Law of Reflection – Fermat's Principle

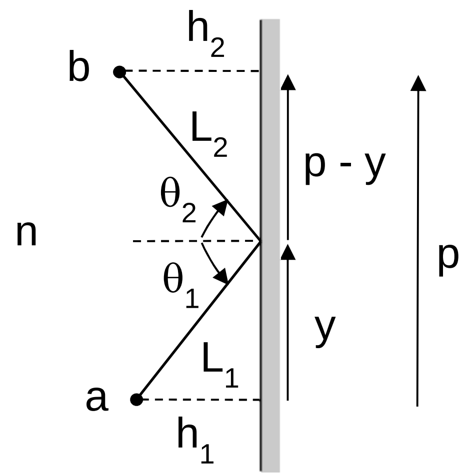
Use Fermat's principle to derive the law of reflection. Use proper sign conventions and reference definitions.

Solution:

Use Fermat's Principle to determine the valid ray path at a reflective boundary. The ray propagates from point a to point b. The variable y defines the ray intersection location at the interface.

As drawn: $\theta_1 > 0 \qquad \theta_2 < 0$

First do the problem assuming that L_1 and L_2 are both positive distances in an index of n.



$$OPL = nL_1 + nL_2$$

$$L_1 = \sqrt{h_1^2 + y^2}$$

$$L_2 = \sqrt{h_2^2 + (p-y)^2}$$

$$\frac{dOPL}{dy} = n\frac{dL_1}{dy} + n\frac{dL_2}{dy} = 0 \quad \text{for a valid ray path}$$

$$\frac{dL_1}{dy} = \frac{y}{\sqrt{h_1^2 + y^2}} = \frac{y}{L_1} = \sin\theta_1$$

$$\frac{dL_2}{dy} = \frac{-(p-y)}{\sqrt{h_2^2 + (p-y)^2}} = \frac{-(p-y)}{L_2} = \sin\theta_2 \qquad \theta_2 < 0 \text{ as drawn}$$

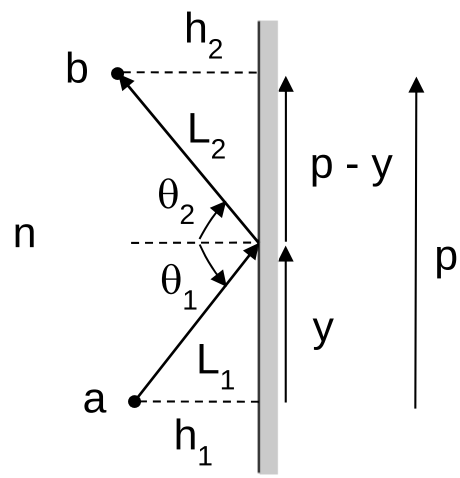
Then

$$n\sin\theta_1 + n\sin\theta_2 = 0$$

$$\sin\theta_1 = -\sin\theta_2$$

$$\theta_1 = -\theta_2$$

This derivation can alternately be done using the sign conventions where the sign of the index of refraction changes upon reflection. In this case, L_2 is negative:



$$n' = -n \qquad\qquad L_2 < 0$$

$$OPL = nL_1 + n'L_2$$

$$L_1 = \sqrt{h_1^2 + y^2}$$

$$L_2 = -\sqrt{h_2^2 + \left(p - y\right)^2}$$

$$\frac{dOPL}{dy} = n\frac{dL_1}{dy} + n'\frac{dL_2}{dy} = 0 \quad \text{for a valid ray path}$$

$$\frac{dL_1}{dy} = \frac{y}{\sqrt{h_1^2 + y^2}} = \frac{y}{L_1} = \sin\theta_1$$

$$\frac{dL_2}{dy} = -\frac{-\left(p - y\right)}{\sqrt{h_2^2 + \left(p - y\right)^2}} = \frac{\left(p - y\right)}{-L_2} = -\sin\theta_2 \qquad \theta_2 < 0 \text{ as drawn}$$

Then

$$n\sin\theta_1 - n'\sin\theta_2 = 0$$

$$n\sin\theta_1 + n\sin\theta_2 = 0$$

$$\sin\theta_1 = -\sin\theta_2$$

$$\theta_1 = -\theta_2$$