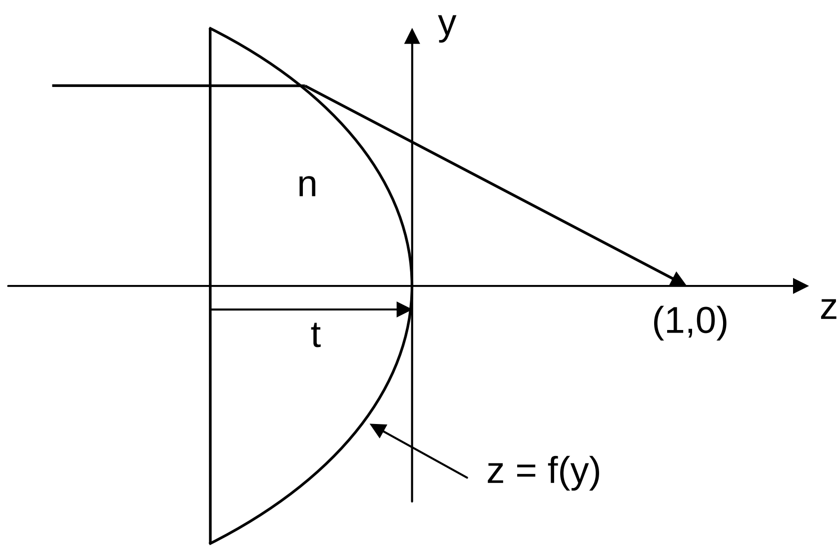
Perfect Plano-Convex Lens

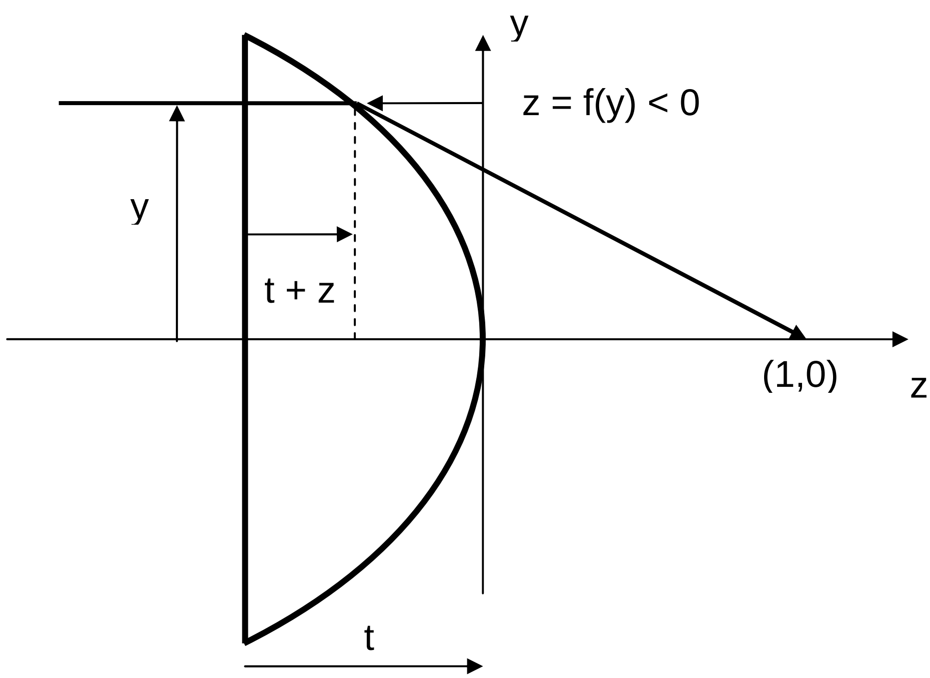
Determine the equation of rotation about the z-axis for a plano-convex lens in air with thickness t and index n that will bring all incident rays parallel to its axis to a common focus at the point (1,0). The equation of rotation z = f(y) gives the z-coordinate or sag of the surface along an arc on the surface. This arc is rotated about the optical axis to produce the rotationally symmetric lens surface. Derive the exact analytic form of this curve. Note that because of the reference definitions, the values for z will be negative.

a) by Fermat's principle.
b) by Snell's law.



One of these methods is relatively easy; the other is very difficult and involves differential equations. Do not become overly frustrated by this problem, the level of math here greatly exceeds anything else we will do this semester.

Solution:



a) Fermat's Principle: Al OPLs are equal.

For y = 0: $OPL = nt + 1$

For arbitrary y: $OPL = n(t+z) + \sqrt{y^2 + (1-z)^2}$

Equating:

$$nt + 1 = n(t+z) + \sqrt{y^2 + (1-z)^2}$$
$$1 - nz = \sqrt{y^2 + (1-z)^2}$$
$$(1-nz)^2 = y^2 + (1-z)^2$$
$$1 - 2nz + n^2 z^2 = y^2 + 1 - 2z + z^2$$

$$\boxed{z^2(n^2-1) - 2z(n-1) - y^2 = 0}$$

Which is in form of a hyperbola about the z-axis:

$$Az^2 + Cy^2 + Dz + Ey + F = 0 \qquad AC < 0$$

This result can also be put in the form:

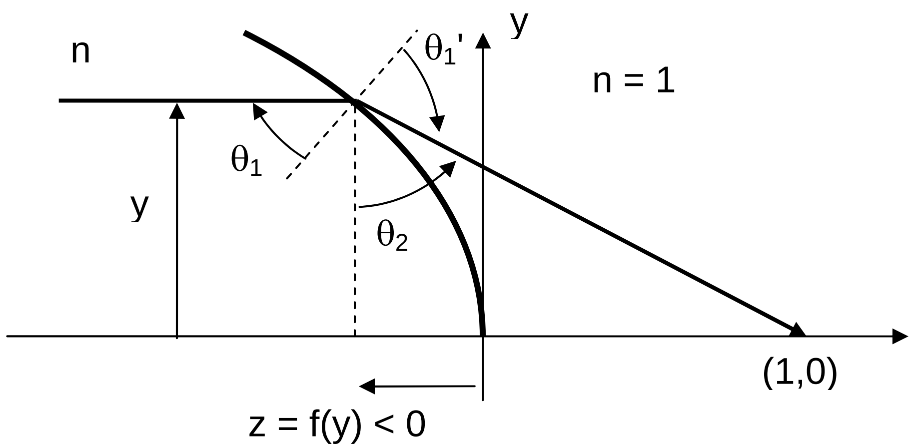
$$z^2\left(n^2-1\right)-2z\left(n-1\right)-y^2=0$$
$$z^2\left(n+1\right)\left(n-1\right)-2z\left(n-1\right)-y^2=0$$
$$z^2\left(n+1\right)^2-2z\left(n+1\right)-y^2\left(\frac{n+1}{n-1}\right)=0$$
$$\left(z\left(n+1\right)-1\right)^2-y^2\left(\frac{n+1}{n-1}\right)=1$$

$$\boxed{\left(n+1\right)^2\left(z-\frac{1}{n+1}\right)^2-y^2\left(\frac{n+1}{n-1}\right)=1}$$

Which is a hyperbola with its center at

$$z=\frac{1}{n+1} \qquad y=0$$

b) Snell’s Law



Note: θ_1 and θ_1' are negative as shown.

$$(90+\theta_1)+\theta_2-\theta_1' = 180$$

$$\theta_1' = \theta_1+\theta_2-90$$

$$\sin\theta_1' = \sin(\theta_1+\theta_2-90) = -\cos(\theta_1+\theta_2)$$

$$\sin\theta_1' = -\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2 \qquad \text{A}$$

Using the triangle formed at the ray intersection with the surface:

$$\tan\theta_2 = \frac{1-z}{y}$$

With the trigonometric identities:

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} \qquad \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}$$

$$\sin\theta_2 = \frac{(1-z)/y}{\sqrt{1+(1-z)^2/y^2}} \qquad \cos\theta_2 = \frac{1}{\sqrt{1+(1-z)^2/y^2}}$$

Directly from the drawing, the slope of the surface normal is

$$m_\perp = -\tan\theta_1$$

Note that as drawn, the slope of the surface normal is positive and θ_1 is negative.

Independently, the slope of the surface at the ray intersection is $m = \frac{dy}{dz}$

Recalling that the product of the slopes of two perpendicular lines is -1, then the slope of the surface normal at the ray intersection is

$$m_\perp = -\frac{1}{m} = -\frac{1}{dy/dz}$$

Equate these two relationships for the slope of the surface normal:

$$m_\perp = -1/(dy/dz) = -\tan\theta_1 \qquad \text{as drawn: } m_\perp > 0 \text{ and } \theta_1 < 0$$

$$\tan\theta_1 = 1/(dy/dz)$$

Applying the same trigonometric identities:

$$\sin\theta_1 = \frac{\tan\theta_1}{\sqrt{1+\tan^2\theta_1}} = \frac{1/\left(dy/dz\right)}{\sqrt{1+1/\left(dy/dz\right)^2}} \qquad \text{B}$$

and

$$\cos\theta_1 = \frac{1}{\sqrt{1+\tan^2\theta_1}} = \frac{1}{\sqrt{1+1/\left(dy/dz\right)^2}}$$

Returning to Equation A from above:

$$\sin\theta_1' = -\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

$$\sin\theta_1' = -\frac{1}{\sqrt{1+1/\left(dy/dz\right)^2}}\frac{1}{\sqrt{1+\left(1-z\right)^2/y^2}} + \frac{1/\left(dy/dz\right)}{\sqrt{1+1/\left(dy/dz\right)^2}}\frac{\left(1-z\right)/y}{\sqrt{1+\left(1-z\right)^2/y^2}}$$

Now apply Snell's Law, use Equation B, and simplify to obtain the differential equation that defines the surface:

$$n\sin\theta_1 = \sin\theta_1'$$

$$n\frac{1/\left(dy/dz\right)}{\sqrt{1+1/\left(dy/dz\right)^2}}=-\frac{1}{\sqrt{1+1/\left(dy/dz\right)^2}}\frac{1}{\sqrt{1+\left(1-z\right)^2/y^2}}$$
$$+\frac{1/\left(dy/dz\right)}{\sqrt{1+1/\left(dy/dz\right)^2}}\frac{\left(1-z\right)/y}{\sqrt{1+\left(1-z\right)^2/y^2}}$$

$$\frac{n}{dy/dz}=-\frac{1}{\sqrt{1+\left(1-z\right)^2/y^2}}+\frac{1}{dy/dz}\frac{\left(1-z\right)/y}{\sqrt{1+\left(1-z\right)^2/y^2}}$$

$$n\sqrt{1+\left(1-z\right)^2/y^2}=-\frac{dy}{dz}+\left(1-z\right)/y$$

$$\boxed{n\sqrt{y^2+\left(1-z\right)^2}=-y\frac{dy}{dz}+1-z}$$

While it is possible to directly solve this equation using several variable transformations, the easiest solution is to use the “assume a solution of the form” method. The obvious choice is to use the solution derived using Fermat’s Principle:

$$\boxed{z^2\left(n^2-1\right)-2z\left(n-1\right)-y^2=0}$$

where

$$\frac{dy}{dz}=\frac{\left[z\left(n^2-1\right)-\left(n-1\right)\right]}{y}$$

As shown below, plugging these two equations into the differential equation will show that this hyperbola the solution to the problem.

Proof :

Equation: $$n\sqrt{y^2+(1-z)^2}=-y\frac{dy}{dz}+1-z$$

Proposed Solution:

$$z^2\left(n^2-1\right)-2z\left(n-1\right)-y^2=0$$

$$y^2=z^2\left(n^2-1\right)-2z\left(n-1\right)$$

$$y=\left(z^2\left(n^2-1\right)-2z\left(n-1\right)\right)^{1/2}$$

$$\frac{dy}{dz}=\frac{1}{2}\left(z^2\left(n^2-1\right)-2z\left(n-1\right)\right)^{-1/2}\left[2z\left(n^2-1\right)-2\left(n-1\right)\right]$$

$$\frac{dy}{dz}=\frac{\left[z\left(n^2-1\right)-\left(n-1\right)\right]}{y}$$

Plug in…

$$n\sqrt{z^2\left(n^2-1\right)-2z\left(n-1\right)+\left(1-z\right)^2}\triangleq-z\left(n^2-1\right)+\left(n-1\right)+1-z$$

$$n\sqrt{z^2n^2-z^2-2zn+2z+1-2z+z^2}\triangleq-zn^2+z+n-z$$

$$n\sqrt{z^2n^2-2zn+1}\triangleq-zn^2+n$$

$$\sqrt{z^2n^2-2zn+1}\triangleq1-zn$$

$$z^2n^2-2zn+1\triangleq\left(1-zn\right)^2$$

$$z^2n^2-2zn+1=1-2zn+z^2n^2$$ Confirmed!