Section 9

Stops and Pupils

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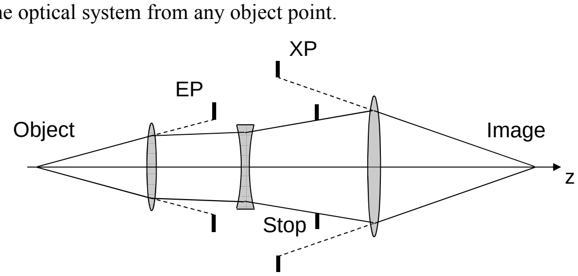
Stops and Pupils

The aperture stop is the aperture in the system that limits the bundle of light that propagates through the system from the axial object point. The stop can be one of the lens apertures or a separate aperture (iris diaphragm) placed in the system, however, the stop is always a physical or real surface. The beam of light that propagates through an axially symmetric system from a on-axis point is shaped like a spindle.

The entrance pupil (EP) is the image of the stop into object space, and the exit pupil (XP) is the image of the stop into image space. The pupils define the cones of light entering and exiting the optical system from any object point.



There is a stop or pupil in each optical space. The EP is in the system object space, and the XP is in the system image space. Intermediate pupils are formed in other spaces.

Aperture Stop

The limiting aperture of the system may not be obvious. There are two common methods to determine which aperture in a system serves as the system stop:

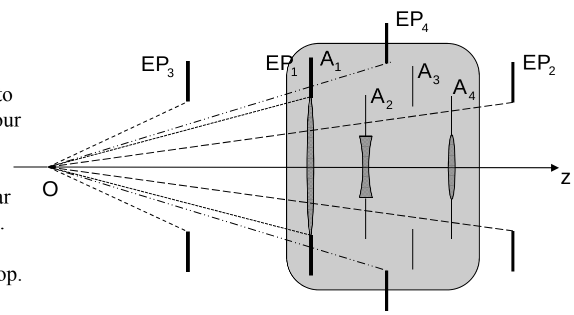
The first method is to image each potential stop into object space (use the optical surfaces between the potential stop and the object). The candidate pupil with the smallest angular size from the perspective of the axial object point corresponds to the stop. An analogous procedure can also be done in image space.

Four apertures exist in this optical system.

Each aperture is imaged into object space to produced four potential EPs.

EP_2 has the smallest angular subtense as viewed from O.

Aperture 2 is the system stop.

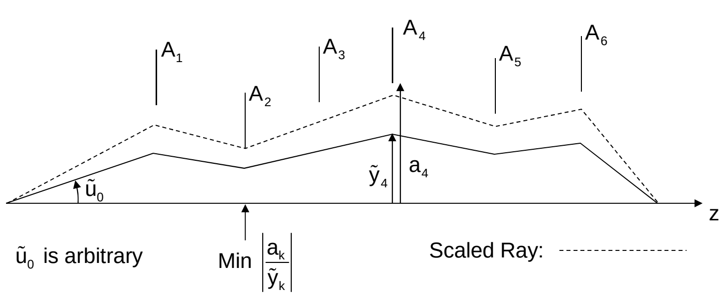


Note: All of the potential EPs are in object space. EP_4 and EP_2 are virtual. Since lens A_1 is the first optical element, its aperture is in object space, and A_1 serves as EP_1.

Aperture Stop

The second method to determine which aperture serves as the system stop is to trace a ray through the system from the axial object point with an arbitrary initial angle. At each aperture or potential stop, form the ratio of the aperture radius a_k to the height of this ray at that surface $\tilde{y}_k$. The stop is the aperture with the minimum ratio:

$$\text{Aperture Stop} \Rightarrow \text{Minimum} \left|\frac{a_k}{\tilde{y}_k}\right|$$



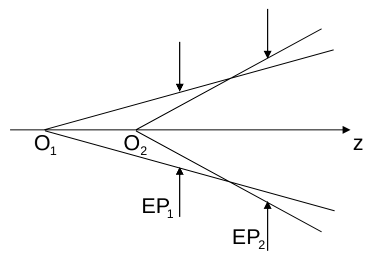
The limiting ray is found by scaling the initial ray:

$$u_j = \tilde{u}_j \left|\frac{a_k}{\tilde{y}_k}\right|_{MIN} \qquad y_j = \tilde{y}_j \left|\frac{a_k}{\tilde{y}_k}\right|_{MIN}$$

Limiting Apertures

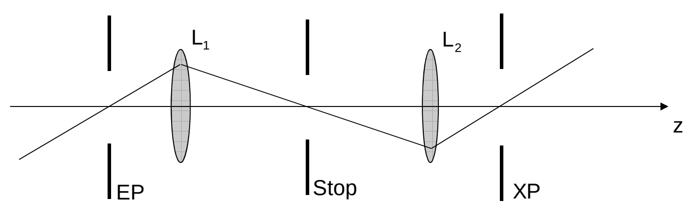
Large changes in object position may cause different apertures to serve as the limiting aperture or system stop. The EP with the smallest angular subtense can change. A different aperture would then become the system stop.

However when designing a system, it is usually critical that the stop surface does not change over the range of possible object positions that the system will be used with.

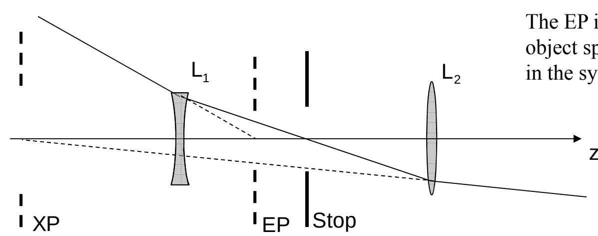


Pupils

The pupil locations can be found by tracing a ray that goes through the center of the stop. The intersections of this ray with the axis in image space and object space determine the location of the exit and entrance pupils.



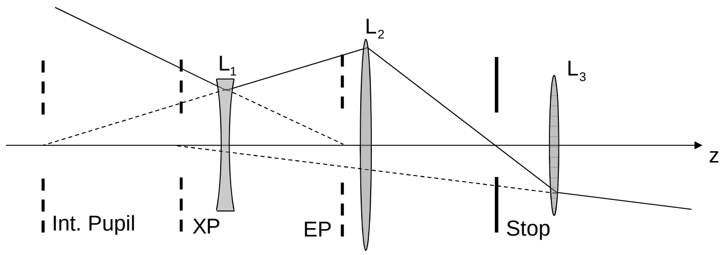
The rays are extended to the axis to locate the pupil. The EP and the XP are often virtual.



The EP is in the system object space, and the XP is in the system image space.

Intermediate Space Pupils

Intermediate pupils are formed in each optical space for multi-element systems. If there are N elements, there are N+1 pupils (including the stop).



Marginal and Chief Rays

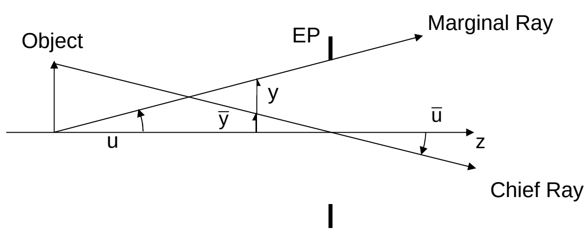
Rays confined to the y-z plane are called meridional rays. The marginal ray and the chief ray are two special meridional rays that together define the properties of the object, images, and pupils.

The marginal ray starts at the axial object position, goes through the edge of the entrance pupil, and defines image locations and pupil sizes. It propagates to the edge of the stop and to the edge of the exit pupil.

The chief ray starts at the edge of the object, goes through the center of the entrance pupil, and defines image heights and pupil locations. It goes through the center of the stop and the center of the exit pupil.

y = marginal ray height $\quad\quad \bar{y}$ = chief ray height

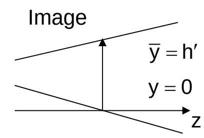
u = marginal ray angle $\quad\quad \bar{u}$ = chief ray angle



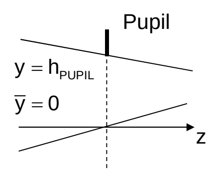
Images and Pupils

The heights of the marginal ray and the chief ray can be evaluated at any z in any optical space.

When the marginal ray crosses the axis, an image is located, and the size of the image is given by the chief ray height in that plane.



Whenever the chief ray crosses the axis, a pupil or the stop is located, and the pupil radius is given by the marginal ray height in that plane.



Intermediate images and pupils are often virtual.

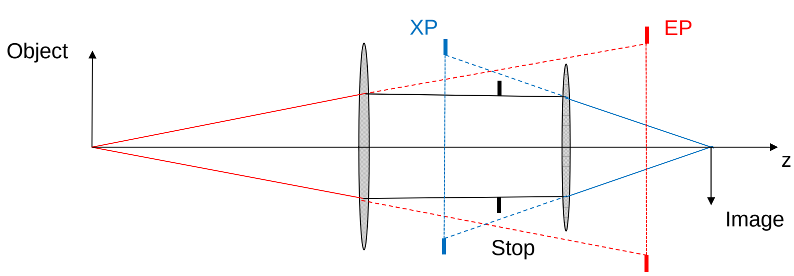
The chief ray is the axis of the unvignetted beam from a point at the edge of the field, and the radius of that beam at any cross section is equal to the marginal ray height in that plane.

On-Axis Ray Bundle

The stop limits the ray bundle from the axial object point. The marginal ray is at the edge of the on-axis ray bundle.

The EP defines the ray bundle in object space from the axial object point. The ray bundle that will propagate through the system fills the EP.

The XP defines the ray bundle converging to the axial image point in image space. The ray bundle emerging from the system appears to come from a filled XP.



The stop and the intermediate image (not shown) define the ray bundle in the intermediate optical space.

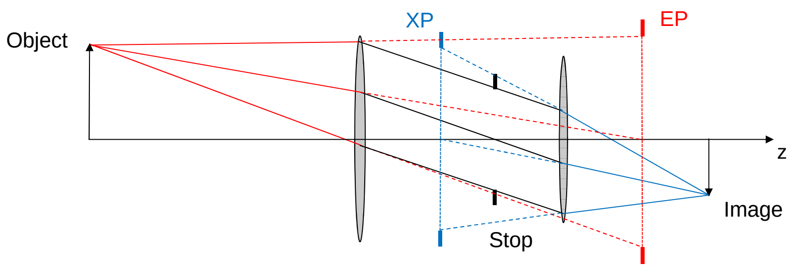
In any additional optical spaces, the intermediate pupils and images will define the ray bundles.

Ray Bundles

The pupils are the image of the stop and do not change position or size with an off-axis object.

The EP and the XP also define the skewed ray bundles that enter and exit the optical system for an off-axis object point. The off-axis ray bundle that will propagate through the system fills the EP. The off-axis ray bundle emerging from the system appears to come from a filled XP.

The skewed ray bundles are centered on the chief ray for that object point.

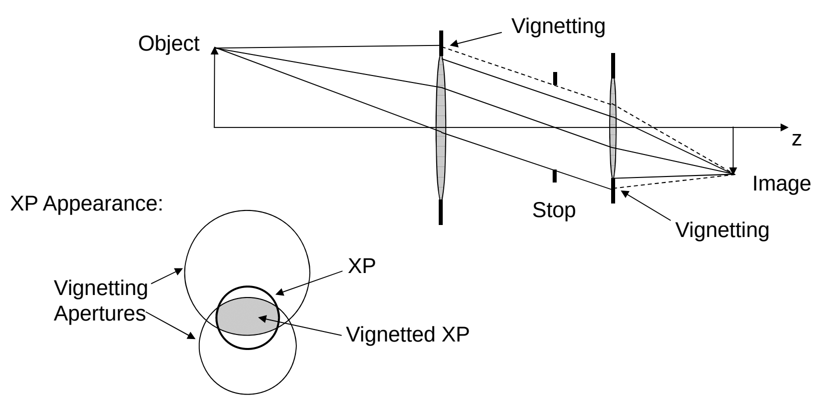


Vignetting

For an off-axis object point, the beam of light through the system is shaped like a spindle of skew cone-shaped sections as long as no aperture other than the stop limits the beam.

If the beam is intercepted by one or more additional apertures, vignetting occurs. The top and or bottom of the ray bundle is clipped, and the beam of light propagating through the system no longer has a circular profile.

For example, reducing the lens diameters in the previous example produces vignetting at the top of the first lens and the bottom of the second lens.



Field of View

The Field of View FOV of an optical system is determined by the object size or the image size depending on the situation:

- the maximum angular size of the object as seen from the entrance pupil
- the maximum object height
- the maximum image height
- the maximum angular size of the image as seen from the exit pupil

Field of View FOV: the diameter of the object/image

Half Field of View HFOV: the radius of the object/image

Full Field of View FFOV is sometimes used instead of FOV to emphasize that this is a diameter measure.

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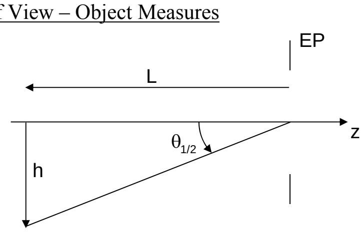
Field of View – Object Measures



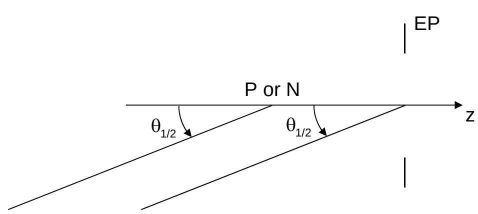
$$HFOV = \theta_{1/2}$$

$$\tan\left(\theta_{1/2}\right) = \frac{h}{L}$$

$$\bar{u} = \tan\left(\theta_{1/2}\right) = \frac{h}{L}$$

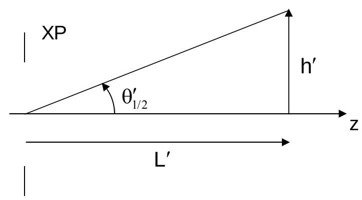
Since the EP is the reference position for the FOV, this defining ray becomes the chief ray of the system in object space.

For distant objects, the apparent angular size of the object as viewed from the EP or from the front principal plane P or the nodal point N are approximately the same.



For close or finite conjugate objects, it is usually better to define the FOV in terms of the object size.

Field of View – Image Measures



$$HFOV = \theta'_{1/2}$$

$$\tan\left(\theta'_{1/2}\right) = \frac{h'}{L'}$$

$$\bar{u}' = \tan\left(\theta'_{1/2}\right) = \frac{h'}{L'}$$

Since the XP is the reference position for the FOV, this defining ray becomes the chief ray of the system in image space.

While it is possible to define the FOV in terms of the angular image size, it is much more common to simply use the image size.

The required image size or the detector size often defines the FOV of the system.

In general, the angular FOV in object space does not equal the angular FOV in image space. The object and image space chief ray angles are also not equal.

$$\theta_{1/2} \neq \theta'_{1/2} \qquad \bar{u} \neq \bar{u}'$$

These quantities will be equal if the EP and the XP are located at the respective nodal points. This is the situation for a thin lens in air with the stop at the lens.

Field of View

The system FOV can be determined by the maximum object size, the detector size, or by the field over which the optical system exhibits good performance. For rectangular image formats, horizontal, vertical and diagonal FOVs must be specified.

The fractional object FOB is used to describe objects of different heights in terms of the HFOV. For example, FOB 0.5 would indicate an object that has size half the maximum. This is used in ray trace code to analyze a system performance at different field sizes.

Common values are

FOB 0 on-axis

FOB 0.7 half of the object area is closer to the optical axis at this angle, and half is farther away ($.7^2 \equiv .5$)

FOB 1 maximum field (edge of object)

Another method for defining angular FOV is to measure the angular size of the object relative to the front nodal point *N*. This is useful because the angular sizes of the object and the image are equal when viewed from the respective nodal points. This definition of angular FOV fails for afocal systems which do not have nodal points. In focal systems with a distant object, the choice of using the EP or nodal point for angular object FOV is of little consequence.

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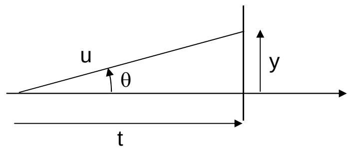
Paraxial Ray Angles

While they are referred to as angles, paraxial ray angles are not angles at all. They measure an angle-like quantity, but these paraxial angles are actually the slope of the ray or the ratio of a height to a distance. As a result, paraxial angles are unitless. If the physical angle in degrees or radians is θ, then the paraxial angle u is given by the tangent of θ.



$$u = \frac{y}{t}$$

$$u = \tan\theta = \frac{y}{t}$$

The use of ray slopes is critical for paraxial raytracing as it results in the linearity of paraxial raytracing. This is easy to see from the transfer equation:

$$y' = y + ut$$

This linear equation for the paraxial ray is the equation of a line, and the constant of proportionality is the ray slope. The need to use the ray slope is also apparent in the above figure. As the physical angle goes from 0 to 90 degrees (or 0 to π/2 radians), the ray height at the following surface goes from 0 to infinity. Since the ray slope also goes from 0 to infinity, the paraxial raytrace equation correctly gives the correct result without any approximations. The use of a physical angle in radians instead of the paraxial ray angle in the transfer equation is only valid by approximation for small angles or by the use of trig functions.

Small Angle Approximations

While it is true that for small angles the tangent of an angle (in radians) is approximately equal to the angle, this is only an approximation – even here the angle loses its units of radians in this conversion to obtain the unitless ray slope.

$$u \approx \theta \quad \text{for small angles } (u \text{ unitless; } \theta \text{ radians})$$

Care must be used in making this approximation as paraxial angles are often used that exceed the small angle approximation. Since the raytrace equations are linear in ray slope and not in ray angle, the ray slope must be used for the paraxial ray angles.

In general, a tangent is required to convert between paraxial angles (or more accurately slopes) and physical angles in degrees or radians.

$$u = \tan\theta = \frac{y}{t}$$

Since the paraxial ray angle is a slope, it is incorrect to determine the paraxial ray angle as if it were a physical angle.

$$u \neq \tan^{-1}\frac{y}{t}$$

This proper conversion to paraxial angles commonly occurs when discussing FOV of a system:

$$\bar{u} = \tan HFOV$$

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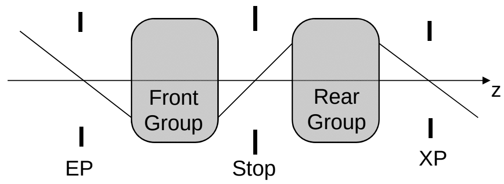
Pupil Positions by Paraxial Raytrace

The stop is a real object for the formation of both the entrance and exit pupil.

The pupil locations can be found by tracing a paraxial ray starting at the center of the aperture stop. The ray is traced through the group of elements behind the stop and reverse traced through the group of elements in front of the stop. The intersections of this ray with the axis in object and image space determine the locations of the entrance and exit pupils. Both pupils are often virtual and are found using virtual extensions of the object space and image space rays.



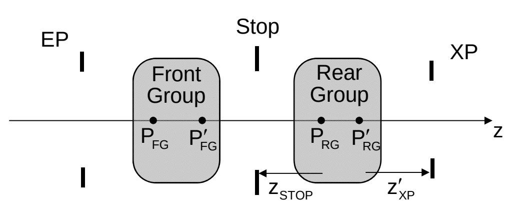
This ray becomes the chief ray when it is scaled to the object or image FOV. The marginal ray height at the pupil locations gives the pupil sizes.

The trial ray used to determine which aperture serves as the system stop can be scaled to the produce the marginal ray.

$$u_j = \tilde{u}_j \left|\frac{a_k}{\tilde{y}_k}\right|_{MIN} \qquad y_j = \tilde{y}_j \left|\frac{a_k}{\tilde{y}_k}\right|_{MIN}$$

Pupil Positions by Gaussian Imagery

The pupil locations and sizes can also be found using Gaussian imagery. Imaging the stop through the rear group of elements to find the XP is straightforward:



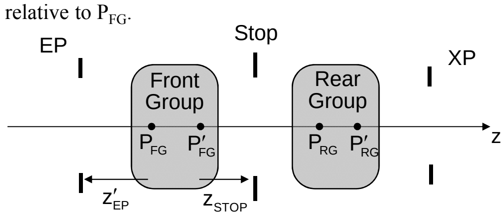
For the XP: $$\frac{1}{z'_{XP}} = \frac{1}{z_{STOP}} + \frac{1}{f_{RG}} \quad \text{(in air)}$$

$$m_{XP} = \frac{z'_{XP}}{z_{STOP}} \qquad D_{XP} = |m_{XP}| D_{STOP}$$

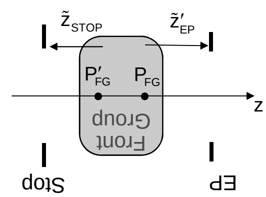
However for the EP, the stop is a real object to the right of the front group, and the Gaussian equations do not directly apply.

Pupil Positions by Gaussian Imagery

The stop is a real object to the right of the front group. Gaussian object and image distances must be measured relative to the principal plane in the same optical space. In this case, the stop (or object) position is measured relative to P'_{FG}, and the EP (or image) position is measured relative to P_{FG}.



Since the Gaussian equations require the light to propagate from left to right, or equivalently that a real object is to the left of the system, the simplest conceptual solution is to flip the problem (turn the paper upside down!). The signs of the flipped distances are opposite the signs of the original distances.

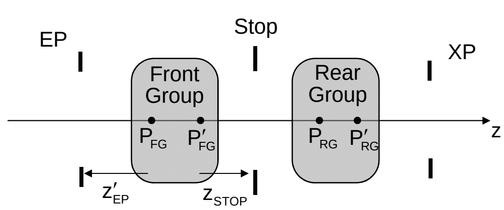


$$\frac{1}{\tilde{z}'_{EP}} = \frac{1}{\tilde{z}_{STOP}} + \frac{1}{f_{FG}} \qquad z'_{EP} = -\tilde{z}'_{EP} \quad z_{STOP} = -\tilde{z}_{STOP}$$

$$\frac{-1}{z'_{EP}} = \frac{-1}{z_{STOP}} + \frac{1}{f_{FG}} \qquad \text{(in air)}$$

Pupil Positions by Gaussian Imagery

Flipping the paper over is effective but awkward. The proper way to determine the EP location is to remember that the light from the real stop propagates from right to left to form the EP, and to assign a negative index to this imagery (just as is done after a reflection). The object and image distances are measured from the principal plane of the front group that is in the same optical space as the object (stop) or image (pupil). The stop (or object) position is measured relative to P'_{FG}, and the EP (or image) position is measured relative to P_{FG}.



For the EP: $$\frac{n'}{z'_{EP}} = \frac{n}{z_{STOP}} + \frac{1}{f_{FG}} \qquad n = n' = -1 \qquad \text{(in air)}$$

$$\frac{-1}{z'_{EP}} = \frac{-1}{z_{STOP}} + \frac{1}{f_{FG}} \qquad m_{EP} = \frac{z'_{EP}/n'}{z_{STOP}/n} = \frac{z'_{EP}}{z_{STOP}} \qquad D_{EP} = |m_{EP}| D_{STOP}$$

Example System

Two thin lenses with a stop halfway between:

$f_1 = 100\,mm$

$f_2 = 75\,mm$

$t = 50\,mm$

Stop at $25\,mm / 25\,mm$

Stop Diameter = $20\,mm$

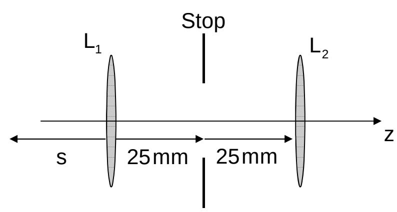
Stop Radius = $a_{STOP} = 10\,mm$

Object Height = $10\,mm$

Two Positions:

$s_A = -100\,mm$ (from L_1)

$s_B = -50\,mm$ (from L_1)



Example System – Gaussian Solution

System: $\phi = .0167/mm$ $\quad f_E = 60\,mm$

$d = 40\,mm$ $\quad d' = -30\,mm$

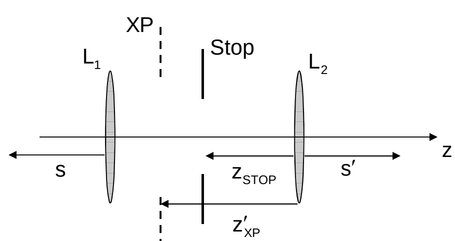
Object/Image: $s_A = -100\,mm$ (from L_1) $\rightarrow$ $s'_A = 75\,mm$ (from L_2)

$s_B = -50\,mm$ (from L_1) $\rightarrow$ $s'_B = 150\,mm$ (from L_2)

Exit Pupil: $\frac{1}{z'_{XP}} = \frac{1}{z_{STOP}} + \frac{1}{f_2}$ $\quad f_2 = 75\,mm$ $\quad z_{STOP} = -25\,mm$

$z'_{XP} = -37.5\,mm$ (to the left of L_2) $\quad D_{STOP} = 20\,mm$

$m_{XP} = \frac{z'_{XP}}{z_{STOP}} = \frac{-37.5\,mm}{-25\,mm} = 1.5$ $\quad D_{XP} = |m_{XP}| D_{STOP} = 1.5(20\,mm) = 30\,mm$



Example System – Gaussian Solution

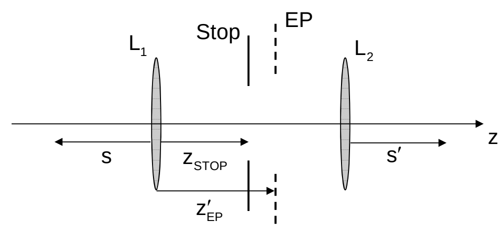
Entrance Pupil (the light is going from right to left):

$$\frac{n'}{z'_{EP}} = \frac{n}{z_{STOP}} + \frac{1}{f_1} \qquad n = n' = -1$$

$$\frac{-1}{z'_{EP}} = \frac{-1}{z_{STOP}} + \frac{1}{f_1} \qquad f_1 = 100\,mm \qquad z_{STOP} = +25\,mm$$

$$z'_{EP} = 33.33\,mm \quad (\text{to the right of } L_1) \qquad D_{STOP} = 20\,mm$$

$$m_{EP} = \frac{z'_{EP}/n'}{z_{STOP}/n} = \frac{33.33\,mm/-1}{25\,mm/-1} = 1.333 \qquad D_{EP} = |m_{EP}|D_{STOP} = 1.333(20\,mm) = 26.7\,mm$$



Both the EP and the XP are virtual.

Example System – Raytrace Solution – Pupil Locations

Start with an arbitrary ray going through the center of the stop. The EP and XP are located where this ray crosses the axis in object space and image space.

		EP	L_1	Stop	L_2	XP		
Surface	0	1	2	3	4	5	6	7
f			100	-	75			
$-\phi$			-.01	-	-.01333			
t		-33.33	25	25	-37.5			
y		0	-2.5	0	2.5	0		
u		.075	.1*	.1*	.06667			

* Arbitrary

Entrance Pupil: 33.33 mm to the right of L_1
Exit Pupil: 37.5 mm to the left of L_2

Example System – Raytrace Solution – Object at -100 mm; h = 10 mm

Trace a potential marginal ray for the image location. Scale this ray to the stop radius. This marginal ray determines the pupil radii. Trace a chief ray to determine the image height.

	Object	EP	L_1	Stop	L_2	XP	Image	
Surface	0	1	2	3	4	5	6	7
f			100	-	75			
$-\phi$			-.01	-	-.01333			
t	133.33	-33.33	25	25	-37.5	112.5		
Potential Marginal Ray:								
$\tilde{y}$	0	1.333	1.00	1.00	1.00	1.5	0	
$\tilde{u}$	.01*	.01	0	0	-.01333	-.01333		
Marginal Ray: Scale to Stop Radius $a_{STOP}/\tilde{y}_{STOP} = 10/1 = 10$								
y	0	13.33	10.0	10.0	10.0	15.0	0	
u	.1	.1	0	0	-.1333	-.1333		
Chief Ray:								
$\bar{y}$	10	0	2.5	0	-2.5	0	-7.5	
$\bar{u}$	-.075	-.075	-.1	-.1	-.06667	-.06667		

* Arbitrary

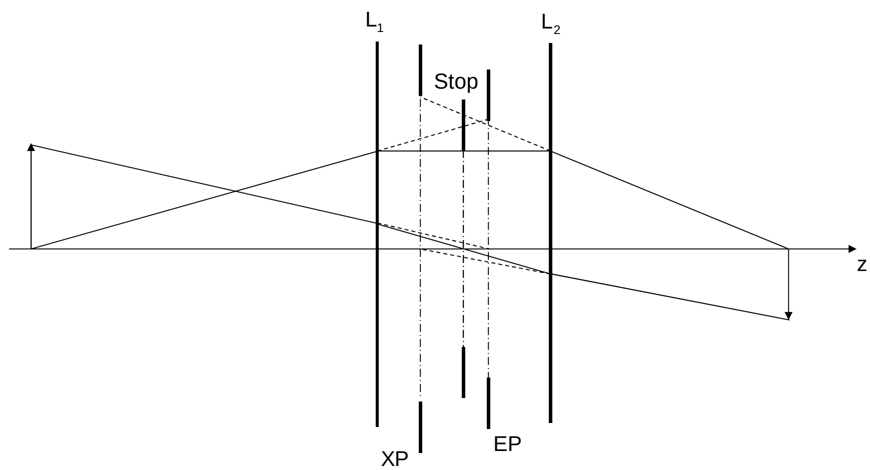
Object to EP = 100 + 33.33

$m = -.75$

Image Location: 112.5 mm to the right of XP
75.0 mm to the right of L_2

Entrance Pupil Radius: 13.33 mm ($D_{EP} = 26.7$ mm)
Exit Pupil Radius: 15.0 mm ($D_{XP} = 30.0$ mm)

Example System – Object at -100 mm



Example System – Raytrace Solution – Object at -50 mm; h = 10 mm

The pupil positions and sizes from the previous analysis can be used. The pupils do not depend on the object location.

	Object	EP	L_1	Stop	L_2	XP	Image	
Surface	0	1	2	3	4	5	6	7
f			100	-	75			
$-\phi$			-.01	-	-.01333			
t	83.33	-33.33	25	25	-37.5	187.5		
Marginal Ray:								
y	0	13.33	8.0	10.0	12.0	15.0	0	
u	.16	.16	.08	.08	-.08	-.08		
Chief Ray:								
$\bar{y}$	10.0	0	4.0	0	-4.0	0	-20.0	
$\bar{u}$	-.12	-.12	-.16	-.16	-.1067	-.1067		

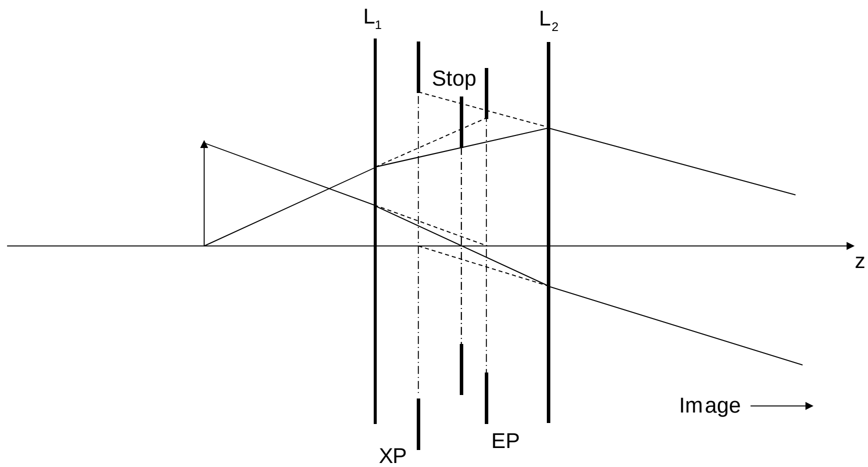
Pupil Radii

Object to EP = 50 + 33.33

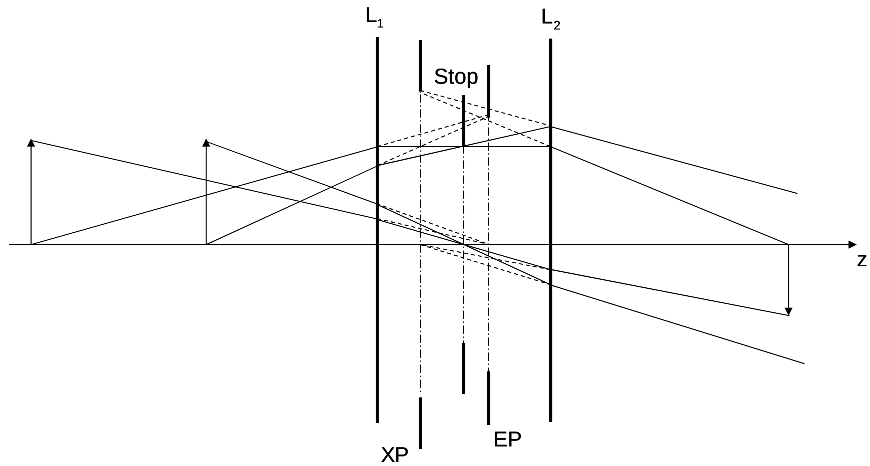
Image Location: 187.5 mm to the right of XP
150.0 mm to the right of L_2

$$m = \frac{\bar{y}'}{\bar{y}} = \frac{-20}{10} = -2.0 \qquad m = \frac{u}{u'} = \frac{.16}{-.08} = -2.0$$

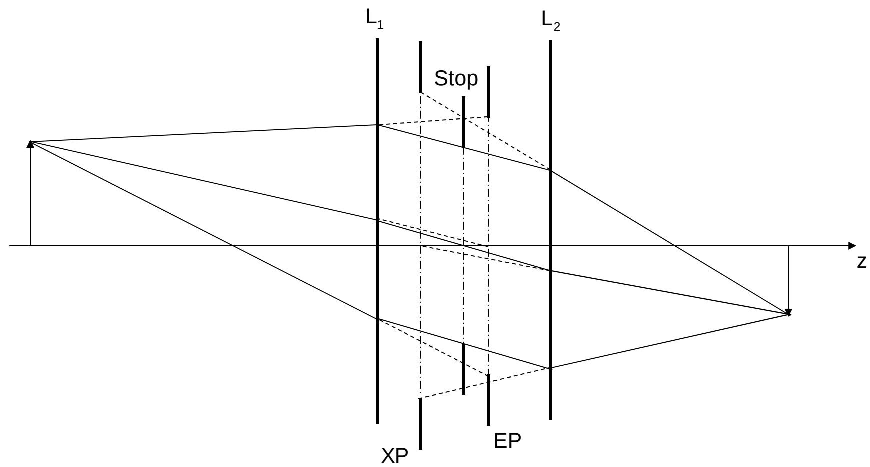
Example System – Object at -50 mm



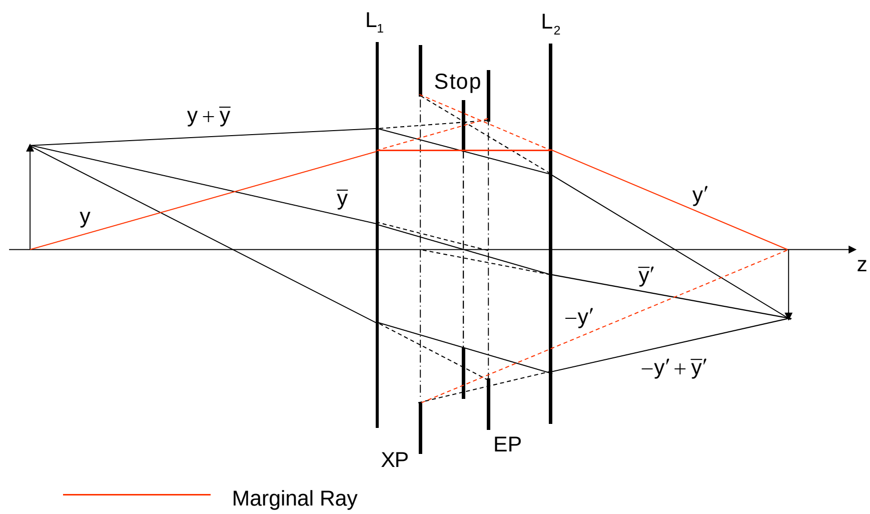
Comparison of Two Object Positions



Example System – Off-Axis Ray Bundle



Example System – Ray Bundle Extent



Numerical Aperture and F-Number

In an optical space of index n_k, the Numerical Aperture NA describes an axial cone of light in terms of the real marginal ray angle U_k. The NA can be applied to any optical space.

$$NA \equiv n_k \left|\sin U_k\right| \approx n_k \left|u_k\right|$$

The F-Number f/# describes the image-space cone of light for an object at infinity:

$$f/\# \equiv \frac{f_E}{D_{EP}} \qquad D_{EP} = \text{Diameter of the EP}$$

Note that there are often inconsistencies in the definition of f/#. Sometimes the exit pupil diameter is used or just the "clear aperture."

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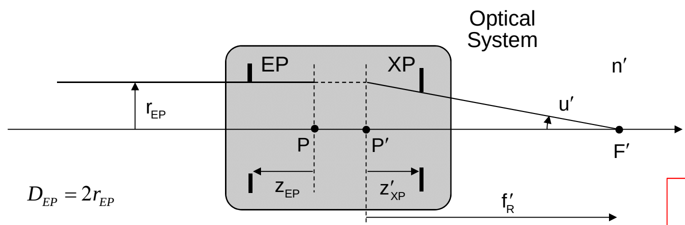
Relating Numerical Aperture and F-Number

In general, the relative locations of the EP and the XP with respect to the front and rear principal planes are unknown. The relative diameters of the EP and XP are also unknown.

Consider the marginal ray for an object at infinity:



$$D_{EP} = 2r_{EP}$$

$z_{EP} = ?$
$z'_{XP} = ?$
$D_{XP} = ?$

$$f/\# \equiv \frac{f}{D_{EP}} = \frac{f'_R}{n'D_{EP}} = \frac{f'_R}{2n'r_{EP}} = \frac{1}{2n'|u'|} \approx \frac{1}{2NA}$$

$$f/\# \approx \frac{1}{2NA} \qquad\qquad \frac{r_{EP}}{f'_R} = |u'| \approx |\sin U'|$$

The only approximation is the small angle approximation. The image space index is expressly included in NA; it is hidden in the effective focal length for the f/#.

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Working F-Number

While the f/# is strictly-speaking an image-space, infinite-conjugate measure, the approximate relationship between NA and f/# allows an f/# to be defined for other optical spaces and conjugates. As a result, an f/# can be defined for any cone of light.

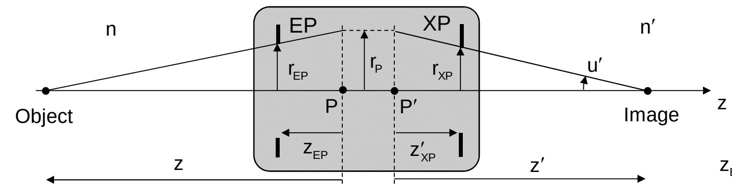
This f/# is often called a working f/# or f/#$_W$.

The previous relationship between NA and f/# becomes a definition:

$$Working\ f/\# = f/\#_W \equiv \frac{1}{2NA} \approx \frac{1}{2n|u|}$$

Image Space Working F-Number

The most common use of working f/# is to describe the image-forming cone for a finite conjugate optical system. This is the cone formed by the XP and the axial image point.



$z_{EP} = ?$

$z'_{XP} = ?$

$$Working\ f/\# = f/\#_W \equiv \frac{1}{2NA} \approx \frac{1}{2n'|u'|} = \frac{z'}{2n' r_P} \qquad |u'| = r_P / z'$$

$$f/\#_W \approx \frac{(1-m) f'_R}{2n' r_P} = \frac{(1-m) f_E}{2 r_P} \qquad z' = (1-m) f'_R$$

Gaussian Equations
m = magnification

A useful relationship requires the EP size, not the ray height at P. Assuming a thin lens with the stop at the lens:

$$r_P = r_{EP} = r_{XP}$$

$$f/\#_W \approx \frac{(1-m) f_E}{2 r_P} \approx \frac{(1-m) f_E}{2 r_{EP}} = \frac{(1-m) f_E}{D_{EP}} \qquad f/\#_W \approx (1-m) f/\#$$

Image Space Working F-Number (continued)

$$f/\#_W \approx (1-m) f/\#$$

Examples (n = 1): m = 0 $f/\#_W = f/\#$

m = -1 $f/\#_W = 2f/\#$ (1:1)

Once again, $f/\#_W$ is almost exclusively used for systems in air.

> This relationship is valid only in situations that approximate a thin lens with the stop at the lens, or more specifically, situations where the pupils are located near their respective principal planes and
>
> $$D_{EP} \approx D_{XP}$$

NAs and f/#s

Fast optical systems have small numeric values for the f/#.
Fast optical systems have large numeric values for the NA.

NA: Range 0 to n
f/#: Range ∞ to 0

Notation:

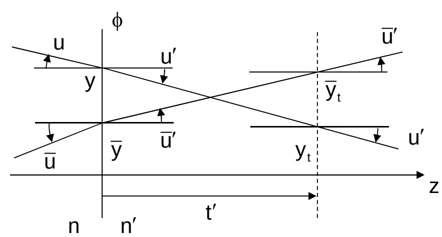
$$f/2 \Rightarrow f/\# = 2$$
$$f/2.8 \Rightarrow f/\# = 2.8$$
$$f/4 \Rightarrow f/\# = 4$$
$$f/5.6 \Rightarrow f/\# = 5.6$$

Most lenses with adjustable stops have f/#'s or f-stops labeled in increments of $\sqrt{2}$. The usual progression is f/1.4, f/2, f/2.8, f/4, f/5.6, f/8, f/11, f/16, f/22, etc, where each stop changes the area of the EP (and the light collection ability) by a factor of 2.

Optical Invariant and Lagrange Invariant

The linearity of paraxial optics provides a relationship between the heights and angles of any two rays propagating through an optical system.

Consider any two rays:



Refraction:

$$n'u' = nu - y\phi$$
$$n'\bar{u}' = n\bar{u} - \bar{y}\phi$$
$$\phi = \frac{nu - n'u'}{y} = \frac{n\bar{u} - n'\bar{u}'}{\bar{y}}$$
$$nu\bar{y} - n'u'\bar{y} = n\bar{u}y - n'\bar{u}'y$$
$$n'\bar{u}'y - n'u'\bar{y} = n\bar{u}y - nu\bar{y}$$

Terms after refraction = Terms before refraction

Transfer: $y_t = y + t'u'$ $\bar{y}_t = \bar{y} + t'\bar{u}'$

$$t' = \frac{y_t - y}{u'} = \frac{\bar{y}_t - \bar{y}}{\bar{u}'} \qquad \frac{t'}{n'} = \frac{y_t - y}{n'u'} = \frac{\bar{y}_t - \bar{y}}{n'\bar{u}'}$$

$$n'\bar{u}'y_t - n'\bar{u}'y = n'u'\bar{y}_t - n'u'\bar{y}$$
$$n'\bar{u}'y_t - n'u'\bar{y}_t = n'\bar{u}'y - n'u'\bar{y}$$

Terms after transfer = Terms before transfer

Optical Invariant and Lagrange Invariant

Invariant on Refraction: $n'\bar{u}'y - n'u'\bar{y} = n\bar{u}y - nu\bar{y}$

Invariant on Transfer: $n'\bar{u}'y_t - n'u'\bar{y}_t = n'\bar{u}'y - n'u'\bar{y}$

Optical Invariant = Invariant on Refraction = Invariant on Transfer

$$I = n\bar{u}y - nu\bar{y} = \bar{\omega}y - \omega\bar{y}$$

If the two rays are the marginal and chief rays, the Lagrange Invariant is formed:

$$Ж = H = n\bar{u}y - nu\bar{y} = \bar{\omega}y - \omega\bar{y}$$

This expression is invariant both on refraction and transfer, and it can be evaluated at any z in any optical space, and often allows for the completion of apparently partial information in an optical space by using the invariant formed in a different optical space. Many of the results obtained from raytrace derivations can also be simply obtained with the Lagrange invariant. The Lagrange invariant is particularly simple at images or objects and pupils.

Image or Object: $y = 0$ $\quad Ж = H = -nu\bar{y} = -\omega\bar{y}$

Pupil: $\bar{y} = 0$ $\quad Ж = H = n\bar{u}y = \bar{\omega}y$

Lateral Magnification and Ж

In an object or an image plane: $y = y' = 0$

$$Ж = H = n\bar{u}y - nu\bar{y}$$

Object: $Ж = -nu\bar{y}$

Image: $Ж = -n'u'\bar{y}'$

$$-nu\bar{y} = -n'u'\bar{y}'$$

$$m \equiv \frac{\bar{y}'}{\bar{y}} = \frac{nu}{n'u'} = \frac{\omega}{\omega'}$$

The lateral image magnification is given by the ratio of the marginal ray angles at the object and image.

A marginal raytrace determines not only the object and image locations, but also the conjugate magnification

---

At the stop or in a pupil plane: $\bar{y} = \bar{y}' = 0$

$$Ж = H = n\bar{u}y - nu\bar{y}$$

Pupil 1: $Ж = n\bar{u}y_{PUPIL}$

Pupil 2: $Ж = n'\bar{u}'y'_{PUPIL}$

$$-n\bar{u}y_{PUPIL} = -n'\bar{u}'y'_{PUPIL}$$

$$m_{PUPIL} \equiv \frac{y'_{PUPIL}}{y_{PUPIL}} = \frac{n\bar{u}}{n'\bar{u}'} = \frac{\bar{\omega}}{\bar{\omega}'}$$

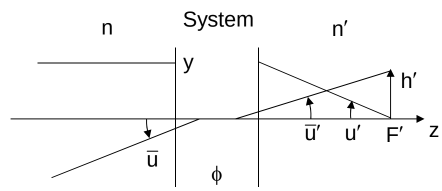
The pupil magnification is given by the ratio of the chief ray angles at the two pupils.

Since these two relationships are derived using only the Lagrange invariant, they are valid for both focal and afocal systems.

Infinite Conjugates and Ж

For an object at infinity, consider the chief ray in object and image space.

The marginal ray is parallel to the axis in object space (u = 0).



At any plane in object space (such as the first vertex):

$$Ж = n\bar{u}y$$

At the image plane F':

$$Ж = -n'u'\bar{y}' = -n'u'h'$$

Equate: $$-n'u'h' = n\bar{u}y$$

$$h' = -\bar{u}\frac{ny}{n'u'}$$

$$h' = \bar{u}nf_E = \varpi f_E$$

In air: $$h' = \bar{u}f_E$$

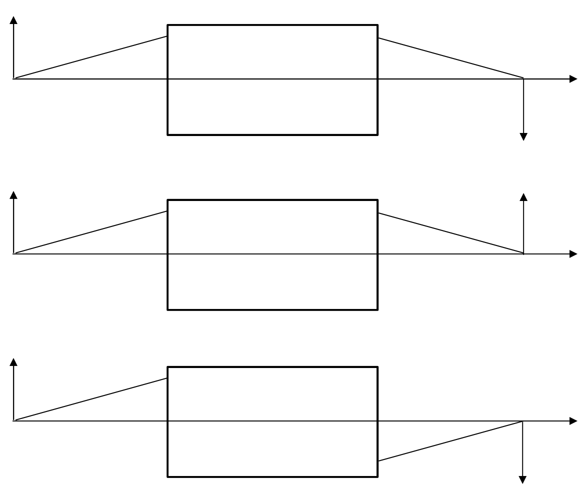
From raytrace of a marginal ray:

$$\phi = -\frac{n'u'}{y} \qquad f_E = -\frac{y}{n'u'}$$

This result can also be derived using the properties of the nodal points.

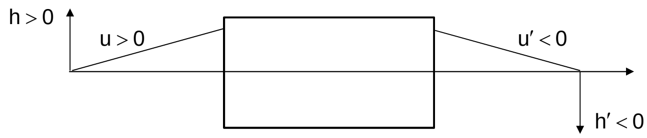
Lagrange Invariant

Possible or Impossible Optical Systems? The marginal ray is shown for the “black box” systems.

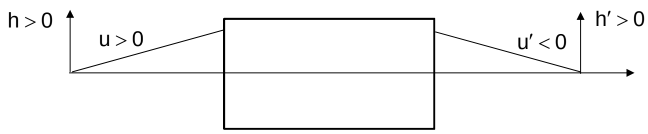


<u>Lagrange Invariant - Answers</u>

$Ж = H = n\bar{u}y - nu\bar{y}$ At an image/object plane: $Ж = -nu\bar{y} = -nuh$



Possible:
Ж does not
change sign.

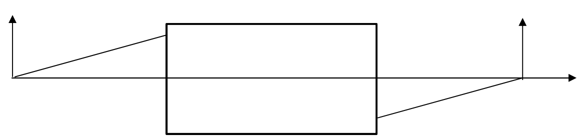


Impossible:
Ж changes
sign.

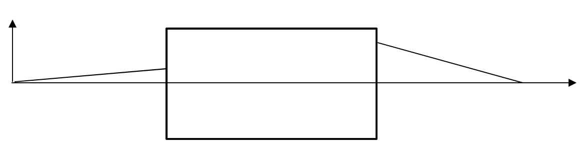
Impossible:
Ж changes
sign.

<u>Lagrange Invariant</u>

Possible or Impossible Optical Systems? The marginal ray is shown for the “black box” systems.



How big is the image?



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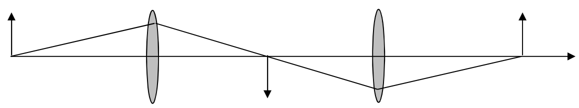
Lagrange Invariant - Answers

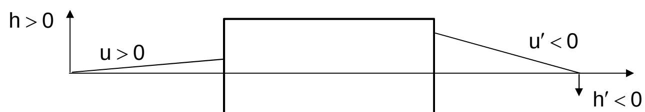
$Ж = H = n\bar{u}y - nu\bar{y}$ At an image/object plane: $Ж = -nu\bar{y} = -nuh$

Possible:
Ж does not
change sign.

A system with an
intermediate image
produces this result.



How big is the image?



The image must be inverted and small so that
Ж does not change sign or magnitude.

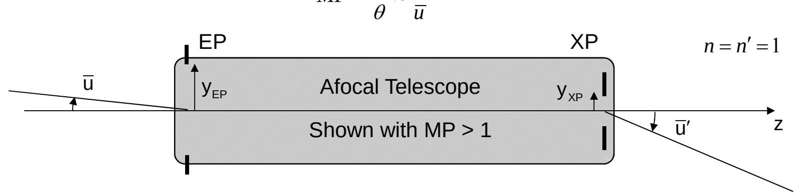
Telescopes and the Lagrange Invariant

The angular magnification of a telescope is also known as the Magnifying Power:

$$MP = \text{Angular Magnification} = \frac{\theta'}{\theta}$$

where the angle subtended by the object is θ, and the image size as seen through the telescope subtends an angle θ'. The angles θ and θ' are often considered to be the paraxial chief ray angles as measured in the Entrance and Exit Pupils.

$$MP = \frac{\theta'}{\theta} \approx \frac{\bar{u}'}{\bar{u}}$$



In the pupils: $Ж = \bar{u}y_{EP} = \bar{u}'y_{XP}$ $\quad MP = \frac{\bar{u}'}{\bar{u}} = \frac{y_{EP}}{y_{XP}}$ $\quad D_{EP} = 2y_{EP}$ $\quad D_{XP} = 2y_{XP}$

The pupils are conjugate and their sizes are related by the afocal lateral magnification m:

$mD_{EP} = D_{XP}$ $\quad MP = \frac{\bar{u}'}{\bar{u}} = \frac{D_{EP}}{D_{XP}} = \frac{1}{m}$

For a telescope that magnifies:

$|m| < 1 \quad D_{EP} > D_{XP}$

Paraxial Raytrace and Linearity

The linearity of a paraxial raytrace leads to the existence of the Optical or Lagrange Invariant. A paraxial system is completely described by the ray data from two unrelated rays. Given two rays, a third ray can be formed as a linear combination of the two rays. The coefficients are the ratios of the pair-wise invariants of the values for the three rays at some initial *z*.

$$y_3 = Ay_1 + By_2 \qquad u_3 = Au_1 + Bu_2$$

$$A = I_{32}/I_{12} \qquad B = I_{13}/I_{12}$$

$$I_{ij} = nu_i y_j - nu_j y_i$$

These coefficients A and B are evaluated at some location where initial ray height and angle data for the third ray are known. The unknown ray height and angle values at other locations can then be found using these coefficients. The expressions are valid at any *z*, in any optical space.

Changing the Lagrange invariant of a system scales the optical system. Doubling the invariant while maintaining the same object and image sizes and pupil diameters halves all of the axial distances (and the focal length).

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Proof of Linearity

$$y_3 = Ay_1 + By_2 \qquad \begin{aligned} u_3 &= Au_1 + Bu_2 \\ u_3' &= Au_1' + Bu_2' \end{aligned} \qquad \begin{aligned} n'u' &= nu - y\phi \\ y' &= y + u't' \end{aligned}$$

Transfer:

$$\begin{aligned} y_3' &= y_3 + u_3't' \\ y_3' &= Ay_1 + By_2 + \left(Au_1' + Bu_2'\right)t' \\ y_3' &= A\left(y_1 + u_1't'\right) + B\left(y_2 + u_2't'\right) \\ y_3' &= Ay_1' + By_2' \end{aligned}$$

Refraction:

$$\begin{aligned} n'u_3' &= nu_3 - y_3\phi \\ n'u_3' &= n(Au_1 + Bu_2) - (Ay_1 + By_2)\phi \\ n'u_3' &= A(nu_1 - y_1\phi) + B(nu_2 - y_2\phi) \\ n'u_3' &= An'u_1' + Bn'u_2' \\ u_3' &= Au_1' + Bu_2' \end{aligned}$$

Linearity holds for both transfer and refraction.

Linearity Coefficients

A: $y_3 = Ay_1 + By_2$ $\quad u_2y_3 = Au_2y_1 + Bu_2y_2$

$u_3 = Au_1 + Bu_2$ $\quad u_3y_2 = Au_1y_2 + Bu_2y_2$

Subtract: $u_2y_3 - u_3y_2 = A(u_2y_1 - u_1y_2)$

$$A = \frac{u_3y_2 - u_2y_3}{u_1y_2 - u_2y_1} = \frac{nu_3y_2 - nu_2y_3}{nu_1y_2 - nu_2y_1}$$

$$A = I_{32}/I_{12}$$

$$I_{ij} = nu_iy_j - nu_jy_i$$

B: $y_3 = Ay_1 + By_2$ $\quad u_1y_3 = Au_1y_1 + Bu_1y_2$

$u_3 = Au_1 + Bu_2$ $\quad u_3y_1 = Au_1y_1 + Bu_2y_1$

Subtract: $u_1y_3 - u_3y_1 = B(u_1y_2 - u_2y_1)$

$$B = \frac{u_1y_3 - u_3y_1}{u_1y_2 - u_2y_1} = \frac{nu_1y_3 - nu_3y_1}{nu_1y_2 - nu_2y_1}$$

$$B = I_{13}/I_{12}$$

Use of the Lagrange Invariant

Consider an f/2 optical system in air with a focal length of 100 mm.
At the conjugates used, the system NA = 0.1.
If the image height is 10 mm, what is the angular field of view of the system in object space?

There is no need to assume $D_{XP} = D_{EP}$, or that thin lenses are used.

$f/2 \Rightarrow \dfrac{f}{D_{EP}} = 2$ $\quad D_{EP} = 50\,mm$ $\quad r_{EP} = 25$

$NA = 0.1 \Rightarrow u' = -0.1$ Marginal ray angle in image space

$Ж = n\bar{u}y - nu\bar{y}$

Image Plane: $y' = 0$ $\quad \bar{y}' = 10$ $\quad n' = 1$

$Ж = -n'u'\bar{y}' = 1$

At EP: $y = r_{EP} = 25$ $\quad \bar{y} = 0$ $\quad n = 1$

$Ж = 1 = n\bar{u}y = 25\bar{u}$

$\bar{u} = .04$ $\quad \theta_{1/2} = \bar{U} = \tan^{-1}\bar{u} = 2.3°$

This example shows that the Lagrange invariant helps relate known quantities in one optical space to unknown quantities in another optical space.