)	E	B - B'	C - C'
(4)	E*	B	C
(5)	F*	B - B'	C - C'

It therefore follows from the formulae of Article 6 that the emergent ray passes thru the points (4) and (5). Passage thru point (4) follows immediately, but passage thru point (5) follows because these formulae are true:

$$B - B' = \frac{i^\circ}{n^\sigma} \quad (E - F) = \frac{i^\circ}{k}$$

$$C - C' = \frac{j^\circ}{n^\sigma} \quad (E - F) = \frac{j^\circ}{k}$$

In the most common case, where $n^* = n^\circ$, it can be seen that $F^* - E^* = E - F$. Then, the construction becomes even simpler, because point (3) becomes superfluous; we need only to determine points (1) (2), and (4), and then draw (4) (5) parallel to (1) (E).

If the direction of the incident ray is such as to pass thru E, then the direction of the emergent ray is such as to pass thru E*. In the case where $n^* = n^\circ$, the incident and emergent rays thru E and E* are parallel to each other. It is customary (in simple lenses) to call such a ray a "principal ray".

The distances of the Second Focal Plane from the Second Principal Plane, and the First Principal Plane from the First Focal Plane, or the values $\frac{-n^*}{k}$, $\frac{-n^\circ}{k}$, could be called the "focal lengths" of the system of media. However, it seems more appropriate to limit the use of this nomenclature to the case where the last medium is identical with the first, in which case these distances are equal to each other. In order to conform to the usual convention, we consider the focal length positive if the first principal point is represented by a larger coordinate than the first focal point; thus, the focal length can always be expressed by $\frac{-n^\circ}{k} = \frac{n^*}{k}$

10.

In the formulae derived in Article 4 for the location of the image, it is obviously permissible to replace N°, N^* by other points if we substitute, instead of g, h, k, l, the corresponding G, H, K, L. By choosing the Principal Points, we get the following expressions:

$$X^* = E^* - \frac{n^* (E - X)}{n^\circ + k (E - X)}$$

$$Y^* = \frac{n^\circ y}{n^\circ + k (E - X)}$$

$$Z^* = \frac{n^\circ Z}{n^\circ \pm k (E - X)}$$

The first formula can be given the following form: -

$$\frac{n^*}{X^* - E^*} + \frac{n^\circ}{E - X} = - k \quad .$$

If we select the focal points, we get: -

$$X^* = F^* + \frac{n^\circ n^*}{k^2(F - X)}$$

$$Y^* = \frac{n^\circ Y}{k\ (F - X)}$$

$$Z^* = \frac{n^\circ Z}{k\ (F - X)}$$

Because of frequent usage, the formulae may be put into the forms which they assume if the first and last media are identical, and the focal length is called f.

$$\frac{1}{X^*-E^*} + \frac{1}{E-X} = \frac{1}{f}$$

$$(X^* - F^*)\ (F - X) = f^2$$

$$(Y^* = \frac{f\ Y}{F-X} = \frac{-Y(X^* - F^*)}{f}$$

$$Z^* = \frac{f\ Z}{(F-X)} = \frac{-Z(X^* - N^*)}{f}$$

11.

The four Cardinal Points (or "Assistance Points"), E. E*, F, F* lose their applicability in the special case where k = o, i, e., where those points are considered as infinitely removed from the refracting surfaces. We can in this case apply the above-mentioned general formulae which here take the following form:

If the equations for the incident ray are

$$\begin{cases} y = \dfrac{i^\circ}{n^\circ} (x - N^\circ) + b^\circ \\ z = \dfrac{j^\circ}{n^\circ} (x - N^\circ) + c^\circ \end{cases}$$

then the equations for the emergent ray are

$$y = \frac{li^\circ}{n^\circ} (x - N^*) + gb^\circ + hi^\circ$$

$$z = \frac{lj^\circ}{n^\circ} (x - N^*) + gc^\circ " hj^\circ$$

If we abbreviate by setting $N^* - \frac{hn^*}{l} = N^{**}$, or, which is equivalent (since $gl = 1$), to

$$N^* - ghn^* = N^{**} ,$$

then these formulae become simpler yet, namely:

$$\begin{cases} y = \dfrac{li^\circ}{n^*} (x - N^{**}) + gb^\circ \\ z = \dfrac{li^\circ}{n^*} (x - N^{**}) + gc^\circ \end{cases}$$

For the position of the image of those points whose coordinates are X, Y, Z, we get the coordinates

$$X^* = N^* - ghn^* - \frac{n^*}{n^\circ} g^2 (N^\circ - X)$$

$$= N^{**} - \frac{n^* g^2}{n^\circ} (N - X)$$

$$Y = g Y$$

$$Z = g Z$$

It then becomes clear, firstly, that the point on the x-axis which by our nomenclature is termed N** represents the image of the point N°, and secondly, that the linear magnification of the parts of the composite image is constant, namely in the ratio $\frac{g}{1}$ or $\frac{1}{I}$.

12.

The case considered in the previous article applies to the telescope where the lenses are adjusted for a normal eye sharply viewing an infinitely distant object. The above formulae indicate that the direction of the emergent ray is dependent only upon the direction of the incident ray. Therefore, incident rays parallel to each other correspond to parallel emergent rays, and the tangent of the inclination of the incident rays to the axis is $\frac{1}{I}$ times the tangent of the inclination of the emergent rays to the axis. The number $1 = \frac{1}{g}$ is therefore what is called the magnification of the telescope and its positive or negative sign indicates erect or inverted appearance. If we interchange the functions of incident or emergent rays by facing the ocular glass toward the objects, then the objects being viewed seem diminished in the same ratio. Upon this fact is based that convenient and exact method for determining the Magnifying Power of a telescope, which I reported in 1823 in Volume II of the "Astronomische Nachrichtringen".

Another method of determining the magnification depends upon the comparison of an object with its image with regard to linear relationship. Ramsden's Dynameter is nothing more than a device to measure the diameter of the image at N** of the circular periphery of the objective. One would naturally first have to ascertain whether this image is real or is possibly of a virtual nature. The image, in order to be real, must satisfy the condition that ghn* be negative. In the Galilean telescope, wherein this image is only a virtual one, an exact result could be obtained only with a micrometric microscope, which would also deserve preference in all cases where greater precision is required. Besides, it becomes clear from the previous article that it would be just as well to use an appropriate object at a finite distance from the objective, as long as the distance does not become so great that the image ceases to be real or attainable with the traveling microscope. Finally, it may be mentioned that the point N** is the one which, in the theory of the telescope, is located at the center of the "eye-ring".

13.

In order to apply the general rules of the second article to the case of a simple glass lens, we denote the refractive index at the transition from air into glass by n, the radii of the first and second surfaces by (n - 1) f° and (n - 1) f', and the thickness of the lens by n e. In this case, we have, therefore, the following:

$$n^\circ = 1$$

$$n' = n$$

$$n'' = 1$$

$$t' = e$$

$$u^\circ = \frac{-1}{f^\circ}$$

$$u' = \frac{1}{f'}$$

and consequently,

$$g = 1 + u^\circ t = \frac{f^\circ - e}{f^\circ} \qquad k + u^\circ + u' + t' u^\circ u' = \frac{(f^\circ + f' - e)}{f^\circ f'}$$

$$h = t' = e \qquad l = 1 + u't = \frac{f' - e}{f'}$$

For the resultant focal length f, we have, according to Article 9:-

$$f = \frac{f^\circ f'}{f^\circ + f' - e}$$

For both Principal Points, here designated by E, E' (according to Article 6): -

$$E = N^\circ + \frac{ef^\circ}{f^\circ + f' - e} = N^\circ + \frac{ef}{f'}$$

$$E' = N' - \frac{ef'}{f^\circ + f' - e} = N' - \frac{ef}{f^\circ}$$

And for the two Focal Points, F, F', we have: -

$$F = E - f = N^\circ - \frac{f^\circ (f' - e)}{f^\circ + f' - e}$$

$$F' = E' + f = N' + \frac{f^\circ (f^\circ - e)}{f^\circ + f' - e}$$

For the point of intersection of the path of the principal ray with the axis, we find:

$$x = N^{\circ} + \frac{nof^{\circ}}{f^{\circ}+f'} = N' - \frac{nef'}{f^{\circ}+f'}$$

This point, which is independent of the inclination of the principal ray, is called by some writers the "Optical Center" of the lens, a term scarcely merited by this point (which possesses no unusual properties). This terminology seems to have led to the error of considering the simple relations occurring at an infinitely thin lens as applicable to thick lenses by means of reference to the optical center. This anology we have shown to be invalid, since only by referring the object to the First Principal Plane and the image to the Second Principal Plane can thick lens formulae be made to resemble thin lens equations. Anyway, we cannot speak of the optical center as previously described, in a system of several lenses, as in an achromatic double objective. If it were desirable to retain the term "Optical Center", then I would consider it more appropriate to apply it to the point lying midway between the two principal points (thereby also midway between the two focal points). If a lens is symmetrical, then the former, common definition of "Optical Center" coincides with mine. The midpoint of the two principal points has the useful property of being easily and accurately determinable by reversal of the lens, because it is obviously the point which retains the same position after reversal if the location of the image is to be unchanged for a final object.

It may be mentioned that the distance of the two principal points from each other becomes

$$E' - E = ne - e\,\frac{(f^{o}+f')}{f^{o}+f'-e} = (n-1)\,e \;-\; \frac{e^{2}}{f^{o}+f'-e}$$

Therefore, inasmuch as e is usually very small in comparison to $f^{o}+f'-e$, the separation E'-E is hardly appreciably different from (n-1)e, which is the same as the thickness of the lens (Ed. note: [ne]) multiplied by $\frac{n-1}{n}$.

14.

In Article 2, general formulae were given by means of which the path of the emergent ray is determined from the path of the incident ray. It is possible, for the case of a system of coaxial lenses to use more convenient equations by introducing the focal lengths of the individual lenses and the intervals between the Second Principal Point of one lens and the First Principal Point of the next successive lens, instead of the radii of curvature of the individual refracting surfaces and their mutual separations. The new formulae now become very similar to those of Article 2, but contain only half as many elements. Since their derivation from the previous material is quite simple, it will be sufficient to state them here in convenient form. We call the focal lengths of the individual, successive lenses f^{o}, f', f'', etc.; their Principal Points (differing from the previous notation), we will denote by two sets of symbols: i.e., the successive First Principal Points will be E°, E', E'', etc., and the successive Second Principal Points will be I°, I', I'', etc. To abbreviate, we write:

$$-\frac{1}{f^\circ} = u^\circ \qquad -\frac{1}{f'} = u' \qquad -\frac{1}{f''} = u'' \text{ , etc.}$$

$$E' - I^\circ = t' \qquad E'' - I' = t'' \qquad E''' - I^* = t''' \text{ , etc.}$$

The last terms in these series may be marked with an asterisk (*).

If we write the equations for the incident ray in the form

$$\begin{cases} y = i^\circ (x - E^\circ) + b^\circ \\ z = j^\circ (x - E^\circ) + c^\circ \end{cases}$$

and the equations for the emergent ray in the following form:

$$\begin{cases} y = i^* (x - I^*) + b^* \\ z = j^* (x - I^*) + c^* \end{cases}$$

and if the four values g, h, k, and l, as determined through the formulae given in Article 2, equation (5), are identical with:

$$b^* = gb^\circ + hi^\circ \qquad c^* = gc^\circ + hj^\circ$$

$$i^* = kb^\circ + li^\circ \qquad j^* = kc^\circ + lj^\circ \quad ,$$

then, for the two Principal Points of the Resultant lens system, we have: -

for the First Principal Point, $x = E^\circ - \frac{l - 1}{k}$

for the Second Principal Point, $x = I^{*} + \frac{1-g}{k}$.

In addition, we have, for the two Focal Points of the lens system,

for the First Focal Point, $x = E^{\circ} + \frac{1}{k}$

for the Second Focal Point, $x = I^{*} - \frac{g}{k}$

The focal length itself $= -\frac{1}{k}$

The formulae for the case where the system consists only of two lenses deserve especially to be written here. In this case,

$$g = \frac{f^{\circ}-t'}{f^{\circ}} \qquad h = t' \qquad k = -\frac{(f^{\circ} + f' - t')}{f^{\circ}f'} \qquad l = \frac{f' - t'}{f'}$$

The values of x for the two Principal Points are:

$$E^{\circ} + \frac{t'f^{\circ}}{f^{\circ}+f'-t'} \qquad \text{and} \qquad I' \quad \frac{t'f'}{f^{\circ}+f'-t'}$$

and the focal length $= \frac{f^{\circ}f'}{f^{\circ}+f'-t'}$.

It is apparent that these formulae are entirely analogous to those given in Article 13 for the determination of the principal points and the focal length of a simple lens, and that t' is merely used in the place of e.

The separation of the 2 Principal Points from one another in the case of 2 lenses = $I' - E^{\circ} - \frac{t'(f^{\circ}+f')}{f^{\circ}+f'-t'}$, which can also be written $I^{\circ} - E^{\circ} + I' - E' - \frac{(t')^2}{f^{\circ}+f'-t'}$

If t' is very small, as is always the case with achromatic double lenses as usually constructed, then the last term becomes negligible, and hence the separation of the two principal points for such a doublet is approximately equal to the sum of the two separations in the individual component lenses.

Further, it is self-evident that all those formulae appearing in this article are applicable without change where, instead of simple lenses, systems of lenses are to be united into one whole system.

15.

The optical properties of a simple lens, as well as of a system of several coaxial lenses, depends, as we have shown, upon three elements, which are firstly the refractive index (or the refractive indices if they are different for different lenses), secondly the positions of the vertices of the refracting surfaces, and thirdly the radii of the refracting surfaces. Since these values usually are not immediately known, mention may be made of the method by which, conversely, these three elements may be determined by means of the optical properties. We will denote the various points on the axis as follows;

X an object

X' its image

F and F' the First and Second Focal Points and finally D, a point that is physically in close connection to the lens (or lens system). As always, we will use the same letters to denote the coordinates of the points in question. Furthermore, we set the focal length = f, and the distance of the point D from the focal points p = D - F and q - F' - D. The three values f, p, q can be regarded as the elements of the lens, and for their determination three investigations will always be necessary, since the distances of object and image from the point D must be measured for three different positions. We wish to solve this problem at first in a very general way.

Let the values of D - X and X' - D be denoted in the first investigation by a, b; in the second by a', b'; and in the third by a", b".

The general equation: $(F - X)\ (X' - F') = f^2$

then will give us

$$(a - p)\ (b - q) = f^2$$

$$(a' - p)\ (b' - q) = f^2$$

$$(a'' - p)\ (b'' - q) = f^2$$

whence, by elimination, we got

$$p = a - \frac{(a'-a)\ (a'' - a)\ (b' - b'')}{R}$$

$$q = b - \frac{(b-b')\ (b-b'')\ (a'' - a')}{R}$$

$$f^2 = \frac{(a'-a)\ (a''-a)\ (a''-a')\ (b-b')\ (b-b'')\ (b'-b'')}{R^2}.$$

wherein the abbreviation is used:

$$R = (a''-a)\ (b-b') - (a'-a)\ (b-b'') .$$

R can also be put into the following form:

$$R = (a''-a')\ (b-b') - (a'-a)\ (b'-b'')$$
$$= (a''-a')\ (b-b'') - (a''-a)\ (b'-b'')$$

Then p and q can also be written as follows:

$$p = a' - \frac{(a'-a)\ (a''-a')\ (b-b'')}{R}$$
$$= a'' - \frac{(a''-a)\ (a''-a')\ (b-b')}{R}$$
$$q = b' - \frac{(b-b')\ (b'-b'')\ (a''-a)}{R}$$
$$= b'' - \frac{(b-b'')\ (b'-b'')\ (a'-a)}{R}$$

16.

To the general solution shown in the previous article, a few remarks must be appended.

I. It is presupposed that in the 3 investigations, the object be on one and the same side of the lens. Should it be convenient to employ the lens in reversed position in one of the examinations, then it is to be imagined that the image is the object and vice versa. Thus, this case can be related to the previous.

II. If viewed by itself, the formula for f^2 leaves it uncertain whether f is to be taken as positive or negative. This is decided through the erect or inverted quality of the image, whence $(X'-F')$ and f must have respectively opposite or like signs. It must also not be forgotten that in spite of the generality of the analytical solution, the practical application remains limited to the case of real images. Otherwise, special auxiliaries must be used for the location of virtual images.

III. Since the performances of the successive investigations permit only a limited degree of precision, it is by no means unimportant which combinations are chosen. In general, it is a good rule that if two of the three examinations are made under only slightly differing circumstances, not all three elements will be accurately determinable.

17.

In a simple lens as well as in one composed of two or more placed close together (e.g. as in the usual construction of an achromatic objective), the 2 Principal Points are only a slight distance from each other. If this distance could be considered a known value $E' - E = d$, then two examinations would be sufficient, because the equation

$$p + q = 2f + d$$

would take the place of the third examination. If we combine with this the two others:

$$(a-p)\ (b-q) = f^2$$

$$(a'-p)\ (b'-q) = f^2$$

and eliminate p and q in order to determine f, we get:

$$\frac{(a'+b' - a - b)^2}{(a'-a)\ (b-b')}\ f^2 + 2\ (a + b + a' + b' - 2d)\ f - (a + b' - d)\ (a'+b-d) = 0$$

This quadratic equation changes into a linear one when $a' + b' - a - b = 0$, that is, when both examinations are so arranged that the distance from object to image is identical and the lens assumes two different positions in this space. Let this distance = c. Then $a = c - b$, and $a' = c - b'$. Hence, $4 (c-d) f = (c-d + b'-b) \quad (c-d-b'+b)$

or $$f = \frac{1}{4} (c-d) - \frac{(b'-b)^2}{4 \, (c-d)} .$$

For each prescribed value of c, F-X must satisfy the equation

$$F-X + \frac{f^2}{F-X} = F-X + X' - F' = c - 2f - d,$$

whose two roots,

$$F-X = \frac{1}{2} (c-2f-d) + \frac{1}{2} \left[(c-4f-d) \, (c-d)\right]^{\frac{1}{2}}$$

$$F-X = \frac{1}{2} (c-2f-d) - \frac{1}{2} \left[(c-4f-d) \, (c-d)\right]^{\frac{1}{2}}$$

are real and unequal as long as c is greater than $(4f + d)$, so that there are always two different positions of the lens for a fixed object X whose image is at the position $X + c$. The product of these two values of F-X, i.e., $(a-p) \, (a'-p)$, equals f^2, whence we see that

$$a'p = b - q \qquad \text{and} \qquad b' - q = a - p .$$

Consequently, $$p = \frac{1}{2} \, (2f + c + d - b - b')$$

$$q = \frac{1}{2} \, (2f - c + d + b + b')$$

$$E = D + \frac{1}{2} (b + b' - c) - \frac{1}{2} d$$

$$E' = D + \frac{1}{2} (b + b' - c) + \frac{1}{2} d$$

18

At that position of the lens where F-X = f, then X'-X = 4f+d, or the image is nearest to the object; it recedes from it as soon as the lens is moved to one side or another of this position, but only slowly at first. From this it follows that if a value is picked for c which only slightly exceeds the value of 4f+d, the examination for determination of the two necessary positions of the lens or the values of b and b' permit only a comparatively low degree of precision. This uncertainty greatly affects the determination of E and E'. Because of this, the method cannot be practically applied under such circumstances. However, if we aim to determine the focal length, the accuracy is not diminished, due to the fact that only the square of b' - b appears in the expression for f. Furthermore, the execution of the method is much more convenient in this case, because it is only necessary to measure the value of the motion of translation of the lens, b' - b, in addition to c (i.e., the separate values of b and b' are unnecessary.)

19.

If d is entirely neglected, we may write $f = \frac{1}{4}c - \frac{(b'-b)^2}{4c}$,

according to the method which Bessel described and proposed (in Volume 17 of "Astronomische Nachrichten"), and later applied to the determination of the focal length of the Königsberg Heliometer. The rigid formula shows that if d is neglected, the focal length is found to be too large by a value of

$\frac{1}{4}d + \frac{d\,(b'-b)^2}{4c\,(c-d)}$, where under the given conditions the second term can be considered as negligible. To attain the degree of precision allowed by the method, appropriate results demand a consideration of the value of d. However, there are several difficulties encountered in acquiring an exact knowledge of this value. For a single lens, it will be sufficiently accurate to calculate the approximation for d (given at the end of Article 13), from the measured thickness and the necessarily known index of refraction. For the case of an achromatic doublet, we can also compute d if we can exactly measure the thicknesses of the individual components and ascertain the values mentioned in Article 14. In order to gain a conception of the influence which the neglect of d may have, we shall consider, for example, an objective wherein the crown glass (of refractive index 1.528) is 7 "lines"# thick, and the flint glass (of refractive index 1.618) is 3 "lines" thick. Hence the distance of the two principal points from each other becomes:

Crown lens	2.42 "lines"
Flint Lens	1.15 "
Combined Lens	3.57 "

Therefore, the focal length becomes 0.89 "lines" too large. In an objective of 8 "feet"# focal length, corresponding to the presupposed dimensions, the error would be about $\frac{1}{13}$ of 1%.

(Ed. note: 1 Prussian Foot = 12 inches = 144 lines = .3138535 me

20.

If it is not possible to apply the method outlined in the previous article for the determination of d, or if it seems unsatisfactory, then direct experiment seems to be the most appropriate means of determining it, as in the following:

We determine the location, on the axis of the lens, of the image of a very distant object relative to the point D. Inasmuch as the object distance can be considered as infinitely large, this image falls at F', and the measured distance F'-D immediately gives q. We repeat the experiment by reversing the lens, whence the image falls at F, and its distance from D gives the value of p. For the third experiment, we bring the object (on the side of F) quite close, determine the distance (=X' - X) of the image from this object, and also the distance D - X = a". Setting X'-D = X' -X-a" = b", we have:

$$(p-a'')\quad(q-b'') = f^2$$

$$d = p + q - 2\left[(p-a'')\quad(q-b'')\right]^{\frac{1}{2}}$$

If these three experiments have been performed with great accuracy, then all three elements p, q, f are adequately determined, and no other experiment is needed. However, if we desire to calculate f to a greater degree of precision, all these experiments must be considered a preparation to the method of Article 18, which gives the value of d.

In order more clearly to understand how the accuracy of the so-gotten determination of d depends upon the various factors involved, we set the above formula in the following form: -

$$d = a'' + b'' + \frac{\left[p-a''-(p-a'')^{1/2}(q-b'')^{1/2}\right]^2}{p-a''},$$

where $p-a'' - \left[(p-a'')(q-b'')\right]^{1/2}$ represents the distance of the object from the first principal point, (in the third experiment), and p-a" represents the distance of that object from the first focal point. It then becomes clear that under the circumstances prevailing, the last part of the formula for d becomes very small. Hence, small inaccuracies in the values of p, q, a", b" scarcely change the calculated value of d, and the degree of precision attained in the determination of d depends primarily upon the exactness of measurement of $X'-X=a''+b''$.

21.

In reference to the method outlined in the previous article, a few remarks are worthy of some space:

I. The technique employed in the third experiment becomes inadequate for a virtual image. However, the following method, combining accuracy and convenience, is suggested: A circle is drawn upon a plane surface, having a diameter as large as, or slightly larger than the protruding edge of the mounting of the lens. The center of this circle is marked by two fine cross-lines. The lens is placed on the plane surface so that the periphery of the mounting and the drawn circle are concentric. Then, a microscope is placed vertically over it. This microscope must be equipped with cross-hairs, and is assembled and fastened to a solid tripod stand. It is moved in its rack until the sharply focussed image of the cross-lines coincides with the cross-hairs of the microscope. Finally, the lens is removed and the microscope is brought closer to the surface by racking it down until the image of the cross-lines again coincides with the cross-hairs of the microscope. The easily and precisely measured value of this racking motion is the distance of the object (the cross-lines) from its image as seen thru the glass. This distance is equal to X'-X. The point on the axis of the lens which lies in the plane of the protruding edge of the mounting which touches the plane surface can be considered as the fixed point D, in which case

$$a'' = o, \quad b'' = X'-X.$$

If a different point D were chosen, this could be located easily with reference to the other point just mentioned, in order to find a".

II. If in the first and second experiments the object distance is not great enough for the image to be considered as coinciding with the focal point, then an approximation is necessary. This is obtained by dividing the square of the focal length by the object distance and subtracting this value from the value of the distance of the image from D, in order to obtain the values of q and p. Obviously, only an approximate knowledge of the focal length and the object distance is necessary, inasmuch as the latter is very great. However, we may just as easily use strictly rigid formulae. For the first experiment, a has the value of D-X, b has the value of X'-D; for the second experiment (where the lens is reversed) the distance D-X = b' and X'-D = a'. In this manner, (which is identical with Article 16, paragraph I), we obtain the same equations as those from which we have proceeded in Article 15, viz.

$$(a-p)\ (b-q) = f^2$$
$$(b'-q)\ (a'-p) = f^2$$
$$(a''-p)\ (b''-q) = f^2 .$$

Therefore, the formulae derived from the solution of these equations retain their value here. If, in the performance of the second experiment, we proceed in such a way that the location of the image in space is in the same plane as in the first experiment (which can easily occur without appreciable change in the object distance), the formulae of Article 15 could be further simplified, because then, a + b = b' + a', which we represent by c. From the second formula for p and the second for q, we would then have:

$$p = a' - \frac{(a'-a'')\ (b-b'')}{c - a'' - b''}$$

$$q = b - \frac{(a'-a'')\ (b-b'')}{c - a'' - b''}$$

III. If the method of Article 20 is not desirable for the complete determination of the elements, we may use the highly accurate method of determining the focal length as given in Article 17. Then, the procedure mentioned in Article 20 remains the most appropriate for the location of the principal points. These are then given by:

$$E = D + \frac{1}{2}(q-p) - \frac{1}{2}d = D + \frac{1}{2}(b-a') - \frac{1}{2}d$$

$$E' = D + \frac{1}{2}(q-p) + \frac{1}{2}d = D + \frac{1}{2}(b-a') + \frac{1}{2}d.$$

22.

For a simple lens, and also, generally speaking, for a system of lenses, a definite value can be allotted to the focal length and to the positions of the principal and focal points, only insofar as we consider light rays of a definite refrangibility (Ed. note: given wave-length); for rays of different refrangibility, these cardinal points will have different positions and the focal length will receive a different value. Hence the heterogeneous light from an object suffers chromatic dispersion by passing through glass. By means of a combination of two or more kinds of glass, this color dispersion can be nullified. For the complete fulfillment of an achromatic objective, it is necessary that parallel beams converge to one point, independent of color. This includes not only such rays as are parallel to the axis, but also those which are inclined to the axis. In other words, the differently colored images of an infinitely distant extended object must not only fall in one plane, but they must be of the same size. The first condition rests upon the identity of the Second Focal Point for differently colored rays; the second is based upon the identity of the focal length. Since the focal length is the distance of the Second Focal Point from the Second Principal Point, we may rephrase the two conditions as equivalent to the statement that both points must be identical for red and for violet rays. If the first condition alone is fulfilled, then the rays inclined to the axis do not give a pure, colorfree image. However, a very slight difference in the focal lengths for different rays may always be deemed harmless. In the theory of achromatic objectives, the first condition only is usually considered. However, in the usual construction of these objectives, where the two lenses either are in contact or are only slightly separated, the position of the principal points is so slightly affected by the variable refrangibility that the second condition is automatically satisfied quite approximately. If it is worth the effort, it is also possible to obtain an exact identity of the Second Principal Point for dissimilarly colored rays.

It is different, however, if the convex crown glass of the lens is separated by a considerable distance from the concave flint glass. It is easy to show that after appropriate arrangements of the separation and focal lengths of the individual lenses so that the Second Focal Point of the system remains identical for differently colored rays, the focal length of this system necessarily becomes greater for violet then for red; the difference is of the same order as that appearing in simple lenses (but in the opposite direction). The same applies also to other constructions (such as occurs in the so-called Dialytical Telescope), where instead of the second lens, a combination of flint and crown either almost or exactly in contact is employed. It will always remain impossible to produce a purely colored image in this manner of an extended object, for the violet image, if it is to lie the same distance from the lens system as the red one, necessarily becomes larger than the latter.

By no means, however, may we conclude that telescopes of the last-named construction must remain more imperfect with reference to achromatism than telescopes with achromatic objectives constructed in the usual manner. We must, on the contrary, admit that the telescope with objectives having imperfect achromatism may still render a purely colored image to the eye by means of a well-calculated arrangement of the ocular.

Indeed, a color-pure image produced by the objective may, because of the color-dispersion produced by the ocular, appear impure to the eye. The so-called color fringe at the periphery can be avoided by means of special construction of the ocular, but the length deviation cannot thus be avoided, and this is aggravated by the fact that the human eye itself is not achromatic. The effect is that the last images, red or violet, have the same apparent size, but not that they simultaneously appear distinct and at equal distances.

The unequal sizes of the first images (red and violet), which are unavoidable with the Dialytic objective, can nevertheless be compensated by appropriate construction of the ocular, so that the colored edge vanishes, as with telescopes of usual construction. However, the second, just mentioned imperfection remains as long as the first red and violet images lie in the same plane.

It should therefore be clear that in order to produce a perfectly achromatic image in the eye, the first image must have a certain length-deviation, depending upon the relations between the ocular and the non-achromatism of the human eye. Theoretically considered, an objective of usual construction can also be calculated so that the prescribed length-deviation takes place, but even if we disregard the difficulty encountered in the technical construction necessary to deal precisely with such small differences, this length-deviation would only fit a particular ocular. In the Dialytic construction, however, the means is provided, due to the adjustability of the second part of the objective, of obtaining the length deviations requisite for every ocular, whereas the ocular can be so constructed that the colored periphery is nullified.

I regret that I must confine myself to this brief dissertation at present and postpone a more detailed development of this interesting subject until another time.

BIBLIOGRAPHY

(1) Wm. B. Carpenter - "The Microscope and its Revelations" Seventh edition (W.H. Dellinger) - 1891 - pp. 108-118. Philadelphia, P. Blakiston, Sons, & Co.

(2) Galileo Ferraris - "Atti R. Accad. Torino" (16), 45, (1880) also Supplement 5 (1881), 500-502.

(3) Moritz von Rohr - "Geometrical Investigation of the Formation of Images in Optical Instruments", London, H.M. Stationery Office, - 1920. pp. 119-121.

(4) S. Rosin and O. Clark - "Combination of Optical Systems" - Journal of the Optical Society of America - 1941, Volume 31, no. 3, pp. 198-201.

(5) J.P.C. Southall - "Principles and Methods of Geometrical Optics (Macmillan 1913), Second edition, pp. 255-262

(6) T. Smith - "Note on Optical Imagery", Proceedings of the Physical Society of London XXV (1912-1913) pp. 239-244.

(7) C.W. Woodworth - Formula appearing in J.P.C. Southall's "Mirrors, Prisms, and Lenses", (Macmillan 1936) Third Edition, page 608.

TRANSLATOR'S APPENDIX

I

At the end of Article 2, in all of Article 3, and in succeeding articles, Gauss makes use of an algorithmic relation introduced in 1762 by the great Swiss mathematician, Leonhard Euler. Euler's mathematical symbolism is basic to Gauss' general treatment of the optical problem, but Gauss' brief exposition of it would fail to clarify the situation to anyone but a professional mathematician. Therefore, I shall try to present an explanation of Euler's Algorithm in order to render the translation more comprehensible and more complete.

The source to which Gauss refers near the end of Article 2 is "Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae," Tom IX, pro annis MDCCLXII et MDCCLXIII, (pp. 53-69), and the title of the essay is "SPECIMEN ALGORITHMI SINGULARIS", AUCTORE L. EULERO. It can be located in large reference libraries under the files of the AKADEMIIA NAUK, PETROGRAD, (LENINGRAD).

II

In dealing with continued fractions of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cdots}}}$$

, Euler discovered several very interesting relations. Let us consider the total values at various positions of the continued fraction. Thus, the value of the continued fraction for the first symbol, a, is simply a. The value of the continued fraction up to the second symbol, b, is $\frac{ab+1}{b}$; the value of the continued fraction up to the third symbol, c, is $\frac{abc+c+a}{bc+1}$; the value of the continued fraction up to the fourth symbol, d, is $\frac{abcd+cd+ad+ab+1}{bcd+d+b}$; and so on. Perusal of this series indicates the following relation: Any given numerator (e.g., abc+c+a) can be formed by multiplying the preceding one (e.g., ab+1) by the last symbol involved (e.g., c) and then adding the value of the numerator just in advance of the preceding one (e.g., a). The identical relation holds true for the successive denominators. If we extend our logical reasoning backwards, we must conclude that the term preceding a must be infinite, and must have the value of $\frac{1}{0}$. We may now place these values together in order to present a clearer picture:

$$\frac{1}{0},\ \frac{a}{1},\ \frac{ab+1}{b},\ \frac{abc+c+a}{bc+1},\ \frac{abcd+cd+ad+ab+1}{bcd+d+b},\ \text{etc.}$$

The reason for the $\frac{1}{0}$ becomes evident when we apply the rule of formation of any given numerator or denominator in its converse sense.

III

In order to simplify working with continued fractions, Euler devised a symbolism to represent the value of the numerator of the continued fraction up to a given symbol. This consisted in enclosing the required symbols, separated by commas, in parentheses. Thus, (a, b, c) represents the value of the numerator abc+c+a. Strictly speaking, we should write

$$(a,b,c) = \left(a+\cfrac{1}{b+\cfrac{1}{c}}\right)\left(b+\frac{1}{c}\right)(c).$$

This is true also in general form:

$$(a,b,\ldots r) = \left(a+\cfrac{1}{b+\cdots+\cfrac{1}{r}}\right)\left(b+\cfrac{1}{c+\cdots+\cfrac{1}{r}}\right)\left(c+\cfrac{1}{d+\cdots+\cfrac{1}{r}}\right)(\ldots)\ (r)\ .$$

We may now repeat the original rule of formation in the new symbolism:

$$\begin{aligned}
(\quad) &= 1\\
(a) &= a\\
(a,b) &= ba+(\ )\\
(a,b,c) &= c(a,b) + (a)\\
(a,b,c,d) &= d(a,b,c) + (a,b)\\
(a,b,c,d,e) &= e(a,b,c,d) + (a,b,c) \qquad \text{etc.}
\end{aligned}$$

In general,

$$(a,b,c,\ldots p,q,r) = r(a,b,c,\ldots p,q) + (a,b,c,\ldots p).$$

We shall now write down these values in expanded form:

$$\begin{aligned}
(\ ) &= 1\\
(a) &= a\\
(a,b) &= ab+1\\
(a,b,c) &= abc+c+a\\
(a,b,c,d) &= abcd+cd+ad+ab+1\\
(a,b,c,d,e) &= abcde+cde+ade+abe+abc+e+c+a
\end{aligned}$$

$$(a,b,c,d,e,f) = abcdef+cdef+adef+abef+abcd+ef+cf+af+cd+ad+ab+1$$

$$(a,b,c,d,e,f,g) = abcdefg+cdefg+adefg+abefg+abcfg+abcdg+abcde+efg+afg+afg+cdg+adg+abg+cde+ade+abe+abc+g+e+c+a$$

Several points of interest may be noted with regard to these expansions:

1. If (,,,) has an odd number of symbols, the last term in the expansion is a.
2. If (,,,) has an even number of symbols, the last term in the expansion is 1 (unity).
3. The first term in the expansion is always equal to the product of the symbols in (,,,).
4. If (,,,) has an odd number of symbols, the terms in the expansion each have an odd number of factors. (e.g., abcde, ade, c).
5. If the (,,,) has an even number of symbols, the terms in the expansion each have an even number of factors (excepting the last term, which is always unity). (e.g., abcd, ad, 1).

From the above, there appears to be no obvious method of writing down the expansion of any form (a,b,c,...r) immediately, by inspection. It is necessary to write successively the expansions of all the preceding forms. It would be possible to express the general expansion in determinant form, but we shall not include it here.

IV

It is also possible to express all the expansions in a fractional series, as follows:-

$$(a) = a\,(1)$$

$$(a,b) = ab\left(1+\frac{1}{ab}\right)$$

$$(a,b,c) = abc\left(1+\frac{1}{ab}+\frac{1}{bc}\right)$$

$$(a,b,c,d) = abcd\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{abcd}\right)$$

$$(a,b,c,d,e) = abcde\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{de}+\frac{1}{abcd}+\frac{1}{abde}+\frac{1}{bcde}\right)$$

$$(a,b,c,d,e,f) = abcdef\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{de}+\frac{1}{ef}+\frac{1}{abcd}+\frac{1}{abde}+\frac{1}{abef}+\frac{1}{bcde}+\frac{1}{bcef}+\frac{1}{cdef}+\frac{1}{abcdef}\right)$$

$$(a,b,c,d,e,f,g) = abcdefg\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{de}+\frac{1}{ef}+\frac{1}{fg}+\frac{1}{abcd}+\frac{1}{abde}+\frac{1}{abef}+\frac{1}{abfg}+\frac{1}{bcde}+\frac{1}{bcef}+\frac{1}{bcfg}+\frac{1}{cdef}+\frac{1}{cdfg}+\frac{1}{defg}+\frac{1}{abcdef}+\frac{1}{abcdfg}+\frac{1}{abdefg}+\frac{1}{bcdefg}\right)$$

Several points of interest may be observed in these fractional expansions: -

1. The factor outside of parentheses in the expansion always equals the product of the symbols in (,,,).
2. The first term in the parenthetic portion of the expansion is always unity, and all other terms in the expansion are fractions with unity in the numerators.

3. The denominators of the fractions always contain an even number of factors. (e.g., ab, abcd, bcdefg).
4. The denominators can be broken up into pairs which are successive or contiguous, such as de, ab/cd, ab/ef, bc/de/fg. It should be noticed that there occurs no such denominator as abdf, because df is not a contiguous pair.
5. The pairs never involve a common symbol and all possible combinations of pairs obeying the previous laws must be exploited.

The above indicates the possibility of writing down the expansion of any form at once, by inspection. We may write the general expansion as follows:-

$$(a,b,c,\ldots p,q,r) = (abc\ldots pqr)\left(1+\frac{1}{ab}+\frac{1}{bc}+\ldots+\frac{1}{qr}\right.$$
$$+\frac{1}{abcd}+\frac{1}{abde}+\ldots+\frac{1}{abqr}+\frac{1}{bcde}+\ldots\frac{1}{bcqr}+\ldots\frac{1}{opqr}$$
$$+\frac{1}{abcdef}+\frac{1}{abcdfg}+\ldots\frac{1}{abcdqr}+\frac{1}{abdefg}+\ldots\frac{1}{abdeqr}$$
$$\left.+\ldots\ldots\ldots+\frac{1}{abcd\ldots pqr}\right)$$

V

In Article 3, Gauss recalls two relations of Euler's Algorithms which should not be difficult to understand. The first of these can be shown to be true, as follows: -

If $(a,b) = b(a)+1$,

then $(a)(b) - 1(a,b) = ab - b(a) - 1 = -1$.

If $(b,c) = c(b)+1$,

and $(a,b,c) = c(a,b) + a$,

then $(a,b)\,(b,c) - b(a,b,c) = \underline{(a,b)c(b)}+(a,b) - \underline{(b)c(a,b)} - (b)\,a$,

But the first and third terms (underlined) in the right hand member of this equation nullify each other. Hence $(a,b)(b,c) - (b)(a,b,c) = (a,b) - (b)(a) = ab+1 - ab = -1$.

In similar manner, we can extend this relation to brackets containing larger numbers of symbols. Collecting our results, we may write: -

$$(a)(b) - 1(a,b) = -1$$

$$(a,b)(b,c) - b(a,b,c) = +1$$

$$(a,b,c)(b,c,d) - (b,c)\,(a,b,c,d) = -1$$

$$(a,b,c,d)(b,c,d,e) - (b,c,d)\,(a,b,c,d,e) = +1$$

or, in general,

$$(a,b,c,\ldots m)(b,c,\ldots m,n) - (b,c,\ldots m)(a,b,c,\ldots m,n) = \pm 1,$$

where the positive sign is used if the number of symbols in the first bracket is even.

The second of Euler's relations reviewed by Gauss is fairly obvious:-

$$(a) = (a)$$

$$(a,b) = ab+1 = (b,a)$$

$$(a,b,c) = abc+c+a = (c,b,a)\ , \text{ etc.}$$

In general, reversing of the order of the symbols does not alter the value of the expansion. We may therefore write: -

$$(a,b,c\ldots m) = (m\ldots c,b,a).$$

Proof of this reversibility follows from a consideration of the method of expansion by means of fractional series.

Thus, $(a,b,c) = abc\left(1 + \frac{1}{ab} + \frac{1}{bc}\right)$

and $(c,b,a) = cba\left(1 + \frac{1}{cb} + \frac{1}{ba}\right)$

These can be seen to be identical.

VI

With this knowledge of Euler's Algorithm, it becomes easy to follow Gauss' procedure in applying it to the general problem of determining the optical constants of a coaxial system of refracting surfaces separated by media of varying refractive indices.

Gauss explains his procedure adequately, and it is therefore not necessary to digress further on the subject. It should be noted, however, that the symbol, k, represents the negative value of the resultant refracting power of the entire system. Hence, it is a simple matter to use the expression $-k = (u^\circ, t', u', t'', u'', \ldots u^*)$, wherein the "u" symbols are the negative refracting powers of the individual surfaces, and the "t" symbols are the "reduced distances" separating their vertices, to determine the refracting power of the entire system. This method compares favorably with all the modern recurrent formulae or determinant systems since derived.#

B. R.

See bibliography on following page.