BANDWIDTH NORMALIZATION BY MOMENTS

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General measurement equation:

$$SIG = \int_0^\infty \Re(\lambda)\, \Phi_\lambda \, \mathrm{d}\lambda \tag{1}$$

Represent $\Re(\lambda)$ by a rectangle of height $\Re_n$ between wavelengths λ_1 and λ_2:

$$SIG = \Re_n \int_{\lambda_1}^{\lambda_2} \Phi_\lambda \mathrm{d}\lambda \tag{2}$$

Let the source function Φ_λ be described as a second-degree polynomial:

$$\Phi_\lambda = A + B\lambda + C\lambda^2 \tag{3}$$

Substitute (3) into (1), divide both sides by $\int\Re(\lambda)\, \mathrm{d}\lambda$ and multiply both sides by $(\lambda_2 - \lambda_1)$ to get:

$$\frac{SIG \cdot (\lambda_2 - \lambda_1)}{\int_0^\infty \Re(\lambda)\mathrm{d}\lambda} = \left[A + B\frac{\int_0^\infty \lambda\Re(\lambda)\mathrm{d}\lambda}{\int_0^\infty \Re(\lambda)\mathrm{d}\lambda} + C\frac{\int_0^\infty \lambda^2\Re(\lambda)\mathrm{d}\lambda}{\int_0^\infty \Re(\lambda)\mathrm{d}\lambda} \right](\lambda_2 - \lambda_1) \tag{4}$$

Next, integrate (3) between the limits λ_1 and λ_2

$$\int_{\lambda_1}^{\lambda_2} \Phi_\lambda \, \mathrm{d}\lambda = \left[A + \frac{B}{2}(\lambda_2 + \lambda_1) + \frac{C}{3}\left(\lambda_2^2 + \lambda_2\lambda_1 + \lambda_1^2\right) \right](\lambda_2 - \lambda_1) \tag{5}$$

Note the similarities between Eqs. (4) and (5). Now apply the following conditions:

$$\frac{\lambda_2 + \lambda_1}{2} = \frac{\int_0^\infty \lambda\Re(\lambda)\mathrm{d}\lambda}{\int_0^\infty \Re(\lambda)\mathrm{d}\lambda} \qquad \frac{\lambda_2^2 + \lambda_2\lambda_1 + \lambda_1^2}{3} = \frac{\int_0^\infty \lambda^2\Re(\lambda\,\mathrm{d}\lambda)}{\int_0^\infty \Re(\lambda\,\mathrm{d}\lambda)} \tag{6}$$

Then

$$\int_{\lambda_1}^{\lambda_2} \Phi_\lambda \, \mathrm{d}\lambda = \frac{SIG(\lambda_2 - \lambda_1)}{\int_0^\infty \Re(\lambda)\mathrm{d}\lambda} \tag{7}$$

Assume that area of response curve = area of equivalent rectangle, i.e.

Then

$$\Re_n \cdot \Delta\lambda = \int_0^\infty \Re(\lambda)\mathrm{d}\lambda \tag{8}$$

and

$$\int_{\lambda_1}^{\lambda_2} \Phi_\lambda \mathrm{d}\lambda = \frac{SIG}{\Re_n} \tag{9}$$

which gives us the desired band-limited power $\Phi_{\text{in-band}}$.

Now we proceed to determine λ_1, λ_2 and $\Re_n$. Substitute:

$$M_1 = \frac{\int_0^\infty \lambda \Re(\lambda)\,\mathrm{d}\lambda}{\int_0^\infty \Re(\lambda)\,\mathrm{d}\lambda} \qquad M_2 = \frac{\int_0^\infty \lambda^2 \Re(\lambda)\,\mathrm{d}\lambda}{\int_0^\infty \Re(\lambda)\,\mathrm{d}\lambda} \tag{10}$$

Then

$$M_1 = \frac{\lambda_2 + \lambda_1}{2} \qquad M_2 = \frac{\lambda_2^2 + \lambda_2\lambda_1 + \lambda_1^2}{3} \tag{11}$$

M_1 is the first moment divided by the area under the curve (zeroth moment) and is the centroid of the response curve, the effective or center wavelength λ_c.

M_2 is the second moment divided by the area under the curve, which is related to the square of the radius of gyration.

Solution of simultaneous Eqs. (11), with the substitution $M_1 = \lambda_c$, yields

$$\lambda_1 = \lambda_c - \sqrt{3(M_2\text{-}\lambda_c^{\,2})}\,, \qquad \lambda_2 = \lambda_c + \sqrt{3(M_2\text{-}\lambda_c^{\,2})} \tag{12}$$

showing the bandpass limits λ_1 and λ_2 of the equivalent rectangle are symmetrically disposed about the center wavelength λ_c.

The quantity (M_2 - λ_c^2) is recognized as the variance σ^2 (square of the standard deviation σ). The bandwidth between wavelength limits λ_1 and λ_2 is:

$$\Delta\lambda = \lambda_2 - \lambda_1 = 2\sqrt{3\sigma^2} = 2\sqrt{3}\,\sigma \tag{13}$$

and the short and long limit wavelengths are then:

$$\lambda_1 = \lambda_c - \sqrt{3}\,\sigma \qquad \lambda_2 = \lambda_c + \sqrt{3}\,\sigma \tag{14}$$

The bandwidth-normalized responsivity is:

$$\Re_n = \frac{1}{2\sqrt{3}\sigma}\int_0^\infty \Re(\lambda)\,\mathrm{d}\lambda \tag{15}$$

Now we have our three necessary parameters, $\Re_n$, λ_1 and λ_2 to completely describe the equivalent rectangular bandwidth.

Note that the coefficients A, B and C of the second-degree polynomial (Eq. 3) used to describe the source have vanished. The implication is significant:

> <u>Any</u> source that can be represented by a second-degree polynomial can be characterized between the wavelength limits λ_1 and λ_2 (which are determined solely by the radiometer) <u>without</u> <u>error</u>.

> There is no ambiguity in any of the normalization parameters; they are all uniquely determined from <u>only</u> the spectral responsivity curve.

> The errors are related to the deviation of the source function from a quadratic.

MOMENTS NORMALIZATION SUMMARY

This is the step-by-step procedure for accomplishing a moments normalization. The starting point is absolute spectral responsivity $\Re(\lambda)$.

Zeroth moment	$M_0 = \int_0^\infty \Re(\lambda)\,\mathrm{d}\lambda$
First moment	$M_1 = \int_0^\infty \lambda\,\Re(\lambda)\,\mathrm{d}\lambda$
Second moment	$M_2 = \int_0^\infty \lambda^2\,\Re(\lambda)\,\mathrm{d}\lambda$
Center wavelength (centroid)	$\lambda_c = \frac{M_1}{M_0}$
Variance	$\sigma^2 = \frac{M_2}{M_0} - \lambda_c^2$
Short wavelength limit	$\lambda_1 = \lambda_c - \sqrt{3}\sigma$
Long wavelength limit	$\lambda_2 = \lambda_c + \sqrt{3}\sigma$
Bandwidth	$\Delta\lambda = 2\sqrt{3}\sigma$
Normalized responsivity	$\Re_n = \frac{M_0}{2\sqrt{3}\sigma}$