FINAL PROBLEM

The students each worked a problem of some complexity during the last few weeks of the semester; they were offered a choice of two problems suggested by Dr. Keith Symon of MURA. One concerned the phase stability condition for a Fixed Field Alternating Gradient Synchrotron; the other approximated an electron gas in one dimension as a collection of parallel planes of charge in motion. In both of these considerable thought on the part of the students was necessary before the programming actually began. Unfortunately, little time was available on the computer during the last two weeks of the semester, so a complete checkout was not possible.

ACKNOWLEDGMENTS

My debt to Douglas Smith (Computer Center, University of Alaska), Dr. Harold Leinbach (Physics Department, State University of Iowa), and Dr. Keith Symon (Midwestern Universities Research Association) has already been mentioned. Other useful suggestions were made by Dr. Robert Simpson (Physics Department, University of Alaska), and Dr. Kenneth Clark (Physics Department, University of Washington). Many others too numerous to mention gave encouragement to the course.

Radiance*

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(Received 22 October 1963)

Wide generality for optical radiometry can be achieved by treating the basic radiometric quantities as field quantities. The treatment is that of classical ray optics, with emphasis on the geometrical relations involved. It is shown that radiance, defined as

$$N \equiv \frac{\partial^2 P}{\partial \Omega \cos\theta \partial A} \; [\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}],$$

the radiant flux or power per unit solid-angle-in-the-direction-of-a-ray per unit projected-area-perpendicular-to-the-ray, has the same value at any point along this ray within an isotropic medium, in the absence of losses by absorption, scattering, or reflection. More generally, the quantity N/n^2 (where n is the index of refraction of the medium) in the direction of a ray is shown to be invariant along that ray, even across a smooth boundary between different lossless media. The usefulness of this invariant property of radiance is illustrated by examples of practical applications.

INTRODUCTION

WHY does a photographic exposure meter give the same reading over a wide range of distances from a uniformly illuminated blank wall with a rough, weathered surface? Of course the indication changes when it is held so close to the wall that its shadow, or that of the supporting arm and hand, reduces the illumination, or when it is held so far away that radiation is also received from the surrounding background beyond the edges of the wall. But between these extremes, the indication will remain constant. Nor is it possible with an external lens, however large or "fast," to focus more radiant power from the wall onto the exposure meter to obtain a higher reading.

Why does a spectroscopist obtain maximum energy with the following procedure? Focus an image of the source onto the entrance slit of the spectroscope. Arrange the source and focusing optics so that the image is just large enough to fill the slit completely (if the source is not uniform, the slit must be filled by the brightest uniform region of the image), and so that the rays which cross to form the image diverge widely enough after passing through the slit that

* This work was supported by Signal Corps Contract DA 36–039 SC-87499.

they completely fill the collimating optics of the spectroscope. Once this condition has been achieved, it is useless to attempt to focus more radiant power through the instrument from the same source.

If it is given that the earth's surface radiates a total (in all directions) of w W/cm^2, how can we quickly estimate the irradiance H, in W/cm^2 due to this source alone without atmospheric attenuation, at a horizontal receiving surface carried on an earth satellite vehicle? For simplicity, assume that the earth is a perfectly diffuse radiator.

These situations all involve extended sources of radiation. It has been my experience that most people find it peculiarly difficult to master the fundamental concepts and relations of radiometry (or photometry) as they apply to extended sources. And I have come to believe that the key to this difficulty lies in the interrelated concepts of an elementary beam of radiation and its radiance (or luminance).

A brief definition of radiance is given in the abstract and is discussed in more detail directly. It is helpful to note here that radiance is analogous to the familiar property of visual brightness, or more exactly to the photometric quantity luminance. Also for convenience, Table I lists the radiometric quantities, symbols, and units used in this paper.

Sometimes radiance is defined as applying only to sources of radiation.[1–3] Frequently it is applied also to images of a source, and it is shown that, in the absence of attenuation, the radiance (or luminance) of an image is equal to that of the source in the direction of any ray reaching it from the source.[3]

The usefulness of defining radiance more broadly as a field quantity which can be evaluated at any point along a ray is also pointed out, particularly with reference to diffuse sources such as a volume of emitting gas.[1] It is established that radiance has the same value everywhere and in all directions, within an isotropic region in thermal equilibrium.[4,5] However, I have not found anywhere a completely general treatment of the invariant property of radiance, defined as a field quantity, although such a treatment can greatly simplify the understanding of many radiometric situations, as well as the computation or estimation of the amount of radiant power or flux incident on a receiver or detector, or passing through some aperture of interest, in a wide variety of circumstances. None of the material presented here is entirely new, rather it is implicit in many publications, but it is not explicit in any that I am aware of.

For generality, we define radiance as a field quantity which can be evaluated at any point on any surface through which radiant power or flux is passing. This includes, but is not restricted to, the surfaces of a source, a receiver, or any intermediate optical element such as a mirror, lens, or stop (aperture limiting a beam of radiation). On this basis, radiance is defined as the radiant flux or power per unit solid-angle-in-the-direction-of-a-ray per unit projected-area-perpendicular-to-the-ray. More precisely, in a given direction from a point on a surface through which radiant energy is passing,

$$N \equiv \partial^2 P / \partial\Omega \cos\theta \partial A \quad [\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}], \quad (1)$$

where N = the radiance at that point in the given direction, P = the radiant flux or power flowing through the surface (within the solid angle Ω and the area A) [W], Ω = the solid angle filled by the rays along which the radiation is propagated (including, of course, the ray extending in the given direction through the given point of the surface) [sr], A = the area of the surface (including, of course, the given point) [cm^2], and θ = the angle between the given direction and the normal to the surface at the given point [dimensionless]. We return to this definition later as the

[1] "Report of WGIRB (Working Group on Infrared Backgrounds)—Infrared Target and Background Radiometric Measurements—Concepts, Units, and Techniques," Report 2389–64-T, NAVEXOS P-2406 (IRIA, Institute of Science and Technology, University of Michigan, Ann Arbor, Michigan, January 1962), pp. 3–4 Contract No. NOnr-1224(12).

[2] E. E. Bell, Proc. Inst. Radio Engrs. **47**, 1432 (1959). Essentially the same material also appears as "Report of the Working Group on Infrared Backgrounds; Part II: Concepts and Units for the Presentation of Infrared Background Information," Report No. 2389-3-S (Engineering Research Institute, University of Michigan, Ann Arbor, Michigan, November 1956), Contract No. NOnr-1224(12).

[3] F. A. Jenkins and H. E. White, *Fundamentals of Optics* (McGraw-Hill Book Company, Inc., New York, 1950), 2nd ed., pp. 104–108.

[4] M. Planck, *Theory of Heat*, translated by H. L. Brose (Macmillan Company, New York, 1957), Vol. 5, Chap. II, pp. 183–196.

[5] F. K. Richtmyer and E. H. Kennard, *Introduction to Modern Physics* (McGraw-Hill Book Company, Inc., New York, 1947), 4th ed., p. 145.

TABLE I[a]. Radiometric quantities, symbols, definitions, and units.

Quantity	Symbol	Defining relations	Units
Radiant energy	U		J
Radiant energy density	u	$u \equiv \frac{\partial U}{\partial V}$	$\mathrm{J \cdot cm^{-3}}$
Radiant power	P	$P \equiv \frac{\partial U}{\partial t}$	watt (W)
Radiant intensity	J	$J \equiv \frac{\partial P}{\partial \Omega}$	$\mathrm{W \cdot sr^{-1}}$
Radiant emittance	W	$\left.\begin{matrix} W \\ H \end{matrix}\right\} \equiv \frac{\partial P}{\partial A}$	$\mathrm{W \cdot cm^{-2}}$
Irradiance	H		
Radiance	N	$N \equiv \frac{\partial^2 P}{\cos\theta \partial A \partial \Omega}$	$\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}$
Wavelength	λ		micron (μ)
Spectral radiant power	P_λ	$P_\lambda \equiv \frac{\partial P}{\partial \lambda}$	$\mathrm{W} \cdot \mu^{-1}$
Spectral radiant intensity	J_λ	$J_\lambda \equiv \frac{\partial J}{\partial \lambda}$	$\mathrm{W \cdot sr^{-1}} \cdot \mu^{-1}$
Spectral radiant emittance	W_λ	$W_\lambda \equiv \frac{\partial W}{\partial \lambda}$	$\mathrm{W \cdot cm^{-2}} \cdot \mu^{-1}$
Spectral irradiance	H_λ	$H_\lambda \equiv \frac{\partial H}{\partial \lambda}$	$\mathrm{W \cdot cm^{-2}} \cdot \mu^{-1}$
Spectral radiance	N_λ	$N_\lambda \equiv \frac{\partial N}{\partial \lambda}$	$\mathrm{W \cdot cm^{-2} \cdot sr^{-1}} \cdot \mu^{-1}$
Radiant emissivity	ϵ	Ratio of "emitted" radiant power to that from an ideal blackbody at the same temperature.	
Radiant absorptance	α	Ratio of "absorbed" radiant power to incident radiant power.	
Radiant reflectance	ρ	Ratio of "reflected" radiant power to incident radiant power.	
Radiant transmittance	τ	Ratio of "transmitted" radiant power to incident radiant power.	

Note:
The spectral radiant emissivity $\epsilon(\lambda) \equiv W_\lambda / W_{\lambda,BB} \neq \partial\epsilon/\partial\lambda$. Hence, the subscript notation ϵ_λ, which could be confused with $\partial\epsilon/\partial\lambda$, is not recommended, although it is often used. Similarly, it is recommended that the spectral absorptance, spectral reflectance, and spectral transmittance be written as $\alpha(\lambda)$, $\rho(\lambda)$, and $\tau(\lambda)$, respectively.

[a] See footnotes 1 and 2.

basic radiometric quantities are presented in a logical sequence leading up to the proof of the invariance property. It is shown that the value of N in the direction of any ray has the same value at all points along that ray within an isotropic medium, in the absence of losses by absorption, scattering, or reflection. More generally, the quantity N/n^2 (where n is the index of refraction of the medium) in the direction of a ray is shown to be invariant along that ray, even across a smooth nonreflecting boundary between different lossless media. The treatment of real situations, where absorption, scattering, and reflection can not be neglected, is also discussed.

ANALYSIS

We define a radiation field as a region in which radiant power is propagated, at a velocity characteristic of the region or medium and independent of direction, along straight noninterfering rays which may pass in any direction through any point within the region. The radiant power or flux may vary with position and direction, but only in a continuous manner, so that a finite amount of power can flow only through a finite area and a finite solid angle. Thus, here and in actuality, there is no such thing as a point source, with all of the power traveling along rays which intersect at a single mathematical point; nor is there such a thing as a perfectly collimated beam with all of the power traveling along rays which are perfectly parallel.[6]

In order to analyze this situation, let us first consider only the distribution of power flow as a function of direction. We define an elementary pencil of rays through a point Q as including all of the rays which pass from Q through an element of area dA at a distance D from Q which is very large in relation to the linear dimensions of dA (see Fig. 1). The solid angle subtended at Q by dA is given by

$$d\Omega = \cos\theta dA/D^2 \quad [\mathrm{sr}],$$

where θ is the angle between the normal to dA and the pencil of rays, i.e., $\cos\theta\cdot dA$, is the projected area of dA normal to this pencil. If we next consider Q not as a mathematical point, but as having dimensions which, however, are very small compared to the dimensions of dA (and, hence, extremely small compared to D), the power which flows along the rays in this pencil from Q to dA can be expressed as

$$dP = Jd\Omega = J\cdot\cos\theta dA/D^2 \quad [\mathrm{W}],$$

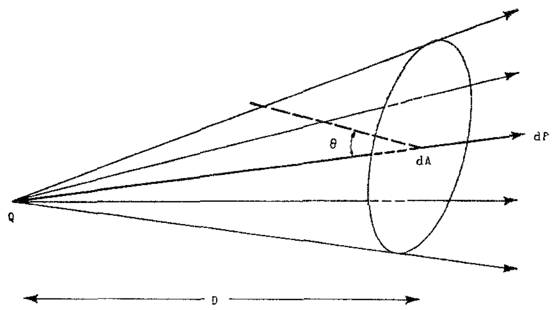

FIG. 1. An elementary pencil of radiation.

where

$$J \equiv \partial P/\partial\Omega \quad [\mathrm{W\cdot sr^{-1}}] \tag{2}$$

is defined as the *radiant intensity* (see Table I) of Q as a "point" source of radiation (e.g., a virtual source, such as the image of an illuminated pin hole) *in the direction of the pencil* (the direction of dA from Q). Note that J may vary with direction and is a constant only for an isotropic source. In general, the power received from a distant "point" source is given by

$$P = \int J d\Omega \quad [\mathrm{W}], \tag{3}$$

where the integration is carried out over the entire solid angle subtended at the source by the receiver.

For completeness, we also look briefly at the purely spatial variation, although this quantity is more easily understood and is not so often a source of difficulty or misunderstanding. If we consider an element of surface dA, situated anywhere in a radiation field, the total amount of radiant power passing through it (either into it from a hemisphere, or out of it into a hemisphere) can be expressed as

$$dP = HdA \quad \text{or} \quad dP = WdA \quad [\mathrm{W}],$$

where

$$H \equiv \partial P/\partial A \quad [\mathrm{W\cdot cm^{-2}}] \tag{4}$$

is the *irradiance* (see Table I), the surface density of radiant power flowing into the surface at a point from a complete hemisphere (or, sometimes, from a stated solid angle which is less than a hemisphere), and where

$$W \equiv \partial P/\partial A \quad [\mathrm{W\cdot cm^{-2}}] \tag{5}$$

[6] The assumptions involved here are discussed with greater rigor by Planck. It can be shown that his analysis does not conflict with the results presented here, although there are apparent differences due to the very different terminology and symbols used. See reference 4, Secs. 94 and 95, pp. 173–177.

is the *radiant emittance* (see Table I), the surface density of radiant power flowing out of the surface at a point into a complete hemisphere. Both of these quantities may vary from point to point over an extended surface, so that the total radiant power flowing into the surface of a receiver is given by

$$P=\int H dA \quad [\mathrm{W}], \tag{6}$$

and that flowing out of the surface of a source is given by

$$P=\int W dA \quad [\mathrm{W}], \tag{7}$$

where the integration is carried out over the entire surface of interest in each case.

We now consider the simultaneous distribution of radiant flux in both space and direction by examining an elementary beam of radiation. The elementary beam of radiation between two elements of area dA_1 and dA_2, situated anywhere in a radiation field where they are separated by a distance D which is very large compared to the linear dimensions of either element of area, is defined as including all of the rays which pass from dA_1 to dA_2 (or from dA_2 to dA_1, since either may be the source and the other the receiver). By inspection of Fig. 2, it can be seen that the cross section of the beam at either end is determined by the projected area of the element at that end, i.e., by $\cos\theta_1 dA_1$ and $\cos\theta_2 dA_2$, respectively. Also, the solid angle subtended at the opposite end by each element is equal to this projected area divided by D^2 in each case, giving

$$d\Omega_1=\cos\theta_1 dA_1/D^2$$

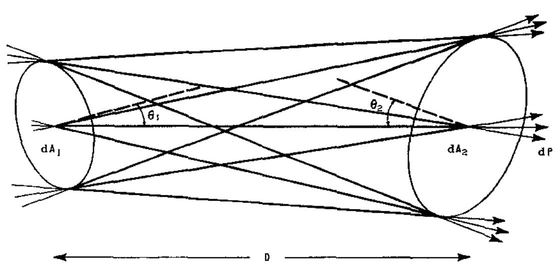

FIG. 2. An elementary beam of radiation between two surface elements dA_1 and dA_2.

and

$$d\Omega_2=\cos\theta_2 dA_2/D^2 \quad [\mathrm{sr}]. \tag{8}$$

If we compute the projected-area-solid-angle-product (which has also been referred to as the "throughput"[7]) at each end we have

$$\left.\begin{aligned} dT_1&=\cos\theta_1 dA_1 d\Omega_2=\cos\theta_1 dA_1\cdot\frac{\cos\theta_2 dA_2}{D^2},\\ dT_2&=\cos\theta_2 dA_2 d\Omega_1=\cos\theta_2 dA_2\cdot\frac{\cos\theta_1 dA_1}{D^2}.\\ \text{But}\quad dT_1&=dT_2=dT \quad [\mathrm{cm}^2\cdot\mathrm{sr}]. \end{aligned}\right\} \tag{9}$$

Next, let us recall the definition of *radiance*, the power per unit projected area per unit solid angle at a point and in a particular direction, as

$$N\equiv\frac{\partial^2 P}{\partial\Omega\cos\theta\partial A} \quad [\mathrm{W\cdot cm^{-2}\cdot sr^{-1}}].$$

If the radiance at dA_1 is N_1 and that at dA_2 is N_2, the power flowing through each surface element is given, respectively, by

$$dP_1=N_1\cos\theta_1 dA_1 d\Omega_2=N_1 dT,$$

and

$$dP_2=N_2\cos\theta_2 dA_2 d\Omega_1=N_2 dT. \tag{10}$$

But the same power is flowing through both of the surface elements that define the beam, since all rays through one also pass through the other and energy is conserved (we have postulated no loss), so

$$dP_1=dP_2 \quad\text{and}\quad N_1=N_2=N. \tag{11}$$

Since the choice of dA_1 and dA_2 is quite arbitrary, and they can define a beam between any widely separated points along a particular ray, it follows that *the value of N in the direction of a ray must be invariant along that ray within an isotropic medium.* The value of N at any point will vary with direction, and the value of N for rays from a particular direction (parallel rays) will vary with position on any surface which they intersect. Hence, in general, the flux passing through a given surface and within a given solid

[7] The term "throughput" appears in the instruction manual issued by Block Associates, Inc., for an interferometer spectrometer. It is not known who originated the term or the precise way in which he would define it, but it appears to agree with the way in which it is used here.

angle is given by

$$P = \int\int N \cos\theta dA d\Omega \quad [\text{W}], \tag{12}$$

where the integration is carried out over the entire surface (with respect to the projected area perpendicular to any given direction) and over all directions included within the given solid angle.

In order to generalize still further, we examine the situation where a beam of radiation passes through a smooth surface separating two media with different refractive indices (see Fig. 3). The power incident on any surface element dA through any element of solid angle $d\Omega$ in the first medium and that from the same beam emerging from the same surface element into a solid angle $d\Omega'$ in the second medium must be the same, if there are no losses by reflection, absorption, or scattering (we are still concerned purely with ray geometry and defer questions of Fresnel reflection losses, etc., until later). This power is given by

$$dP = N\cos\theta dA\, d\Omega = N'\cos\theta' dA\, d\Omega' \quad [\text{W}]. \tag{13}$$

By Snell's law of refraction, we write

$$n\sin\theta = n'\sin\theta' \tag{14}$$

$$\therefore n\cos\theta d\theta = n'\cos\theta' d\theta'. \tag{15}$$

Also,

$$d\Omega = \sin\theta d\theta d\phi \quad \text{and} \quad d\Omega' = \sin\theta' d\theta' d\phi \tag{16}$$

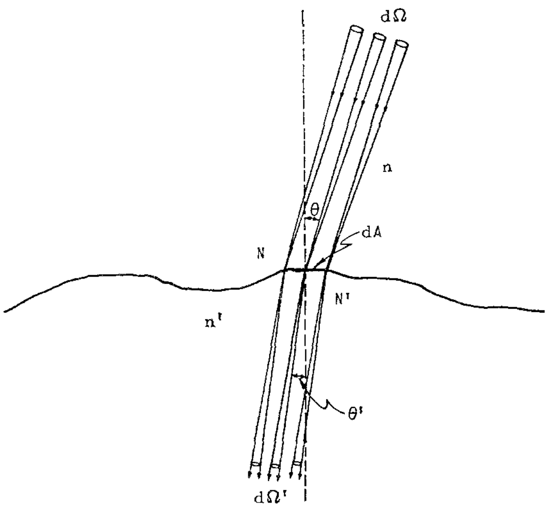

FIG. 3. Refraction at a smooth boundary between media of different refractive indices (n and n').

(the azimuth angle ϕ of a ray is not changed by refraction). Hence,

$$\frac{NdA\ \cos\theta d\Omega}{N'dA\ \cos\theta' d\Omega'} = \frac{N\cos\theta\ \sin\theta d\theta d\phi}{N'\cos\theta'\ \sin\theta' d\theta' d\phi} = \frac{Nn'^2}{N'n^2} = 1.$$

$$\therefore N/n^2 = N'/n'^2. \quad [\text{W}\cdot\text{cm}^{-2}\cdot\text{sr}^{-1}]. \tag{17}$$

Thus, *the quantity N/n^2 in the direction of a ray is invariant along that ray*, even as it passes through a smooth boundary surface between media of different refractive indices, if there are no losses by reflection, absorption, or scattering.[8] It is also apparent from Eq. (17) that the value of N in the direction of a ray will be the same at any point along such a ray which lies in a medium with the same index, regardless of passage through other media, such as refractive lenses, at intermediate points. Total reflection from a smooth surface merely changes the direction of a beam and does not alter N. This can be verified in detail by an analysis similar to that just given for refraction at a smooth boundary.

A "smooth" surface, as the term is used here, is defined to include any surface where it is possible everywhere to construct a tangent plane, i.e., where every surface element dA can be treated as common to the surface and to a plane tangent to the surface at that point.

It should be recognized, of course, that all real situations involve some losses by absorption, scattering, or reflection, although it is frequently possible to keep the losses to negligible amounts by careful design. Also, in real situations we are concerned with sources and receivers, not hypothetical elements of surface.

In the foregoing analysis, we have attempted to achieve as much generality as possible by considering radiation fields and surface elements placed in those fields with as few restrictions as possible. In this way, the results of the analysis can be extended widely to describe the radiometric quantities at the surfaces of almost any receiver of radiation in terms of those quantities at any source, or at any intermediate location where it may be convenient to specify or measure

[8] L. C. Martin, *Technical Optics* (Sir Isaac Pitman & Sons, Ltd., London, 1960), 2nd ed., Vol. 2, pp. 266–268. Martin's proof is given only for the luminance B of an image, but with only slight modification it, too, is easily generalized to apply to any point along a ray.

them. This has been done purely in terms of ray geometry, neglecting losses by absorption, scattering, and reflection. Such losses, where they are not negligible, can be accounted for by multiplying the radiometric quantities, determined from ray geometry only, by the appropriate factors. For example, if the radiance along a particular ray at a source is N_s, and the radiant transmittance (see Table I) along its path (taking into account the source spectrum and the spectral transmittance (see Table I) of the intervening medium) from source to receiver is τ, and source and receiver both lie in the same medium (same index of refraction), then the radiance at the receiver in the direction of this same ray is given by $N_r = \tau N_s$. If, in addition, this same ray has also been imperfectly reflected by a mirror at some point along its path to the receiver, the radiance at the receiver becomes $N_r = \rho\tau N_s$, where ρ is the radiant reflectance (see Table I) of the mirror.

A situation of even more importance, perhaps, is the effect of reflection loss on the radiance along a ray which is refracted at a boundary between two media of different refractive indices. Fresnel's equations require that there be some reflectance at any simple boundary between two such media. Here, if the radiance along the incident ray is N, and if the radiant reflectance for the particular angle of incidence is ρ, the radiance along the refracted ray in the second medium will be given by

$$N' = N(n'^2/n^2)(1-\rho) \quad [\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}]. \tag{18}$$

Note, however, that it may be possible to reduce ρ to a negligible value, at least for a limited range of wavelengths and angles of incidence, by the use of so-called antireflection coatings, thus approaching the condition described by Eq. (17).

PRACTICAL APPLICATIONS

Before describing some situations which illustrate the usefulness of the invariant property of radiance, we consider briefly the limitations governing the application of the concepts of radiance and radiant intensity to real sources. To look first at the extremes, it is obviously meaningless, with respect to a receiver at the earth's surface, to speak of the radiance of a star or the radiant intensity of a clear sky. Even the nearest star, in spite of its huge size, subtends such a small solid angle at the earth that it defines a pencil, rather than a beam, of radiation. Hence it is characterized by its radiant intensity. Conversely, the sky subtends so large a solid angle that, for most purposes it is necessary to recognize the variations in its radiance in different directions from a receiver at the ground.

Most sources fall into an intermediate category where they may be characterized by their radiance or their radiant intensity, depending on the distance of the receiver and the size of the smallest solid angle or resolution element which is considered significant. Thus a planet, like a star, will ordinarily define a pencil of radiation, i.e., its radiant intensity is the quantity of importance in determining the total radiant power reaching the aperture of a telescope on earth from the entire planet. However, the amount of radiant power in various portions of a highly magnified image of the planet Mars, for example, is a function of the radiance of the corresponding portions of the planet's surface in the direction of the earth. Similarly, the plume of a large missile in powered flight is clearly an extended source, with a complicated spatial distribution of varying radiance, with respect to a receiver at a distance of a few hundred feet. However, from high altitudes such a plume may be treated as a point source, characterized by its radiant intensity in

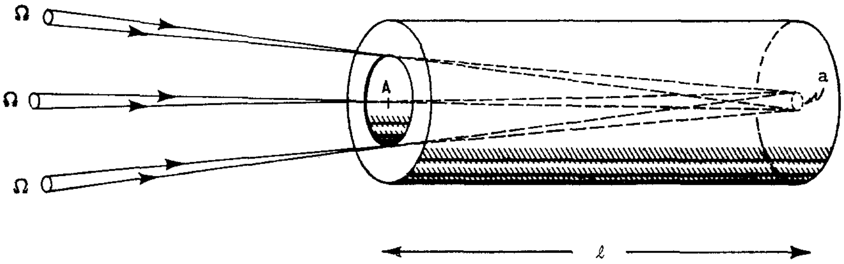

FIG. 4. A simple radiometer. A sensitive receiver (detector) of area a is located at one end of an opaque tube of length ℓ with an aperture of area A in the opposite end of the tube to limit the beam of rays incident upon the receiver.

the direction of a detection device on the ground in most instances where the field of view of the device is large enough to permit acquisition and tracking of the moving target without unusually complex and expensive equipment.

The usefulness of the invariant property of radiance can be illustrated in connection with a very simple radiometer consisting of a detector with a flat sensitive surface of area a, such as a common lead sulphide cell, mounted in a tube of length ℓ, with a limiting aperture of area A at the far end to define the beam of radiation incident on the detector (see Fig. 4). It is assumed that the ratio of ℓ to the linear dimensions of the detector is large enough so that, to the desired degree of accuracy, the detector subtends the same solid angle $\Omega = a/\ell^2$ at all points of the entrance aperture. If such an instrument is placed close enough to an extended source of uniform radiance N, the total radiant power incident on the cell (measured by its electrical response) can be written immediately as

$$P = \tau N A \Omega = \tau N A a/\ell^2 \quad [\mathrm{W}], \tag{19}$$

where τ is the total radiant transmittance over the path from source to cell. This is possible because we know that, in the absence of loss, the radiance along each ray at the entrance aperture is N, the same as at the source along the same rays. And at this position the rays fill an area A and a solid angle $\Omega = a/\ell^2$ at each point within that area. In practice, by measuring P from the response of the cell, and knowing τ, A, a, and ℓ, we can compute the radiance of an unknown *uniform* source as

$$N = \frac{P\ell^2}{\tau A a} \quad [\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}]. \tag{20}$$

If τ is unknown, as is frequently the case for atmospheric transmittance, we will measure instead the *apparent* radiance (at the radiometer):

$$N' = \tau N = \frac{P\ell^2}{A a} \quad [\mathrm{W \cdot cm^{-2} \cdot sr^{-1}}]. \tag{21}$$

Probably, in a practical situation, the values of A, a, and ℓ, will not be determined individually. Instead, the factor

$$T \equiv A\Omega = Aa/\ell^2 = P/N' = P/\tau N \quad [\mathrm{cm^2 \cdot sr}] \tag{22}$$

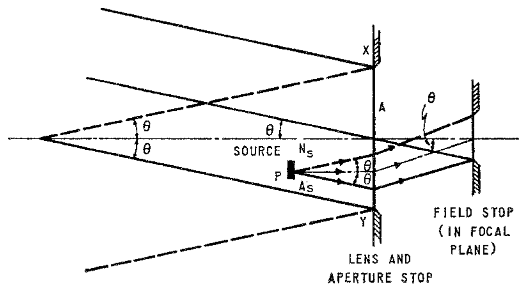

FIG. 5. Near-small-source (Jones method) calibration. Source completely within region bounded by XZ and YZ, which make an angle θ with the optical axis. θ = half-field angle. P = arbitrary point of source. Rays from any such point P within a cone of half-angle θ will uniformly irradiate the field stop as shown.

which has been called the "throughput," will be determined by a calibration measurement of a source of known radiance at a distance where τ is known or is practically unity (negligible attenuation).

Another illustration of useful application of the general property of invariance, at points where rays do not cross to form an image of the source, is in connection with the so-called Jones Method[9,10] for calibrating a radiometer. As shown in Fig. 5, a small source (area small compared with the area of the radiometer aperture) is placed close to the radiometer where it will uniformly irradiate the field stop of the instrument. In order to do this, the source must be small enough and close enough so that the radiometer aperture subtends, at every point of the source, an angle greater than the field angle of the instrument. As can be seen in Fig. 5, this means that the source must lie entirely within the region enclosed by the cone represented by the lines XZ and YZ, which meet at Z to form an angle equal to the field angle of the radiometer, as shown.

In the following approximate treatment, it is assumed that the optics are ideal, so that a parallel bundle of rays incident on the aperture stop of the instrument is sharply focused at a

[9] This method was first used by Dr. R. C. Jones in the fall of 1951 on Navy Contract NObsr 42179. The method became known by word of mouth, and was first described in writing in "An Unusual Method for Calibrating an Infrared Radiometer" (Polaroid Corporation, 19 September 1955, Memorandum 614) under Navy Contract NObsr 63175.

[10] "The Jones Method of Radiometer Calibration" *Techniques* (Barnes Engineering Company, winter, 1957). Reprints of this article have also been issued as Barnes Engineering Infrared Bulletin 030.

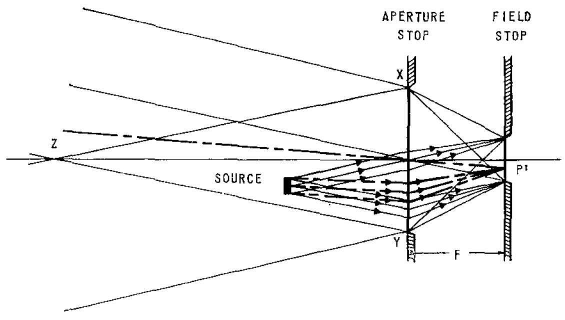

FIG. 6. Alternative illustration of Jones method of calibration. P' is an arbitrary point within the field stop.

single point lying in the field stop. Also, it is assumed that the field angle is small so that for the angle θ shown in Fig. 5, $\cos\theta \approx 1$.

Consider any point P on the radiating surface of the source. It is apparent in Fig. 5 that the rays emitted from P within a solid angle Ω equal to the field angle of the radiometer will uniformly irradiate the field stop. If the radiance of the source is N_s, and its projected area (normal to the optic axis) is A_s, the total radiant power thus radiated from all points of the source through the field stop (in the absence of absorption, reflection, and scattering losses) is given by

$$P = N_s A_s \Omega \quad [\text{W}]. \tag{23}$$

Alternatively, as illustrated in Fig. 6, we may consider any point P' within the field stop. All rays reaching P' from the source must travel parallel to each other in a single direction from the source to the radiometer aperture. There they will fill an area of the aperture equal to A_s, the projected area of the source. They are then focused onto P'. The effective aperture area A_s subtends a solid angle at P' given by

$$\omega \approx A_s / F^2 \quad [\text{sr}], \tag{24}$$

where F is the focal length of the radiometer optics. Since the radiance at the field stop in the direction of any of the rays from the source is equal to the source radiance N_s, we can write

$$P = N_s A_F \omega = N_s A_F A_s / F^2 \quad [\text{W}], \tag{25}$$

where A_F is the area of the field stop. Since the field solid angle is given by

$$\Omega = A_F / F^2 \quad [\text{sr}], \tag{26}$$

it can be seen that

$$P = N_s A_s A_F / F^2 = N_s A_s \Omega \quad [\text{W}], \tag{27}$$

in agreement with Eq. (23). Note that in this case the source is not imaged at the field stop where the value of radiant power P is being evaluated.

DISCUSSION OF INTRODUCTORY EXAMPLES

In the Introduction we described three situations involving extended sources. It should be clear from Eq. (13) that the exposure meter must give the same response, regardless of distance or orientation, as long as all of the radiation entering it has the same value of radiance. If the wall is perfectly diffuse, all of the rays from it will have the same radiance. An external lens cannot change either the area or angle through which the meter accepts radiation, whether it comes directly from the wall or after it passes through the lens, and the value of radiance also cannot be changed (except by attenuation, which is assumed negligible). In the same way, once the full receiving area (entrance slit) and solid angle (subtended at the slit by the collimating optics) of the spectrometer are filled with rays of the maximum available radiance, there remains no way to increase any of these quantities with additional lenses. In the last example, we must first determine the radiance N of the earth's surface. From Eq. (1) and Eq. (5) we can write for a uniform plane diffuse radiator for which both N and W are constants:

$$P = \int W dA = \int\int N \cos\theta dA d\Omega \quad [\text{W}],$$

$$\therefore W = N \int \cos\theta d\Omega \quad [\text{W}\cdot\text{cm}^{-2}]. \tag{28}$$

If we choose spherical coordinates with the z axis perpendicular to the radiating surface,

$$d\Omega = \sin\theta d\theta d\phi \quad [\text{sr}]$$

and

$$W = N \int_0^{2\pi} \int_0^{\pi/2} \sin\theta \cos\theta d\theta d\phi$$

$$= N[\phi]_0^{2\pi} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \pi N \quad [\text{W}\cdot\text{cm}^{-2}]. \tag{29}$$

Similarly, from Eq. (1) and Eq. (4), the irradiance at a plane surface due to uniform radiation of radiance N, arriving within a cone of half-angle θ, is given by

$$H=N\int\cos\theta d\Omega=\pi N\sin^2\theta \quad [\mathrm{W\cdot cm^{-2}}]. \qquad (30)$$

This gives the desired result for an earth of uniform radiance $N=W/\pi$ which subtends a cone of half-angle θ at the receiver ($H=W\sin^2\theta$).

SUMMARY

It has been established that in any radiation field radiance is invariant along a ray, in the direction of the ray, within an isotropic medium, and that the quantity N/n^2 is invariant along a ray, in the direction of the ray, across smooth boundaries between media with different refractive indices, so that N has the same value at all points along the ray lying in media of the same index, regardless of passage through other media at intermediate points. Practical applications of the usefulness of this invariant property have been presented. They show that it facilitates the evaluation of the radiant power flowing through any surface where it is possible to determine the cross section of a beam (the projected area of its intersection with the surface) and the solid angle from which rays are flowing through each point of that surface, if the value of radiance is known at any point along each of the rays. In practical optical systems, such surfaces are usually found at the stops (aperture stop and field stop) and their images (e.g., the entrance and exit pupils and windows), and at the surfaces of sources and receivers, and their images.

Evaluation of radiant power becomes a very simple matter for a beam passing through a well-defined plane surface (θ is not a function of position) of area A and within a well-defined solid angle Ω that is the same at all points of the surface (no vignetting) whenever it is possible to assume a uniform value of radiance N throughout the beam. The general expression in Eq. (12) can then be simplified as follows:

$$P=\int\int N\cos\theta dAd\Omega$$

$$=NA\int\int\cos\theta d\Omega=NA\Omega' \quad [\mathrm{W}]. \qquad (31)$$

The expression $\Omega'=\int\int\cos\theta d\Omega$ has been called the "weighted solid angle" or "projected solid angle".[11] The area-solid-angle-product, "optical invariant,"[12] "throughput,"[7] or "étendue"[13] can be given for the unvignetted beam through a plane surface or aperture of area A as

$$T=A\Omega'=A\int\int\cos\theta d\Omega \quad [\mathrm{cm^2\cdot sr}]. \qquad (32)$$

Furthermore, in most cases the solid angle is a circular cone of half-vertex-angle θ, with its axis perpendicular to the plane. Then, using spherical coordinates with the z axis perpendicular to the plane, the integration can be carried out thus:

$$T=A\int_0^{2\pi}\int_0^{\theta}\sin\theta\cos\theta d\theta d\phi$$

$$=\pi A\sin^2\theta \quad [\mathrm{cm^2\cdot sr}], \qquad (33)$$

and

$$P=NT=\pi NA\sin^2\theta \quad [\mathrm{W}]. \qquad (34)$$

ACKNOWLEDGMENT

In this report, the author has made extensive use of the ideas of others, gained from many stimulating and helpful discussions and from published books and papers, over an extended period. To attempt to list and acknowledge all of these sources would be impossible, and it has not been attempted except in the few instances where specific reference has been made to published items. However, the development of this treatment of the subject would have been impossible without such help and encouragement.

[11] R. C. Jones, J. Opt. Soc. Am. **50**, 1058 (1960).
[12] R. C. Jones, Appl. Optics **1**, 607 (1962). Although not flagged out, the concept of invariance of the $A\Omega$ product is also used by R. C. Jones in two other papers: J. Opt. Soc. Am. **43**, 138 (1953); and J. Opt. Am. **52**, 747 (1962).
[13] P. Connes, J. Phys. Radium **19**, 262 (1958). (I am indebted to Dr. R. Clark Jones for calling my attention to this reference.)