The Lock-In : Noise Reduction and Phase Sensitive Detection

The two essential reasons for using a Lock-In amplifier in a scientific experiment are its ability to **„reduce noise“**, i.e. to improve the Signal-to-Noise ratio of a signal to be measured, and to do **phase sensitive detection**, the latter being mainly deployed in technical applications, as e.g. in phase locked loops and control circuits.

Fundamentally, the Lock-In is sensitive to **ac signals**. So, whatever you do, you have to care for an ac signal at the input of the Lock-In. How this can be done in an experiment, you will learn in **Part Three : Scientific Applications** (for a short introduction, see **below**). Furthermore, it needs a **„reference“**, i.e. a periodic ac voltage, preferably but not necessarily with the signal frequency, fed into the reference input. Most Lock-Ins have a **built in reference generator** you can work with. In this case you need not apply an external reference signal, of course.

To see how our Project Lock-In works, have a look at the **Schematic I** page. You will find there the internal electronic components of the instrument and their interconnections. The most essential parts to make the Lock-In work as such, are a **multiplier** circuit and an **integrator**, which together form a „Phase Sensitive Detector“, or PSD. For reasons you will immediately understand, a **phase shifter** is needed in the overall circuitry.

Since the „experiment“ is being „modulated“ by a sine function with frequency ω_o, the signal from the experiment has the same frequency, or, at least, a strong component with frequency ω_o.

Under these conditions, the **output** of the Lock-In is a **dc voltage**

$$U(\Delta) = U_{out} = \frac{ab}{2} \times \cos(\Delta) \qquad (1)$$

where a and b are the **rms amplitudes** of the signal and the reference ac input voltages, respectively. In general, the reference amplitude is **set internally to two** so that the **output voltage** is proportional to the **input signal amplitude times the cosine of the phase difference**, Δ, between the „signal“ and the „reference“ inputs. For the **origin** of this equation, please refer to **Part Two**, eq.(3).

For **maximum output**, you have to **adjust the phase difference to 0° (or 180°)**, a task not very convenient to perform with very weak, noisy input signals. There are, however, standard methods to do so experimentally, as you will see in Part Three. Alternatively, you may take the **Lock-In Analyzer** (see **Part Four**, for reference) with which you can get rid of any manual phase settings.

With eq. (1) almost everything is told about **phase sensitive detection** : When you have two signals with known amplitudes, you can easily calculate the phase difference between the two signals from the output voltage. This feature has found many interesting applications in technical problems. In Part Three we will go a bit more into some details of psd.

The next, and for some people : the most important and most useful, feature of the Lock-In is its ability to **improve the Signal-to-Noise ratio of noisy signals up to 60 dB** (in fact even more under certain conditions). The theory for this will be presented in Part Two, together with the opportunity to do yourself measurements which will strongly support this theory ...

To explore this noise reduction feature, set the Phase Angle for maximum output level and add noise to your input signal by setting the S/N value to a „very noisy“ value, say 0 dB, or less. As you can watch on the „input“ screen, the original sine becomes more and more „chopped“ by the noise. The output is also noisy, around its dc „mean value“, i.e. the rms value of the input signal, as set by the „Signal Level“ slider, times the „Gain“ setting of the Lock-In.

Note : For reasons of easier visualization, the **rms value** of an ac input signal (measuring signal and reference) has been set, throughout our Project, **equal to the amplitude** of this signal. So, „rms“ value and „amplitude“ are **synonymous in our Project**.

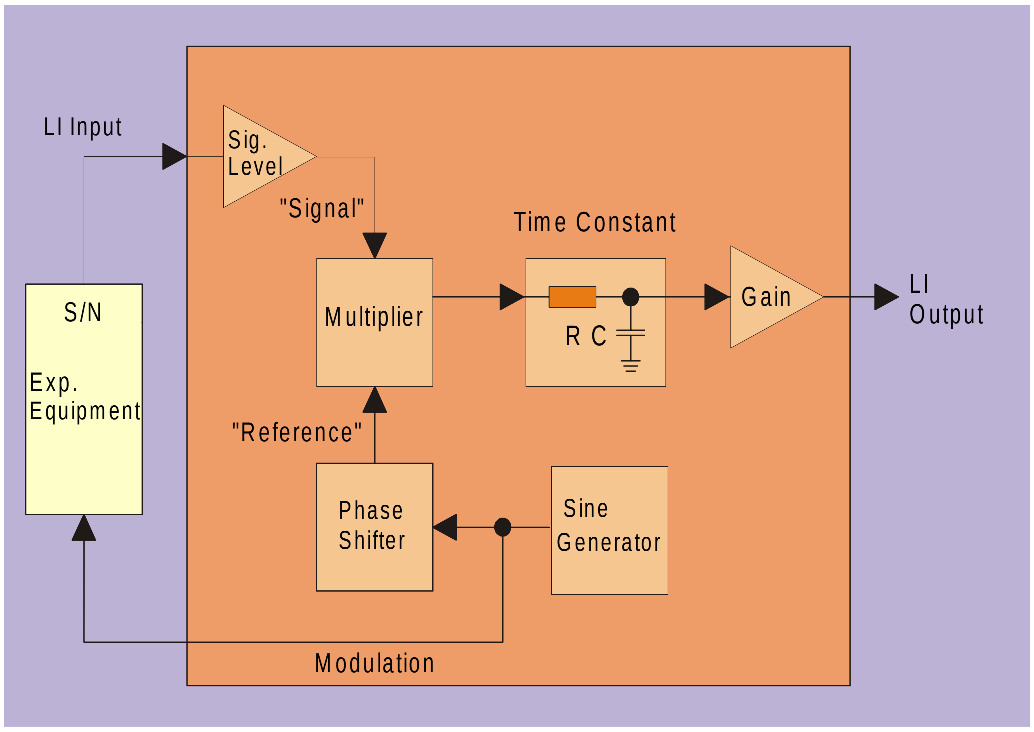
By raising the time constant TC, you will notice that the output noise becomes more and more lower.

To perform **quantitative measurements** of the dependence of the noise rms value on TC, you will need the password mentioned in the Introduction. But also without this password, you can **qualitatively** explore this behavior very nicely.

As will be pointed out in more detail in **Part Three**, under realistic conditions of - say - a physical experiment in which a Lock-In is employed, the **System under Investigation** is typically **stimulated** by some kind of periodic signal (not necessarily an electric one ...). As a consequence, as we will see, the interesting **response** of the System will contain the **same frequency** (and possibly some **harmonics** if the System‘s behavior is **nonlinear**). But, due to the **"travelling time"** of the signals within the System, the modulated **output signal from the System** - which is our **input signal into the PSD** - is generally **delayed** with respect to the stimulating ac signal, which itself (or it's electric "representative") may be used as the **reference input** into the PSD. In this case, the two signals are **automatically "locked" to one another**, the Lock-In amplifier further processing these two "locked" signals.

In short, in response to a **harmonic input signal**, $f(t) = a \sin \omega t$, to a **linear system**, you will get an output of the form $g(t) = A(\omega) \cdot \sin[\omega t + \Delta(\omega)]$. In a **nonlinear** system, there will also occur **harmonics** (more in **Part Three**).

Schematic I



The **principal components** of the Lock-In are :

(1) a multiplier,
(2) an RC circuit, or integrator,
(3) a phase shifting circuit.

In most cases, like in the present one, a **sine generator** as the reference source is built in.

The **multiplier** in fact „multiplies“ both, the input and the reference signals. The resulting output signal is integrated, by an RC circuit, or, preferably, an **OpAmp integrator** over a time set by the time constant, TC.

As you know from your basic electronics course, this RC circuit generally serves as a **low pass filter**. The combination multiplier unit plus RC circuit, however, acts completely different, as you will see during the course of your investigations.

In our experiment, the RC circuit is "responsible" for the upper integration limit, the Time Constant **RC = nT**.

The **Phase Shifter** circuit shifts the phase of the "reference" with respect to the "signal", has been added (Remember: The phase difference Δ has to be adjusted to 0° or 180°, respectively, to get maximum output).

An external "experiment" is shown, some parameter of which is being **modulated** by the reference signal, with frequency ω_o, as described in the previous section. Consequently, the output signal from the experiment also contains the stimulating frequency. This experiment also **adds noise** to the Lock-In input.

The schematic represents the **"final configuration"** of a **common Lock-In** amplifier, i.e. it contains all of the **essential components** of a real-world Lock-In instrument.

When you were to **build yourself** a LockIn amplifier with hardware components, the Multiplier / Integrator module could be made of a **Multiplier IC**, followed by an **integrator** circuit.

Technically, there exist **other construction principles for the PSD**, too, to name those where a **square wave** reference signal is used to **switch a transistor circuit** with the frequency of the input signal.

Both principles have their advantages and their drawbacks which shall not be discussed here (There will be, however, a **similar question to be answered**, anywhere during this course ...)

Experiments and Exercises:

(1) Add noise to your input signal; most impressive will be a **S/N of -20 dB.**

Improve the S/N ratio by increasing the integration time constant, TC.

To help you with this task, a click on the **"NAD"** ("Noise Amplitude Distribution") **button** invokes a routine which takes a predefined number of noise amplitude data. At the end of the routine, a plot is shown with an amplitude distribution of the samples taken. For very large numbers this distribution will finally approach a "Gaussian Distribution" curve. Even the allowed maximum of 1000 samples is not sufficient to this end. Generally some useful **rms amplitude** value, however, can be taken from the plot. (In this context, recall the meaning of the Standard Deviation of a **Gaussian distribution**.)

Play a bit around with this feature !

Plot your noise amplitude data vs. TC.

You can invoke an appropriate **log-log plotting grid** by clicking on the **N(TC) button**.

Note : In contrast to the Voltage Output display, which shows an additional average of **ten times** the „Integrator Output" signal, the data you get by means of the Gaussian distribution routine is the data from the **original** „Integrator Output" !

(2) (a) What is the functional dependence, S/N(TC) ?

(b) What improvement of S/N do you expect for 10^4 T ?
You may roughly check this expectation by watching the Voltage Output display for some data at TC = 900 ... 1000 T. Add these data to your N(TC) plot.

(c) At what phase shifts, Δ, between reference and signal will you get the maximum output amplitudes ?

(d) When will the output be zero ?

(3) (a) Why do you need a **continuous range** of phase shift settings ?

(b) Do you expect to find any difference when measuring the **noise amplitude**, at fixed input S/N, at **different phase settings** ?

Verify your opinion experimentally and try an explanation of your findings.

(c) If you use a **square wave reference signal**, what do you think are the differences compared with a sine ?

(d) Can one use the Lock-In for the **detection of time varying input signals** - besides noise, too ? If yes: Under what conditions ?

(4) Prove eq.(1) from **Part Two, eq.(3)**.

Gaussian or Normal Distribution

Electronic noise is a **continuous random process**, the **amplitude probability density** function of which is described by the well known **Gaussian or „normal" distribution**

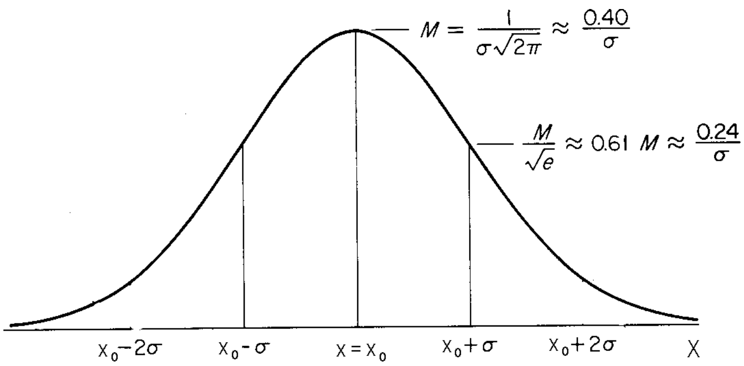
$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-x_o)^2}{2\sigma^2}} \qquad \text{for} \quad -\infty < x < \infty$$

where *p(x)* denotes the probability density of the noise amplitude *x* to occur.

Since the amplitude certainly lies between $-\infty$ and $+\infty$, we have $\int_{-\infty}^{+\infty} p(x)\,\mathrm{d}x = 1$.

Therefore, the **maximum probability** of the Gaussian distribution function occurs at $\mathbf{x_o}$, and has the value

$$p_{\max} = p(x_o) = \frac{1}{\sigma\sqrt{2\pi}}.$$



From the diagram it can be seen that the **probability density** of the noise amplitude is **greatest about x = $\mathbf{x_o}$** and symmetric about that point. The smaller σ, the greater the maximum density and the less the „spread" of the density distribution.

The larger σ, the more the probability density is spreading out, with less concentration near x_o.

With **larger σ,** the values of the random noise amplitude become **„less predictable“** on any trial (see below for the meaning of σ).

The **first moment** of p(x) is defined as

$$m_1 = \int_{-\infty}^{+\infty} x\, p(x)\, \mathrm{d}x$$

which can be shown to yield, after some calculation, $m_1 = x_o$.

I.e. the first moment is equal to the **average value, x_o**

Similarly, the **second moment** of p(x)

$$m_2 = \int_{-\infty}^{+\infty} x^2\, p(x)\, \mathrm{d}x$$

can be calculated to give

$$m_2 = x_o^2 + \sigma^2 .$$

The quantity $$\mu_2 = m_2 - x_o^2 = \sigma^2$$

is named the **variance** of the random signal, and σ is called **standard deviation**.

The **standard deviation** is equal to the **rms (root mean square)** value of the **noise voltage** (or current), whereas the **variance** represents the **mean square** value of the noise, corresponding to its **power**.

Correlation Function and Fourier Transform

In **Part Two / experiment #1** , you will investigate the **band pass** behavior of the Lock-In: you have to plot the output voltage from the Lock-In as a function of the frequency ω_1, or the frequency ratio ω_1/ω_0, respectively, where ω_0 is the (fixed) frequency of the „reference“, and ω_1 is the variable frequency of the „signal“.

The power, or voltage, as a **function of the frequency**, P(ω), or U(ω), is generally called a **spectrum**. In our Correlation Function we had, however, to deal with **time** dependent functions.

From your math lessons you know how to **transform a time dependent function, f(t)** , into its **frequency domain „counterpart“, F(ω)** : This is done by means of the **Fourier Transform**

(1)
$$\mathrm{F}(\omega) = \int_{-\infty}^{+\infty} \mathrm{f}(t)\, e^{i\omega t} \mathrm{d}t$$

If we now simply take

$$\mathrm{f}(t) = \sin(\omega t + \Delta)$$

then we get

(2)
$$\mathrm{F}(\omega) = \int_{-\infty}^{+\infty} \mathrm{f}(t)\, e^{i\omega t} \mathrm{d}t = \int_{-\infty}^{+\infty} \sin(\omega t + \Delta)\, [\cos(\omega t) + i \sin(\omega t)]\, \mathrm{d}t$$

the **imaginary part** being equal to

(3)
$$\Im\mathrm{m}[\mathrm{F}(\omega)] = \int_{-\infty}^{+\infty} \sin(\omega t + \Delta)\, \sin(\omega t)\, \mathrm{d}t$$

If we now **switch on** $\mathrm{f}(t) = \sin(\omega t + \Delta)$ at t = 0, and **switch it off** at t = τ, then we get

(3a)
$$\Im\mathrm{m}[\mathrm{F}(\omega)] = \int_{0}^{\tau} \sin(\omega t + \Delta)\, \sin(\omega t)\, \mathrm{d}t$$

which corresponds, except for any amplitudes or „normalizing factors“, exactly to our **Correlation Function, Part Two / eq.(3)**

$$R(nT,\omega,\Delta) = \frac{a\,b}{nT}\int_0^{nT} \sin(\omega t)\,\sin(\omega t+\Delta)\,dt$$

This means that the **Correlation Function**, in our case with the **special functions**, $f(t) = \sin(\omega t)$ and $g(t) = \sin(\omega t + \Delta)$, and the **imaginary part of the Fourier transform** of $g(t) = \sin(\omega t + \Delta)$, for t = 0 ... nT are **identical**.

So what you will measure in **experiment #1 / Part Two**, is in fact the (imaginary part of the) **Fourier spectrum of a switched sine wave** (with arbitrary amplitude and phase factors, respectively), depending on the **length of the switching interval, nT**. Note that the **phase** Δ introduces a **phase shift** also in $F(\omega)$; for $\Delta = 0$, it can be shown that the **real part** of $F(\omega)$ **vanishes**. In this case (Δ was set to zero by default in Part Two !), eq.(3a) represents the **complete Fourier transform** of $g(t) = \sin(\omega t)$, for t = 0 ..... nT.

The Correlation Function

The key for the basic understanding of the **Lock-In amplifier** lies in the behavior of the so-called **Correlation Function.**

(1) $$R(\delta) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} f(t) \times g(t+\delta)\, dt$$

Here we have a definite integral over the product of **generally any** two time dependent functions, f and g, with a „delay“ parameter, δ. If there is any „correlation“ between f and g , F(δ) ≠ 0. In the case of **very weak signals from an experiment, g(*t*)**, with possibly very large noise superimposed, this relation allows for **looking for a correlation of g(*t*)** with a **"known" signal, f(*t*)**, conveniently called **"reference"** (see below).

If you don't like waiting until infinity for the value of the integral, then you have to stop the integration before. F then also becomes a function of the upper integration limit, τ:

(2) $$R(\delta,\tau) = \frac{1}{\tau} \int_0^{\tau} f(t) \times g(t+\delta)\, dt$$

If the Correlation Function, according to eq.(2), yields a value **unequal to zero**, then there must be a **nonzero signal, g(*t*)**, coming from the experiment.

The experimental problem, in this view of affairs, is to make the experiment **respond** with some **characteristic of the reference**, typically its **frequency**. This can be done by some kind of **modulation** (or stimulation) of an interesting parameter of the experimental system with the reference frequency.

The simplest case is to use a **harmonic function**,

$$f(t) = a \sin(\omega t),$$

as the **reference**. Here, a is the amplitude of the reference, and ω is its frequency.

If the "right" parameters of the experimental system are being modulated in this way, then the experiment will **respond** with signals which also **contain the frequency of the modulation**, i.e. simply

$g(t) = b \sin(\omega t + \Delta)$ (+ „harmonics“).

Here b is the amplitude of the signal you are looking for, Δ is a **phase shift** in the measurement signal with respect to the modulation function, caused by various **„travelling times“**, or **delays**, within the experimental system. The amplitudes a and b are considered to be **time independent** for the moment.

This signal content can now be **detected**, in principle, by applying the **Correlation Function**. Exactly this procedure is being employed in the Lock-In amplifier: In this way it is able to detect very **small periodic electric signals** which may even be **completely buried in noise**.

[Note: We have taken here the two simplest **harmonic** functions just for the **sake of simplicity**. In principle, you may take **any other periodic function**, like square wave, triangle, ... In the „technical realization of the Correlation Function“, the Lock-In amplifier, a **square wave** function is often used as the "reference" function, f(t). The "signal" function, g(t), in many practical cases also **deviates considerably** from the pure **sine** taken above. These variations may, of course, change the value of the correlation function more or less significantly. We will further consider this problem in Part Three of this series.]

The Correlation Function now takes the form (called **autocorrelation** function, due to the same arguments of the two basis functions)

$$\text{(3)} \qquad \mathrm{R}(nT,\Delta) = \frac{a\,b}{nT}\int_0^{nT} \sin(\omega t)\times\sin(\omega t+\Delta)\,\mathrm{d}t$$

Here, the upper integration limit is **nT**, where T is the **period** of the frequency ω, and n is an integer. Since, as said above, no one likes waiting to infinity for an experimental value, we have to cut our measurement interval at **finite values of n**. Therefore, we will never know the „exact value“ of the Correlation Function. In Part One, you have already investigated this "approximation" as a function of nT for a very noisy input signal.

Eq.(3), now, is the key to all the favorable properties of the Lock-In.

In Part One, you have investigated the **noise reduction** of a signal as a function of the **integration time** , TC. As you may already know, a certain reduction in high frequency noise is possible with any **low pass filter**, or integrator, like the one used in our Lock-In circuit. This filter reduces the intensity of the noise spectrum above the boundary frequency of the filter with a certain slope per octave, or decade, according to the „order“ of the filter. By **increasing RC**, or nT, respectively, you simply **lower the boundary frequency** of the filter, surely reducing the noise in this way, but thereby **attenuating** the signal itself, too. According to one of the fundamental laws of communication theory, the **Wiener-Khinchin theorem**, the **reduction** of the noise voltage imposed upon a useful signal with frequency fo, is proportional to the **square root of the bandwidth** of a bandpass filter, with center frequency fo.

Generally, the **bandwidth** is defined as **„full width at half maximum"** of the **power** spectrum passed through the filter as a function of frequency. Normally, however, you do not measure the power, but a **voltage** (or current). In this case, the bandwidth is measured as **„full width at 1/√2 maximum"** of the (voltage) filter response curve (since U ∝ √P).

As you know, the „bandwidth of a filter" alone does not tell very much about its ability to suppress unwanted signals with different frequencies. A bandwidth of, say, 1kHz is excellent for a center frequency of 100 MHz, but rather poor for the center at 5 kHz. A much better measure, however, of the „quality" of a filter is its **„Quality"** (whow ...) **factor**, **Q = fo / Δf**, where fo is the center frequency, and Δf the bandwidth as defined above. In the following, we define „bandwidth“ always as the „relative bandwidth“, BW = Δf / fo = 1/Q.

So, to understand our results from Part One, we have to look for a **band pass filter** in our measuring setup, with varying, and **very narrow**, at the end, **bandwidth**. But when you look at the schematic I, you will not detect any circuit which resembles such a filter ...

It is the objective of this Part to give you an idea, both **experimentally and theoretically**, of how the Lock-In manages to behave as such a very narrow bandpass filter.

To explore the action of the Correlation Function with respect to this goal, we will first examine the behavior of eq.(3) regarding its **frequency response**. This means that we will **vary the frequencies** of the **"signal" function**, **$g(\omega_1+\Delta)$**, with respect to the **"reference" function, $f(\omega_0)$**, or vice versa.

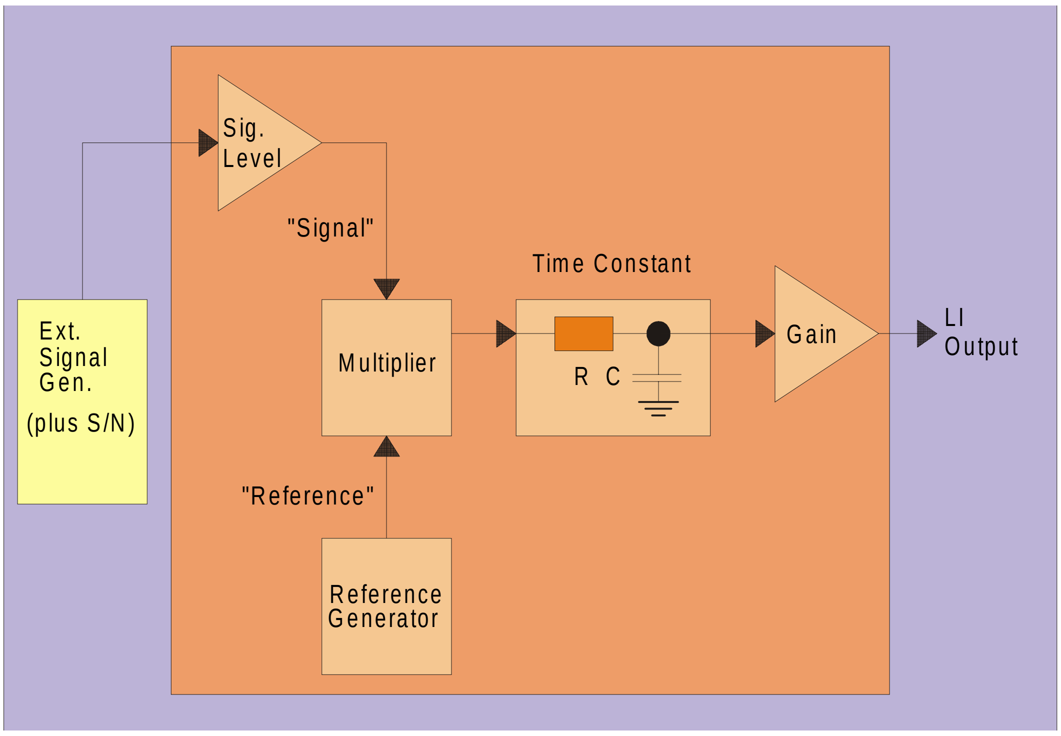
I.e., we will investigate the Correlation Function in the form

$$\text{(3a)} \qquad R(nT,\omega_1,\omega_0,\Delta) = \frac{ab}{nT}\int_0^{nT} \sin(\omega_0 t)\times\sin(\omega_1 t+\Delta)\,dt$$

(which is called **cross-correlation** function because of the different arguments of the basis functions.)

First, however, you should have a look at the **Schematic II** of our "specimen" Lock-In.

Schematic II



According to the requirements of **eq.(3a)**, we need **two sine generators** with **variable frequencies** (or one with fixed frequency, and the other with variable frequency ...), called "Ext. Signal Gen.", with frequency ω_1, and "Reference Generator", with frequency $\omega_o = 1$ (Hz, kHz, MHz ...), respectively. In our virtual Lock-In below, we will vary the **frequency ratio**, ω_1/ω_0.

Furthermore, there must be a **Multiplier** with two inputs and an **Integrator**. These are the **two essential electronic components** in any Lock-In.

For the sake of simplicity, we have omitted here the phase shifter since it isn‘t needed in the present context, as you may see from the Correlation Function : If $\omega_1 \neq \omega_o$, then the value of the Correlation Function is **independent** of Δ

In order to make the input signal **noisy** we have a noise generator (with **"white" noise** in this case) built into the signal source.

Remember :

$f(t) = a \sin(\omega_0 t)$ is the **reference** function, and

$g(t) = b \sin(\omega_1 t + \Delta)$ is the **input voltage** from the External Signal Generator.

Note, that we now have **two different and independent signal sources**. The Lock-In input from the „experiment“ does not result from a „modulation“ with the reference. The main experimental task of this Part will be to take a **plot of the output voltage** of the Lock-In as a **function of the frequency ratio** ω_1/ω_0 .

Experiments and Exercises:

(1) Measure the output voltage from the Lock-In as a **function of the frequency ratio f/f_o** for different integration times, TC.
Plot the measured values of the output voltage (display) over f / f_o (frequency slider) in an appropriately divided grid for each TC (see below).

Your V(f) diagrams will show the behavior of a **band pass filter**, the **bandwith** depending on TC.

Determine the bandwidth (full width at -3 dB (= $1/\sqrt{2}$) of the maximum) of the Lock-In as a function of TC.

For this experiment, you will need "No Noise" (S/N choice).
Set the Input Level at your convenience (1V recommended).

You may use the "plotting grid" (an octave-logarithmic grid) offered for this purpose which can be invoked with the **"V(f)" button**.
In the V(f) window you find a **„Function Fit“** button. Clicking (twice for the first time) on this button will make appear „a function“ of f/f_o, the x- and y- parameters of which may be adjusted to fit your experimental data.

(2) What function does the „Function Fit“ curve represent ? Give an explanation about the mathematical origin of this function.
What parameters do you adjust by varying the x- and y-sliders, respectively ?

(3) Make a plot of the bandwidth vs. TC (use the log-log grid invoked by the BW(nT) button).
What would be the (extrapolated) BW value for 10^4 T ?

Note for your work with the drawing frames:
By clicking the left mouse button, a data point is generated with the coordinates shown in the "Coords" fields. Any one of these points canbe removed indiviually with the right mouse button (you'll have to aim precisely ...). Clicking the "Reset" button will delete all data in the frame.

Please keep in mind that printing from within applets is not (yet) allowed. So, to get your diagrams on paper you should use a screen dump routine (available e.g. on **http://www.winsite.com** or via many local ftp servers on the Internet).

(4) To examine the behavior of the product of two harmonic functions, i.e. eq.(3a) **without integration**, you may choose **"zero" TC** from the Time Constant choice.

What is the **frequency of the output** signal for $f = f_o = 1$?
What happens when you vary the frequency ?
What would the output look like, when $0 < TC < 1T$?
Try an explanation of the "undershoots", i.e. the appearance of negative output voltages, in your V(f) plots above.

Note:
For more exact answers to the above questions you should consider the **Fourier transform** of the **Correlation Function**, eq.(3a) and the **functions involved herein**.

(5) Evidently, the **output from the Lock-In**, as described by the Correlation Function, is a **dc** voltage.
How does this come about, with the fact in mind that the **output from the multiplier** is still an **ac** signal ?

Noise Reduction by Narrow Band Filters

As you are investigating in **Part Two**, due to the special functions involved in the correlation function, the Lock-In behaves like a **band pass filter** with respect to its reference frequency, the **bandwidth** of the filter depending on the **integration time**. Intuitively this mechanism makes it clear how the Lock-In is capable to improve the Signal-to-Noise ratio significantly: The band pass filter „cuts“ simply out a more or less broad part of the noise power spectrum in the vicinity of the input signal, hereby reducing the average relative noise power at the output, or the relative rms noise voltage, respectively. „Relative“ means: with respect to the input conditions.

How can we get further information about the noise reduction numbers in our Lock-In ?

Especially:

How does the noise reduction depend on the integration time, or bandwidth, respectively ?

First, we have a random function of time, r(t), (the noise) added by the „experiment“ to our „wanted signal“, $g(t) = b\sin(\omega t + \Delta)$

The correlation function now has the form

(1) $$R(nT,\Delta) = \frac{a}{nT}\int_0^{nT} [b\sin(\omega t+\Delta) + r(t)]\sin(\omega t)\,dt = R_o(nT,\Delta) + \frac{a}{nT}\int_0^{nT} r(t)\sin(\omega t)\,dt$$

The behavior of the first part, $R_o(nT,\Delta)$, is known from **Parts One** and **Two**. What can we do with the second part, now ?

Since we don‘t „know“ r(t) analytically, we **can‘t evaluate the integral directly**. What we **do know**, however, about r(t) is its **rms value** (we can **measure** it with an **rms voltmeter**, or we can **analyze the noise spectrum**, as we did in **Part One**) and its spectrum (it is **white noise** with a **constant power spectral density**, independent of frequency, in principle ...)

The rms value of r(t) may be calculated by $\sqrt{\langle r^2(t)\rangle}$ where $\langle \ldots \rangle$ means the **time average**.

Instead of the **cross-correlation function** from **Part Two**, we now consider the ***autocorrelation function*** which is simply the correlation function of Part Two / eq.(1) with only **one function**, f(t), with the same argument :

(2) $$R(\delta) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} f(t)\,f(t+\delta)\,dt$$

which can be interpreted as a **time average** : $\langle f(t)\,f(t+\delta)\rangle$.

The correlation function characterizes the **statistical relation** between the values of f(t) at time t and (t+δ), where δ can be both positive and negative as well as zero.
If f(t) and f(t+δ) are **statistically independent (real noise)**, then

(3) $$R(\delta) = \langle f(t)\,f(t+\delta)\rangle = \langle f(t)\rangle\langle f(t+\delta)\rangle = 0$$

For δ = 0, the autocorrelation function becomes

(4) $$R(0) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} f^2(t)\,dt = \langle f^2(t)\rangle$$

which may be considered as the **mean power of the random process f(t)**, also known as its **variance** (see also in Part One : **Gaussian or Normal Distribution**). The mean amplitude of the **noise voltage** becomes $U_{rms} = \sqrt{\langle p\rangle R} = \sqrt{\langle f^2(t)\rangle R}$, where R is the resistance across which the „voltage drop" U_{rms} occurs.

The **power spectral density** , **S(*ω*)** (dimension: W/Hz), of an **aperiodic funtion f(t)** is defined by

(5) $$S(\omega) = \lim_{\tau\to\infty}\left\langle \frac{|F(\omega)|^2}{2\pi\tau}\right\rangle$$

where F(ω) is the **Fourier transform** of f(t).

Combining eqs.(4) and (5) yields the **average power** of the process expressed by f(t):

(6) $$R(0) = \langle f^2(t)\rangle = \frac{1}{2\pi}\int_0^{\infty} S(\omega)\,d\omega = \langle p\rangle$$

This is a special form of the famous **Wiener-Khinchin Theorem** which permits the determination of the power spectral density from a given correlation function, and conversely.

Generally, the **Wiener-Khinchin Theorem** states that the **autocorrelation function R(*δ*)** and the **spectral power density S(ω)** are the **Fourier transform**, or the **inverse Fourier transform**, respectively, of **each other** :

(7) $$R(\delta) = \frac{1}{2\pi}\int_0^\infty S(\omega)\,\exp(i\omega\delta)\,d\omega$$

and

(7a) $$S(\omega) = \int_0^\infty R(\delta)\,\exp(-i\omega\delta)\,d\delta$$

(The proof of this theorem is not difficult to perform but a bit lengthy, so that it will be omitted here.)

We will use this theorem to **calculate the noise reduction by the Lock-In** a little bit later.

Band Limited White Noise

White noise is defined as a **random process** with **constant spectral power density**, S(ω) = N_o [W/Hz], independent of frequency. This is, of course, a purely **mathematical fiction**, since

(8) $$\langle p \rangle = R(0) = \frac{1}{2\pi}\int_0^\infty S(\omega)\,d\omega = \frac{1}{2\pi}\int_0^\infty N_0\,d\omega \rightarrow \infty$$

which makes **no sense, physically**.

It is, however, useful to introduce the concept of **„band limited white noise“** whose spectral density is given by

S(ω) = N_o for $\omega_1 < \omega < \omega_2$ and zero outside this range.

In this case

(9) $$\langle p \rangle = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} N_0 \, \mathrm{d}\omega = N_o B$$

remains certainly finite; $B = (\omega_2 - \omega_1)/2\pi$ is the **bandwidth** of the band limited noisy process.

The Transfer Function

When a „signal“ is **applied to the input of a linear** network, the **same signal** may appear **different at the output**. If we apply, for example, a simple harmonic function

$$\mathrm{f}(t) = V_o \cos(\omega t)$$

at the input, the response at the output will generally be of the form

$$\mathrm{g}(t) = V_o \, A(\omega) \cos[\omega t + \Delta(\omega)]$$

or

$$\mathrm{g}(t) = \mathrm{H}(\omega) \, \mathrm{f}(t),$$

with

$$\mathrm{H}(\omega) = A(\omega) \exp[i \, \Delta(\omega)]$$

being the **Transfer Function** of the network.

It can be shown that the **output spectral density $\mathbf{S_o(\omega)}$** is related to the **input spectral density $\mathbf{S_i(\omega)}$** by

(10) $$\mathrm{S_o}(\omega) = |\mathrm{H}(\omega)|^2 \, \mathrm{S_i}(\omega).$$

Bandfilter

We have now all the tools together to **calculate the noise reduction by our Lock-In „filter“**. For this purpose we consider the application of white noise to a band filter of bandwidth $\Delta\omega \ll \omega_o$ (ω_o is the center frequency). For the sake of simplicity, the filter characteristic is assumed to be of the **ideal rectangular shape**. In this case, the transfer function

$$\mathrm{H}(\omega) = K = const. \qquad \text{for} \qquad \omega_o - \frac{\Delta\omega}{2} < \omega < \omega_o + \frac{\Delta\omega}{2}$$

and

$$\mathrm{H}(\omega) = 0 \qquad \text{elsewhere.}$$

The output spectral density is now related to the input density by eq.(10) :

$$\mathrm{S_o}(\omega) = |H(\omega)|^2 S_i(\omega) = N_0 K^2 \quad \text{for} \qquad \omega_o - \frac{\Delta\omega}{2} < \omega < \omega_o + \frac{\Delta\omega}{2}$$

and zero elsewhere.

By using the **Wiener-Khinchin Theorem**, the autocorrelation function becomes now

$$\text{(11)} \qquad \mathrm{R}(\delta) = \frac{1}{2\pi}\int_o^\infty \mathrm{S_o}(\omega)\exp(-i\omega\delta)\,\mathrm{d}\omega = \frac{N_o K^2}{2\pi}\int_{\omega_o - \Delta\omega/2}^{\omega_o + \Delta\omega/2} \exp(-i\omega\delta)\,\mathrm{d}\omega$$

$$= N_o K^2 \frac{\Delta\omega}{2\pi} \frac{\sin(\Delta\omega\,\delta/2)}{\Delta\omega\,\delta/2} \times \cos(\omega_o\delta)$$

which reduces to

$$\text{(11a)} \qquad \mathrm{R}(0) = N_o K^2 B = \langle p \rangle$$

for δ = 0, which is basically the same relation as eq.(9).

Eq.(11) again contains the familiar **sin(x) / x** relation, as **known from the experimental work in Part Two**, and is **maximum for δ = 0**.

In the above calculations, the **transfer function** was assumed to be a **constant, K**. In the Lock-In we have, however, a **different „transfer profile"** which must be taken into account in a more exact calculation. If you will do so, using again the Wiener-Khinchin Theorem, you will notice that the **essential results** of our above calculation remain **unaltered** :

The auotocorrelation function, and hence the **transmitted average noise power**, is directly **proportional to the bandwidth** BW = $\Delta\omega/2\pi$. Hence the **rms noise voltage** at the output is **proportional to the square root of the band width**.

This is the **answer to our question at the beginning of this chapter**. You should **check your experimental data** whether or not they fulfill these theoretical considerations.

Resonance Absorption

In the previous sections of this course you - hopefully ... - learned a lot about the basic function principles of the Lock-In Amplifier. Now, in Part Three, we will see how the device works within a typical **experimental setup.**

In our Advanced Physics Lab at the University of Konstanz we use the Lock-In in four experiments:

(A) the "classical" (cw) Electron Spin Resonance experiment,
(B) detection of Surface Plasmons in an evaporated metal film,
(C) in a "phonon assisted" tunnelling experiment, and
(D) in our "Noise Analysis" experiment where the Lock-In is demonstrated "in praxi".

(We could have added here the Raman effect, but here we actually use photon counting instead of Lock-In detection.)

In the first two experiments, the physical systems respond to an **external stimulation** by an **electromagnetic radiation field** with a **resonant absorption of radiation power**. Without going too much into detail, let us only state that a **surface plasmon** is generated for a **definite angle of incidence** onto the glas prism carrying a gold film on its base. Under these conditions, the reflected light beam is more or less **attenuated**. In the case of the **ESR**, the absorption process occurs **at a certain magnetic field**, the frequency of the electromagnetic radiation being kept constant (given, e.g., by the dimensions of the resonant cavity).

In both cases, the desired response of the system occurs only under **special conditions** :

These conditions are experimentally searched for, in general, by **slowly sweeping** one of the variable input parameters of the experiment, e.g. the **angle of incidence** in the case of the **plasmon resonance**, or the **magnetic field** in the case of the *ESR*, respectively. When the correct conditions are encountered, the response of the physical system can be detected e.g. as **loss of energy** of electromagnetic radiation applied to the system which can be detected by rectifying the output radiation. This loss can be **very small**, resulting in a very small variation of the rectified dc output. Very small variations of relatively large dc signals, however, can be **extremely difficult to detect** due to possibly **strong noise** components and **inevitable drifts** within the system.

So, with our knowledge of the Lock-In we can **improve our situation** considerably : We simply superimpose a small **periodic variation** onto our slowly variable ("dc") "search parameter", i.e. we **modulate** this input parameter to the system.

As a consequence, the output signal from the physical system will show the periodicity of the modulating signal (plus **„harmonics“** of it, as you will see in **Part Three/2**), and can now further be processed with the known advantages of the Lock-In, namely the significant **improvement of the Signal-to-Noise ratio** of ac signals as shown in the previous Parts of this series.

The **actual experimental situation** is shown on the „user interface“ of the applet.

Again, you have **two screens** representing now a **"System" window** and an **output window**, respectively.

On the system screen, labelled **Experiment**, the **response of the system** under investigation is depicted in the form of a **"System Function"**.

In the present version of the experiment, you have the **choice** between a **Lorentzian resonant absorption curve**, $1/[(x - x_o)^2 - fwhm^2/4]$, where fwhm = full width at half maximum of the resonance curve is a convenient measure of the **"quality" of the resonance process** under investigation (remember the quality factor of a band pass filter in **Part Two**) , and a **series of straight lines**, interconnected by **adjustable "kinks"** (or "bends", if you prefer ...).

In order to generate the above mentioned **surface plasmon**, we have to **vary the angle of incidence** onto the glas prism which carries the gold film on its "back" side. In our real lab experiment this is done by a **mirror** mounted on a small **rotation table** which reflects the beam of a laser diode onto the prism. A **mechanical vibration** is added to the **continous rotation** movement by fixing the mirror at the end of a ***piezoelectric bimorph*** which is driven **sinusoidally** by the built-in **reference generator** of the Lock-In Amplifier.

When we approach the **resonance condition** by the **continuous variation** of the angle of incidence, we **"scan"** across the (Lorentzian) resonance curve by the **mechanical vibration** of the mirror.

This situation is shown, as an example, on the **Experiment** Screen: The **light input** to the plasmon generating system - the gold film - is visualized by a **red vibrating "beam"** at the bottom of the screen. A **sinusoidal angular vibration** is superimposed as the beam **scans continuously** across the resonance curve. The actual **output** of the system,the reflected light intensity as detected by a photodiode, is shown in yellow at the right side of the input screen.

As you will easily notice, this output is **everything but sinusoidal**, in general, due to the **nonlinearity** of the scanned **system response curve**.

This **yellow** output signal is now **fed into the input** of a Lock-In and processed by the PSD, exactly in the same way as the input signal in the Lock-In Schematic I in **Part One** of this series.

The **output from the Lock-In** is shown on the **righthand "Lock-In Output"** screen.

In the **tunnelling experiment** mentioned above we don't have a resonant absorption curve, but, instead, a series of **"straight lines"**, with **different slopes**, embedded in the **I(V) curve of a silicon backward diode**. The most interesting features with this situation are best investigated in Part III/2 of this series with the **"2f-mode"** of the Lock In, but, for **comparison with the resonance line**, you should have a look at the output in this case, too.

The Lock-In settings:

The following parameters are invariable and pre-set :

the phase $\Delta = 0$, i.e. maximum output from the PSD,
Input Signal Level = 1V,
Time Constant = 1T.

The **adjustable parameters** are as follows:

The vibration, or **modulation amplitude** (the **red** sine curve at the bottom of the Input Screen), adjustable from 0.1 to 4*fwhm (see above),
The Lock-In **output gain**, from 0.5 to 30.
You can add "static" **noise** to the system's response function.
You may change the **system function** between **"Lorentzian"** and **"Straight Lines"**.

In the **Straight Lines** system, you can **adjust the situation at will by "dragging"** the **markers at the kinks**, with the **left mouse key pressed**.

For our actual **Lorentzian** (green on the left screen) the **fwhm is set to 20 pixels**. The **peak-to-peak amplitude** of your **modulation signal** in the applet can be **adjusted in units of fwhm**.

The actions of the buttons on the applet's front are, hopefully, self explaining.
Try them out !

Note: **At first glance**, the **output** from the Lock-In looks like the **derivative** of the original function. But, in general, this is **not** the case. You can test it by **integrating the output curve** and visual **comparison with the original function**, the Lorentzian. As you will see, the **shape of the output plot** depends very much on the **amplitude** of the modulation. This behavior can also be investigated very well by using the **Straight Lines** as the system function.

Exercises and Questions:

(1) At first glance, the output curve from the Lock-In looks like the **derivative of the "System Function"**. In general, this is, however, **NOT** the case, as you may explore by variation of the (red) "modulation" amplitude.

In order to see this behavior, take several output plots for **different modulation amplitudes**, **integrate** them, and determine the fwhm‘s of these curves.

Question : At what conditions do you really get the "derivative" of the original system function ? What are the reasons ?

(2) Generate a system function with noise. With diminishing modulation amplitude, the output signal becomes smaller and smaller, whereas the relative output noise , i.e. the N/S ratio, is rising.

Exercise : Find an"optimum" modulation amplitude setting in order to get a reasonable output, a maximum S/N ratio, and a good reproduction of the system function (after integration).

Consider a real situation : You are looking for a very small, very noisy signal.

Questions :
(a) What modulation amplitude would you begin your search with ? Why ?
(b) Can you use this ampliude reasonably for your measurements ?
(c) What criteria do limit your maximum usable modulation amplitude ?

(3) As you know from the previous Parts, at the "zero crossing point" in the output function, the input signal into the LI has to be either zero or have a phase difference of 90^o with respect to the reference sine.

As you can easily verify from the simulation (watching the **yellow input signal** on the lefthand screen, by **stepping** through the zero crossing region of the output), these conditions are evidently **NOT** met in this case (i.e. the signal is certainly not zero, and the phase is set to zero by default).

Question : Why do you get this zero output signal, without the above mentioned conditions being fullfilled ?

(4) Since the output of the system (the yellow curve on the lefthand screen) represents some kind of **"mirroring" of the modulation amplitude at the system function**, this output contains all nonlinearities of the system function in the form of admixtures of harmonics to the fundamental frequency.

<u>Question :</u> What would happen if you fed the PSD of the Lock-In with a **harmonic** of the input frequency as the reference ? What would the Lock-In be sensitive for in this case ?

(5) When looking at the Lock-In output with the "Straight Lines" system function:
How does the output look like at the **kinks** ?
On what does the height of the output steps depend ?
If you **differentiate** the output signal, what kind of curve do you expect ?

The „2f“ Mode

Before proceeding, you should be familiar with **Part III/1**.

As we have seen there, the **output from the experiment** (the input signal into the Lock-In) results from a **"mirroring"** of the sinusoidal **modulation** of the experimental "scan" parameter at the **"system function"** of the experiment. In the case of a nonlinear system function, as is the Lorentz function, or the "straight lines" system at the kinks, this process leads to a **deformation** of the output signal with respect to the original sine; in other words, there will be a **characteristic amount of admixture** of **harmonics** to the fundamental modulating sine, the amplitudes of the harmonics clearly depending on the **nonlinearity** - or **curvature** - of the system function.

These **harmonics may be detected by phase sensitive detection**, just as it was the case with the fundamental frequency. In this case, of course, you have to apply a **harmonic** frequency to the **reference input** of the PSD. For the detection of the first harmonic content, you feed the double frequency, 2f, into the reference input of the PSD, for the second harmonic 3f, and so on.

The effect of **harmonics generation** can best be seen in the previous applet (**Part III/1**) at the **minimum point** of the model system function, the Lorentzian absorption curve, and at the **kinks** of the Straight Lines system. Here, as one can see from a **Fourier analysis** of the yellow experimental output signal on the left screen, there is no longer any contribution from the fundamental, but, instead, a **very strong signal with the double frequency**. This effect leads to a **zero output** signal - the "zero crossing" - in the **"1f"** Lock-In output curve.

If you apply, however, **2f** to the reference input of the PSD, you will get a **maximum** Lock-In signal output at this special point.

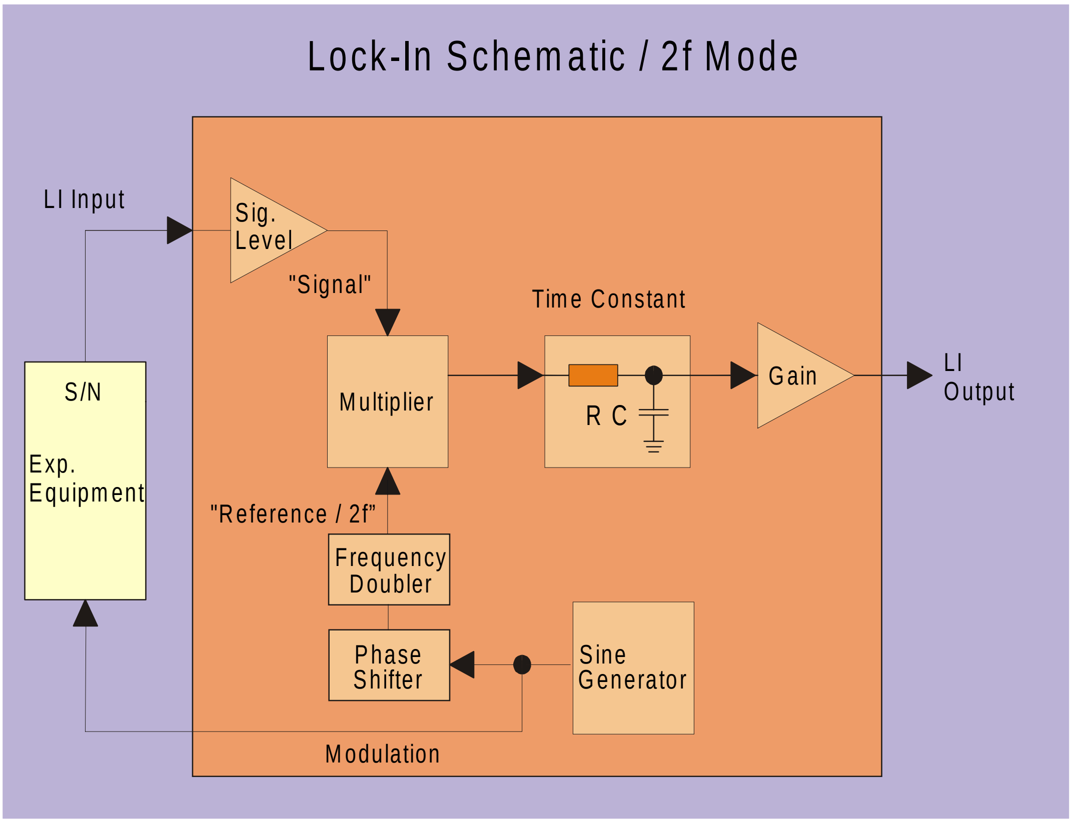
As one may argue from this experiment - and can show exactly by analytical methods - the **2f output** corresponds to the **second derivative (curvature) of the system function**, under the ***same conditions*** that hold for the experimental parameters to make the fundamental Lock-In output to be proportional to the **first derivative** (i.e. for very small modulation amplitudes).

This feature is of special interest in the case of system functions which exhibit **kinks**, or points where the slope of the function changes. By differentiation you will get a "step" in the first derivative and a **spike** in the second one (as you probably already know from **question #5** in the previous Part ...).

In our Physics Lab, we encounter this case in the **"Phonon Assisted Tunneling"** experiment, where electrons in Silicon, at a temperature of 4.2 K, are accellerated by an applied voltage and gain an appropriate momentum to "tunnel" through the energy gap between the valence and conduction band in Si. At specific energies, eU, there opens a new tunneling, or conductivity, channel which **steepens the slope** of the I(U) function of the backward diode. These situation can favorably be investigated with the **2f mode** of a Lock-In amplifier, resulting in "peaks" where the changes in conductivity occur.

This situation is nicely simulated with the „Straight Lines“ system function in this Part.

As usual, you should first have a look at the **2f mode schematic**.



Here, the **2f reference** is generated **between the phase shifter and the PSD input** by **doubling** the output from the Sine Generator. At the output of the frequency doubler, the 2f reference is by default set to be "in phase" with the 1f signal input and to have the same amplitude ($1V_{eff}$, to name it).

The **experiment** is **modulated** - as before - with frequency **1f !**

Note that the Phase Shifter acts on the 1f sine from the Sine generator. This means that a phase shift of , e.g., π in 1f corresponds to a shift of 2π in the frequency doubled signal. For maximum LI output in the present 2f case, the phase on the Lock-In panel could have to be re-adjusted. (The **correct phase setting**, of course, **depends on the place where the frequency doubler is located: before or behind the phase shifter**; in our case, it's behind).

This situation is **not very easy to understand**, so you may think a little bit about it ...

Experiments and Questions:

(1) Why does the maximum Lock-In output signal appear at a phase shift of 45º and NOT at 0º, as in the "1f" case ?
(Hint : Compare the - yellow - input into the Lock-In at the minimum of the Lorentzian with the Fourier series expansion of f(x) = | sin(x) | ....)

(2) What would be the phase setting for maximum output if, in the Lock-In schematic above, the phase shifter and frequency doubler were interchanged ?

(3) Under what circumstances will the LI output truly represent the 2nd derivative of the system function ? Would you like to work, in reality, with these conditions ?

(4) Simulate a "phonon assisted" conduction curve, as described above and watch the output signal. Compare the result with your considerations from question #6 in Part III/1. Integrate twice (with adjustment of IC1) and try to reproduce the lefthand lines configuration.

(5) Repeat the scan with the phase reversed (i.e. -45º, instead of +45º). Why will you not be able to reconstruct the original lefthand lines configuration in this case ?

What parameters determine the height of the "spikes" in the output signal ?

(6) Suppose you measured a Lock-In signal in the normal 1f mode. Now, you simply switch over to „2f“ and repeat the measurement.
What **two** parameter settings of the instrument should be readjusted, and why ?

The Lock-In Analyzer

As we have seen in **Part Two**, the output of the Lock-In amplifier depends on the cosine of the pase difference between the "signal" and the "reference" input into the Phase Sensitive Detector (assuming the two signals to have the same frequency). So, in order to get the maximum output amplitude, you have to adjust the setting of the phase shifter very carefully to $\Delta = 0$. But, given a very small, noisy input signal, how do you get this special setting ?

Well, of course, you can **try** (and **hope** ....).

More promising, however, is the following **procedure** :

Choose an arbitrary phase setting and watch the output signal. Then vary the phase until the output becomes zero. Now you have a phase difference of $90°$ or $270°$ between the input and the reference signals. The rest is easy: Make a phase change of $\pm 90°$, and you have got it ....

This was in fact the proper, old fashioned, method to find out the optimal phase setting .... until the invention of the Lock-In Analyzer. This instrument **doesn't need any phase setting from the user's side**.

How does the Analyzer work ?

In **Part Two**, we used the auto-correlation function

$$\text{(1)} \qquad R_1(\Delta) = \frac{ab}{nT}\int_0^{nT} \sin(\omega t) \times \sin(\omega t + \Delta)\, dt$$

with the result

$$\text{(2)} \qquad R_1(\Delta_1) = U_1 = \frac{ab}{2} \times \cos(\Delta_1)$$

If we now **shift** the phase angle Δ_1 in eq.(2) by an amount of $\pm \pi/2$, then we can use this **new phase setting** $\Delta_2 = \Delta_1 \pm \pi/2$ equivalently in our Correlation Function

$$\text{(3)} \qquad R_2(\Delta_2) = \frac{ab}{nT}\int_0^{nT} \sin(\omega t) \times \sin(\omega t + \Delta_2)\, dt = \frac{ab}{nT}\int_0^{nT} \sin(\omega t) \times \sin(\omega t + \Delta_1 \pm \frac{\pi}{2})\, dt$$

with the result

$$\text{(4)} \qquad R_2(\Delta_1) = U_2 = \pm\frac{ab}{2} \times \sin(\Delta_1)$$

Now, we have **two output voltages**, U_1 and U_2, depending on Δ_1 and Δ_2, respectively, which we now may **add as polar coordinates** to get the **output voltage**

$$(5) \qquad U_{out} = \sqrt{U_1^2 + U_2^2} = \sqrt{(\frac{ab}{2})^2 \times [\cos^2(\Delta) + \sin^2(\Delta)]} = \frac{a}{\sqrt{2}} \times \frac{b}{\sqrt{2}} = a_{rms} \times b_{rms}$$

Similarly, we now can calculate the **phase difference** between the input and the reference signals:

$$(6a) \qquad \tan(\Delta) = \frac{\sin(\Delta)}{\cos(\Delta)} = \frac{U_2}{U_1}$$

or

$$(6b) \qquad \Delta = \tan^{-1}(\frac{U_2}{U_1})$$

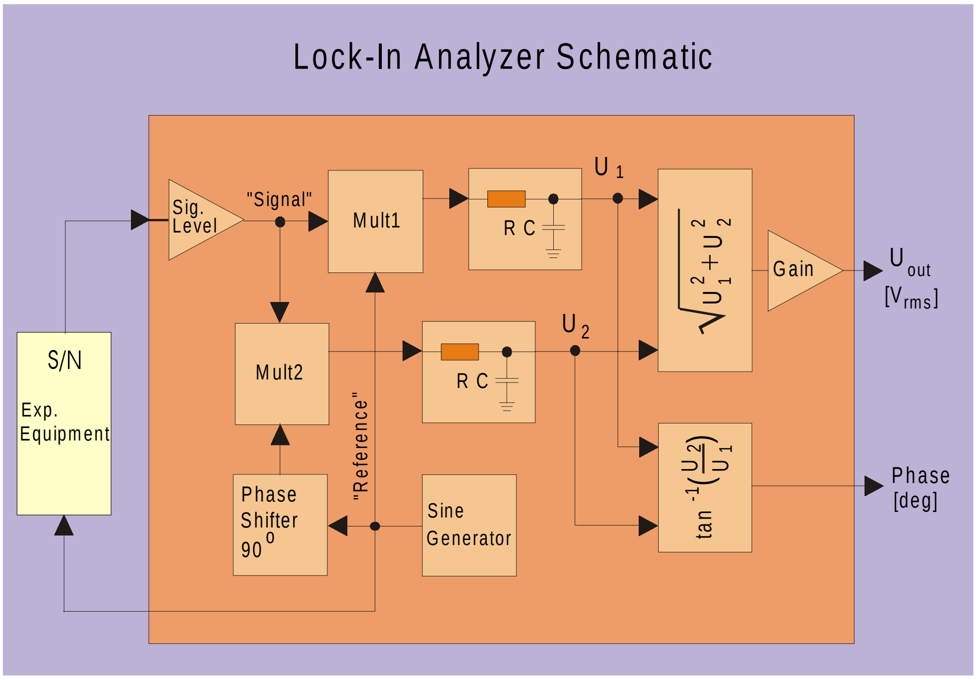
If you adjust the reference voltage, $a_{rms,}$ to unity, then the **output reading U_{out}** will be b_{rms}, that is: the **rms signal** you are looking for, without any phase adjustments.

In this way, the Lock-In Analyzer gives you **two** output voltage values :

one for the **rms value of the input signal**, and the other one for the **phase difference** between signal and reference.

"In reality", the necessary mathematical operations in the above calculations are performed by specialized integrated analog circuits,the so-called "Multifunction Converters" (eg. Burr-Brown 4302).

Now, at the latest, you should have a look at the **Schematic IV** (below).



Questions:

(1) What is the reason of the **instability of the phase reading** in the vicinity of ± 90° ?

Does the phase reading cover the whole phase range 0 ... 360° ?
If not so : why ?

(2) How would the outputs of the Lock-In Analyzer look like if you scan a "system function", like the ones in **Part Three** ?

(3) For what kind of measurements would you prefer to use the Lock-In Analyzer, for what the conventional Lock-In amplifier ?