BANDWIDTH NORMALIZATION BY MOMENTS

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General measurement equation:

\[ SIG = \int_{0}^{\infty} \Re(\lambda) \Phi_{\lambda} d\lambda \]  \hspace{1cm} (1)

Represent \( \Re(\lambda) \) by a rectangle of height \( \Re_n \) between wavelengths \( \lambda_1 \) and \( \lambda_2 \):

\[ SIG = \Re_n \int_{\lambda_1}^{\lambda_2} \Phi_{\lambda} d\lambda \]  \hspace{1cm} (2)

Let the source function \( \Phi_{\lambda} \) be described as a second-degree polynomial:

\[ \Phi_{\lambda} = A + B\lambda + C\lambda^2 \]  \hspace{1cm} (3)

Substitute (3) into (1), divide both sides by \( \int \Re(\lambda) d\lambda \) and multiply both sides by \( (\lambda_2 - \lambda_1) \) to get:

\[ \frac{SIG \cdot (\lambda_2 - \lambda_1)}{\int \Re(\lambda) d\lambda} = \left[ A + B \frac{\int \Re(\lambda) d\lambda}{\int \Re(\lambda) d\lambda} + C \frac{\int \lambda^2 \Re(\lambda) d\lambda}{\int \Re(\lambda) d\lambda} \right] (\lambda_2 - \lambda_1) \]  \hspace{1cm} (4)

Next, integrate (3) between the limits \( \lambda_1 \) and \( \lambda_2 \)

\[ \int_{\lambda_1}^{\lambda_2} \Phi_{\lambda} d\lambda = \left[ A + B \frac{\lambda_2 + \lambda_1}{2} + C \frac{\lambda_2^2 + \lambda_1^2}{3} \right] (\lambda_2 - \lambda_1) \]  \hspace{1cm} (5)

Note the similarities between Eqs. (4) and (5). Now apply the following conditions:

\[ \frac{\lambda_2 + \lambda_1}{2} = \frac{\int \Re(\lambda) d\lambda}{\int \Re(\lambda) d\lambda} \quad \frac{\lambda_2^2 + \lambda_1^2}{3} = \frac{\int \lambda^2 \Re(\lambda) d\lambda}{\int \Re(\lambda) d\lambda} \]  \hspace{1cm} (6)

Then

\[ \int_{\lambda_1}^{\lambda_2} \Phi_{\lambda} d\lambda = \frac{SIG (\lambda_2 - \lambda_1)}{\int \Re(\lambda) d\lambda} \]  \hspace{1cm} (7)

Assume that area of response curve = area of equivalent rectangle, i.e.

Then

\[ \Re_n \cdot \Delta\lambda = \int \Re(\lambda) d\lambda \]  \hspace{1cm} (8)

and

\[ \int_{\lambda_1}^{\lambda_2} \Phi_{\lambda} d\lambda = \frac{SIG}{\Re_n} \]  \hspace{1cm} (9)

which gives us the desired band-limited power \( \Phi_{\text{in-band}} \).
Now we proceed to determine $\lambda_1$, $\lambda_2$ and $\Re_n$. Substitute:

\[
M_1 = \frac{\int_0^\infty \lambda \Re(\lambda) \, d\lambda}{\int_0^\infty \Re(\lambda) \, d\lambda} \quad \text{and} \quad M_2 = \frac{\int_0^\infty \lambda^2 \Re(\lambda) \, d\lambda}{\int_0^\infty \Re(\lambda) \, d\lambda}
\]

Then

\[
M_1 = \frac{\lambda_2 + \lambda_1}{2} \quad \text{and} \quad M_2 = \frac{\lambda_2^2 + \lambda_1^2}{3}
\]

$M_1$ is the first moment divided by the area under the curve (zeroth moment) and is the centroid of the response curve, the effective or center wavelength $\lambda_c$.

$M_2$ is the second moment divided by the area under the curve, which is related to the square of the radius of gyration.

Solution of simultaneous Eqs. (11), with the substitution $M_1 = \lambda_c$, yields

\[
\lambda_1 = \lambda_c - \sqrt{3(M_2 - \lambda_c^2)} \quad \text{and} \quad \lambda_2 = \lambda_c + \sqrt{3(M_2 - \lambda_c^2)}
\]

showing the bandpass limits $\lambda_1$ and $\lambda_2$ of the equivalent rectangle are symmetrically disposed about the center wavelength $\lambda_c$.

The quantity $(M_2 - \lambda_c^2)$ is recognized as the variance $\sigma^2$ (square of the standard deviation $\sigma$). The bandwidth between wavelength limits $\lambda_1$ and $\lambda_2$ is:

\[
\Delta \lambda = \lambda_2 - \lambda_1 = 2\sqrt{3\sigma^2} = 2\sqrt{3}\sigma
\]

and the short and long limit wavelengths are then:

\[
\lambda_1 = \lambda_c - \sqrt{3}\sigma \quad \text{and} \quad \lambda_2 = \lambda_c + \sqrt{3}\sigma
\]

The bandwidth-normalized responsivity is:

\[
\Re_n = \frac{\Re}{2\sqrt{3}\sigma} \int_0^\infty \Re(\lambda) \, d\lambda
\]

Now we have our three necessary parameters, $\Re_n$, $\lambda_1$ and $\lambda_2$ to completely describe the equivalent rectangular bandwidth.
Note that the coefficients A, B and C of the second-degree polynomial (Eq. 3) used to describe the source have vanished. The implication is significant:

**Any** source that can be represented by a second-degree polynomial can be characterized between the wavelength limits λ₁ and λ₂ (which are determined solely by the radiometer) **without error**.

There is no ambiguity in any of the normalization parameters; they are all uniquely determined from **only** the spectral responsivity curve.

The errors are related to the deviation of the source function from a quadratic.

**MOMENTS NORMALIZATION SUMMARY**

This is the step-by-step procedure for accomplishing a moments normalization. The starting point is absolute spectral responsivity \( \mathcal{R}(\lambda) \).

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<tr>
<th>Moment</th>
<th>Expression</th>
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<tr>
<td>Zeroth moment</td>
<td>( M_0 = \int_0^\infty \mathcal{R}(\lambda) , d\lambda )</td>
</tr>
<tr>
<td>First moment</td>
<td>( M_1 = \int_0^\infty \lambda \mathcal{R}(\lambda) , d\lambda )</td>
</tr>
<tr>
<td>Second moment</td>
<td>( M_2 = \int_0^\infty \lambda^2 \mathcal{R}(\lambda) , d\lambda )</td>
</tr>
<tr>
<td>Center wavelength (centroid)</td>
<td>( \lambda_c = \frac{M_1}{M_0} )</td>
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<tr>
<td>Variance</td>
<td>( \sigma^2 = \frac{M_2 - \lambda_c^2}{M_0} )</td>
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<td>Short wavelength limit</td>
<td>( \lambda_i = \lambda_c - \sqrt{3}\sigma )</td>
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<tr>
<td>Long wavelength limit</td>
<td>( \lambda_z = \lambda_c + \sqrt{3}\sigma )</td>
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<td>Bandwidth</td>
<td>( \Delta \lambda = 2\sqrt{3}\sigma )</td>
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<tr>
<td>Normalized responsivity</td>
<td>( \mathcal{R}_n = \frac{M_0}{\sqrt{2}2\sqrt{3}\sigma} )</td>
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