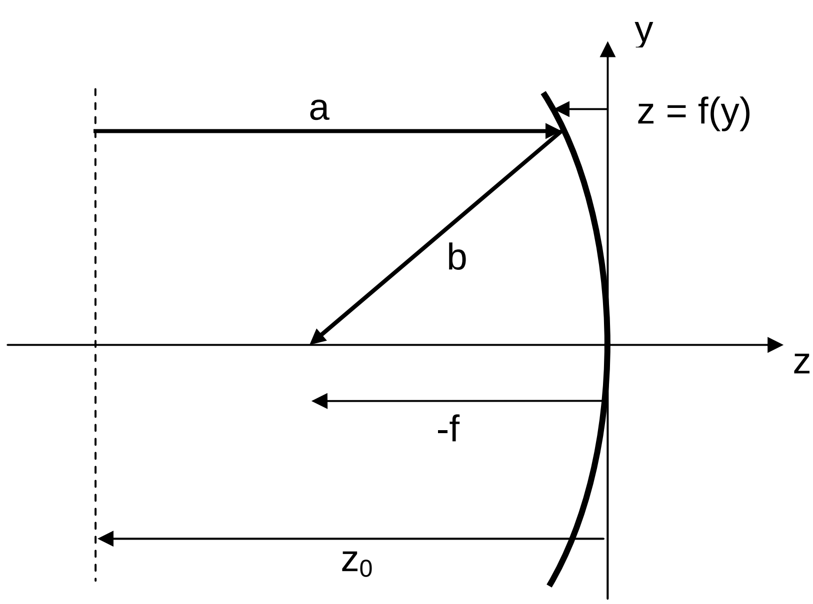
Fermat's Principle – Concave Mirror

Use Fermat's principle to determine the shape (equation) of a concave mirror in air that produces a perfect image of a distant (at infinity) point source a distance f to the left of the mirror vertex. The focal length f of a concave mirror is given as a positive value.

Remember that for an imaging situation, the optical path lengths (OPLs) for all rays connecting the object and image points are equal. Since rays from infinity are parallel, a reference plane at an arbitrary z can be used to define the OPL.

Solution:



z_0 represents an arbitrary reference plane and is negative as shown. The surface sag z is also negative.

$$a = -z_0 + z$$

$$b^2 = y^2 + (f + z)^2 \qquad f > 0$$

With the sign conventions for "b"; b is negative as shown:

$$b = -\sqrt{y^2 + (f+z)^2}$$

Determine the two optical path lengths. Remember that the index changes sign after a reflection: (n = 1 in air)

$$OPL_a = na = -z_0 + z$$
$$OPL_b = (-n)b = (-1)\left(-\sqrt{y^2 + (f+z)^2}\right) = \sqrt{y^2 + (f+z)^2}$$

Fermat's principle requires that the total OPL be constant:

$$OPL_a + OPL_b = Constant$$

This sum can be evaluated exactly along the optical axis (z = 0 at y = 0):

$$OPL_a + OPL_b = -z_0 + f$$

Equate this value with the total OPL for an arbitrary y:

$$-z_0 + f = -z_0 + z + \sqrt{y^2 + (f+z)^2}$$
$$(f-z)^2 = y^2 + (f+z)^2$$
$$f^2 - 2fz + z^2 = y^2 + f^2 + 2fz + z^2$$
$$y^2 = -4fz$$
$$\boxed{z = \frac{-y^2}{4f}}$$

This is a parabola. The result equals the expected expression for the surface shape or sag of a parabola ($y^2/2R$) with a radius of curvature R = 2f.

Note that this problem can also be done without using the sign convention for "b" by recognizing that both component OPLs must be positive. The values could simply be written down:

$$OPL_a = -z_0 + z$$
$$OPL_b = \sqrt{y^2 + (f + z)^2}$$

The remainder of the derivation follows as above.