Tangents Shedding some light on the f-number

The f-stops here

by Marcus R. Hatch and David E. Stoltzmann

The f-number has peen around for nearly a century now, and it is certainly one of the fundamental parameters of most optical systems. It is used chiefly to characterize a level of illumination on or at a particular surface, but it also determines a number of other optical properties.

In a typical incoherent optical system, the f-stop affects or controls:

- the amount of light falling on the image plane:
- the depth of focus and depth of field;
- the system resolution, optical transfer function and cutoff frequency:
- the level of any residual aperture dependent aberrations;
- the level of vignetting as a function of field angle.

The use of the f-number in the above areas is treated sufficiently in standard optical texts and related literature. But the actual computation of the f-number for a general optical system needs more attention than it has been given in the past.

Rules don't always work

The rule of thumb that the f-number is equal to the focal length of the optical system divided by the diameter of its entrance aperture generally suffices for a large class of problems. But there are times when variations of the standard rule are needed, and there are other times when the rule and its variations simply don't apply.

To compute the f-number of an actual system, you must know its focal length. Consider the situation in Figure la. where a simple lens in air is forming an image of an infinitely distant object on a piece of film.

Depicting the optical system, for simplicity, as a thin lens, the figure shows the focal length f' (called the second or posterior focal length), which is the axial distance measured from where the refracted axial ray crosses the optical axis at the film to the point where this ray. when extended back to the left, crosses the incoming axial ray. This is just the distance from the thin lens to the focal plane, as we would expect.

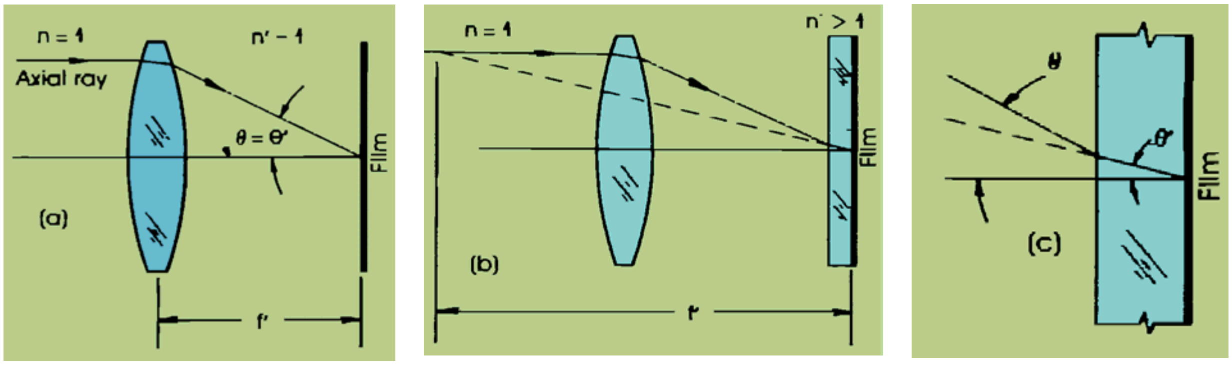

Figure 1. Posterior focal length for an image in air, and in a thin glass plate.

Figure lb depicts the same lens as Figure la, but includes a thin glass plate placed over the film. An enlargement of the axial ray in the glass plate is shown in Figure 1c and illustrates the refraction taking place at the air-glass interface. Note, however, that the extension of this axial ray back to the left now gives a focal length f', as shown in Figure 1b, that is much greater than the focal length f' given by Figure 1a.

Some confusion about the EFL

Many computer ray-trace programs print out the focal length f' as shown in Figure 1b and call it the effective focal length (EFL) as a matter of convention. This can be confusing at times, especially knowing that the "real" focal length of the lens in Figure lb (that is, the focal length that determines the size of the image at the film) has not been changed by the addition of the glass plate.

The anterior, or first, focal length, designated by f, is what remains constant with the insertion of the glass plate in Figure 1b, and this is the focal length that determines the image size. The two focal lengths are related to the refractive indices in object and image space (n and n' respectively) by the following equation:

$$\frac{f}{f'} = \frac{n}{n'} \tag{1}$$

where the assumed sign convention implies that a positive lens has a positive focal length regardless of the direction of travel of the light.

In this example, the f-number of the lens, and thus the illumination level at the film, should not be changed by the insertion of the glass plate, just as the focal length was not expected to change. Therefore, according to the rule of thumb for computing the f-number, the first focal length, when divided by the entrance aperture, will yield the correct f-number.

From a radiometric point of view, it is customary to relate the f-number to the angular subtense of the imaging cone of light formed by the optical system for an axial object point. In this sense, the f-number is usually considered an image space quantity, and many computer ray-trace programs compute it as the reciprocal of twice the product of the image space refractive index and the paraxial axial ray slope at the image plane. The factor of the image space index compensates for the change in the axial ray slope, as depicted in Figure 1c.

This definition of the f-number is convenient to use because it allows the f-number to be computed from paraxial raytrace quantities. But it can lead to difficulties if the object space medium is other than air or if the system has a very low f-number. The latter condition results from the inherent errors of the paraxial approximation, while the former condition simply fails to take into account the object space refractive index.

The general equation that should be used to define the f-number for critical systems, is:

$$F/\# = \frac{n}{2n'\sin\theta'} = \frac{n}{2(NA)} \tag{2}$$

where the numerical aperture (NA) is defined by NA = n' sinθ', where θ' is the real axial ray angle in image space, and n and n' refer to the object and image space indices, respectively.

The following are acceptable relationships for the particular cases noted:

$$f/\# \cong \frac{n}{2n'\tan\theta'} \tag{3}$$

paraxial approximation

$$f/\# \cong \frac{1}{2n'\tan\theta'} \tag{4}$$

paraxial approximation, object in air: used by many computer ray-trace programs

$$f/\# = \frac{f}{\text{entrance aperture}} \tag{5}$$

(infinite object conjugate: aplanatic system)

where f is the first focal length of the lens.

If the lens is in air, the first and second focal lengths are the same, according to Equation 1. The paraxial approximation replaces sinθ' with tanθ' because it considers the final refraction from the optical system to arise at a plane, a principal plane, instead of the actual refracting locus, which, in the case of a well-corrected (aplanatic) system, is a sphere.

Obeys Abbe

Because an aplanatic lens obeys Abbe's sine condition and has its equivalent refracting locus centered about the focal point, the maximum possible aperture for an aplanat in air equals f/0.5. This is seen by examining Equation 1, where the maximum value of sine θ' is 1.0 for θ' = 90°.

But the immersion of the optical system in a medium with a higher refractive index than air alters this theoretical limit on the f-number. Oil-immersion objectives for microscopes use this attribute to increase their numerical aperture, or decrease their f-number, and obtain greater resolution. And in an analogous situation, a camera used under water has its effective aperture reduced by the value of the refractive index of the water.

Equation 5 might appear to allow any arbitrarily small f-number to be produced simply by further increasing the entrance aperture for a given focal length. In fact, a fresnel lens can easily be imagined to have an aperture much greater than twice its focal length. Similarly, an extremely deep-dish reflector could be made with any arbitrary aperture. Closer inspection, however, reveals that neither of these examples are aplanatic optical systems and so do not meet the requirements of Equation 5.

The "speed" of the fresnel lens is determined by sin θ' from Equation 1, and regardless of its aperture, sin θ' is limited to a maximum value of 1.0.

In the case of the reflector, if the reflector is used in an image-forming capacity, sinθ' is also limited to 1.0 (θ' has a maximum value of 90°), because a ray having θ' greater than 90° would strike the image from the wrong side. And if the reflector is used in a non-image-forming capacity, such as a solar concentrator, the concept of f-number becomes meaningless and is replaced by concentration ratios.

Annotating the f-number with a ∞ subscript ($f/\#_\infty$) avoids possible confusion about what the f-number of the lens system is when the system is used at finite conjugates.

Consider, for example, a 1:1 copy lens whose barrel markings, or design specifications, are simply f/5.6. Does this mean the lens is f/5.6 for an infinitely distant object but in copy use is effectively working at f/11? Remember that at 1:1, the object and image distances are both 2f. Or does the marking mean that the lens is operating at f/5.6 in copy use and is really f/2.8 for an infinite conjugate?

To determine the $f/\#_e$ when a lens with a known $f/\#_\infty$ is used at finite conjugates, the following formula is usually applied:

$$f/\#_e = f/\#_\infty (1+m) \tag{6}$$

where the magnification of the system (m) is equal to the image size divided by the object size and is a positive value. This thin-lens formula can generally be applied when the lens is reasonably symmetrical, but it can also lead to significant errors when used with lenses having a substantial difference in the size and orientation of the entrance and exit pupils.

Equation 6 can be made more general by including a factor of the pupillary magnification (m_p), as shown in Equation 7:

$$f/\#_e = f/\#_\infty \left(1 + \frac{m}{m_p}\right) \tag{7}$$

where m_p equals the exit pupil diameter (the aperture stop as seen from the image) divided by the entrance pupil diameter (the aperture stop as seen by the object). Equivalently, m_p equals the ratio of the slope of the entering chief ray multiplied by the object space index to the slope of the merging ray times the image space index.

Some pupils get inverted

In Equation 7, m is called positive for an inverted image and negative for an erect image, such as one that would be produced by an optical system with an additional relay lens. The term m_p is called positive for a system that does not invert the pupils.

Although most normal lenses do not invert the pupils, some optical systems do. This typically happens in systems having intermediate images, but it can also occur in some ordinary lenses used at finite conjugates.

In this latter case, where m could be positive and m_p negative, the system can have the peculiar property that the $f/\#_e$ is actually smaller than the $f/\#_\infty$; that is, the system becomes faster when used at finite conjugates.

It is worth noting that the normal rule of thumb, Equation 6, would fail to predict the correct $f/\#_e$ in situations like this.

Can't always measure

In a practical sense, it is not always simple to measure the pupil diameters for use in Equation 7. They cannot simply be measured by holding a ruler to the front or back of the lens while viewing the aperture stop, as has been suggested by some sources. Also, some optical methods, such as placing a point source at the focus of the lens and measuring the diameter of the emerging collimated beam, yield correct diameters only for the exit or entrance pupils for an infinite conjugate.

Errors can result when the lens is used at finite conjugates if an element or retainer limits the beam of light and effectively becomes the aperture stop. But using Equation 4, which is based on the axial ray's final slope angle, or a variation of that equation, to compute the f-number, $f/\#_e$ is automatically computed for finite conjugates, alleviating the need for using Equation 7. Note, however, that while Equation 7 is not necessary if Equation 4 has been used to compute $f/\#_e$, it helps understand certain peculiarities that can arise with $f/\#_e$.

With regard to the entrance and exit pupils as discussed above, the lens diaphragm doesn't always function as the aperture stop when the lens is used at various object and image conjugates. Figure 2 depicts a situation in which, when the lens is used at a close conjugate, the edge of one of the lens elements becomes the effective aperture stop.

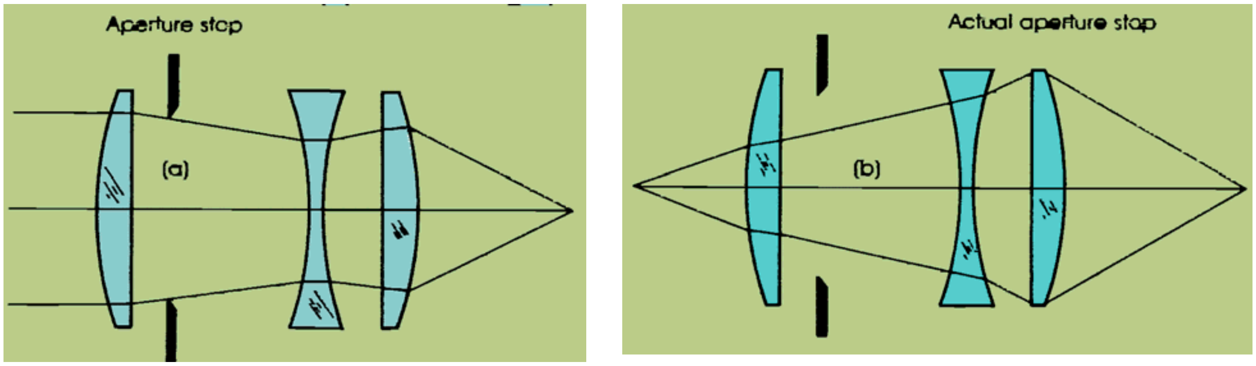

Figure 2. The effective aperture stop of an optical system can change when the system is used with a long conjugate (a) and a short conjugate (b).

A similar complication arises when lenses are turned end-for-end, as is sometimes done in microphotography. In such a situation, a smaller diameter lens element is probably functioning as the effective aperture stop rather than the diaphragm.

In cases like this, where the actual aperture stop is determined by something other than the diaphragm, no simple equation can be used to determine $f/\#_e$ from $f/\#_\infty$. Raytracing the design or using a through-the-lens light meter is the best way to solve the problem.-
When applying Equation 7 to a lens that has been turned end-for-end, you must remember that if the diaphragm still controls the aperture, what used to be the exit pupil now becomes the entrance pupil, and vice versa.

Shapes other than circles

Entrance pupils generally are assumed to be circular. But arbitrarily shaped pupils can be dealt with as well by modifying Equation 5 to produce Equation 8:

$$f/\#_\infty = \frac{f}{2(A/\pi)^{1/2}} \tag{8}$$

where A is the area of the entrance aperture. Therefore, the f-number for an arbitrarily shaped pupil is equivalent to the f-number for a circular aperture having the same area.

In finite conjugate systems, the $f/\#_e$ is obtained from the $f/\#_\infty$. in the manner previously described. But in systems that have appreciable transmission losses, ones coming from beamsplitters or inefficient antireflection coatings, for example, a more reliable estimate of the level of illumination produced by the optical system is given by what is called the T-number:

$$T/\# = \frac{f/\#}{(t_o)^{1/2}} \tag{9}$$

where the transmission factor of the optical system, t_o, includes any of the above transmission losses as well as any losses due to absorption and scattering.

The T-number of an actual lens can be thought of as equal to the f-number for an equivalent circular opening in an imaginary lens that has no transmission losses and gives the same axial image illumination as the actual lens at the specified aperture.

Some high-precision applications, like the motion-picture industry, have required lenses calibrated in T-stops rather than f-stops. In such cases, where the level of illumination at the film plane had to be known very accurately, each lens was individually calibrated photometrically. This procedure tended to eliminate problems inherent in mechanical diaphragms, which could be in error by as much as 25 percent of the area of the effective aperture for a particular f-stop setting.

Although many f-number marking systems for lenses have been used, today's generally accepted one is:

$$f/\# = 2^{N/2} \tag{10}$$

where integer values of N give the familiar "click stops" found on many lenses today and non-integer values of N can be used to define intermediate f-numbers. Table 1 gives a few examples. If the lens aperture changes by a whole stop (N changes to N±1), the level of illumination on the image increases or decreases by a factor of 2. The change in illumination due to an arbitrary change in the f-number is:

$$\begin{aligned} &\text{Illumination after} = \\ &\text{Illumination before} \times \\ &\left(\frac{\text{f/\# before}}{\text{f/\# after}}\right)^2 \\ &= \text{Illumination before} \times 2^{\mathrm{M}} \end{aligned} \tag{11}$$

where M is equal to the number of stops between the two f-numbers.

Table 1 Common f-stops given by equation 10

N	**0**	**1/2**	**1**	**3/2**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
f/#	**1**	**1.2**	**1.4**	**1.7**	**2**	**2.8**	**4**	**5.6**	**8**	**11**	**16**	**22**	**32**

Try raytracing a simple concave reflector immersed in water, whose object and image conjugates are in water as well, and see if the computed f-number and focal length are what you would expect.

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