Section 6

Object-Image Relationships

The purpose of this study is to examine the imaging properties of the general system that has been defined by its Gaussian properties and cardinal points.

Different combinations of front and rear focal lengths can be studied:

- $f_r < 0$  $f'_r > 0$  Positive Focal System
- $f_r > 0$  $f'_r < 0$  Negative Focal System
- $f_r < 0$  $f'_r < 0$  Positive Focal System; Net Reflective
- $f_r > 0$  $f'_r > 0$  Negative Focal System; Net Reflective
Object-Image Relationships – In Terms of the Front and Rear Focal Lengths

Newtonian Equations (Origins at F, F’):
\[
\frac{z_r}{f_r} = \frac{1}{m}, \quad \frac{z_r'}{f'_r} = -m
\]
\[
z_r z_r' = f_r f'_r
\]
\[
\frac{\Delta z'}{\Delta z} = \left( -\frac{f'_r}{f_r} \right) m_1 m_2
\]

Gaussian Equations (Origins at P, P’):
\[
\frac{z}{f_r} = 1 - \frac{1}{m}, \quad \frac{z'}{f'_r} = 1 - m
\]
\[
f_r + f'_r = 1, \quad \frac{z'}{z} = \left( -\frac{f'_r}{f_r} \right) m
\]
\[
\frac{\Delta z'}{\Delta z} = \left( -\frac{f'_r}{f_r} \right) m_1 m_2
\]

Afocal Systems:
\[
m = \frac{f_{12}}{f_{11}} = -\frac{f_2}{f_1}
\]
\[
m = \frac{\Delta z'}{\Delta z} = \left( -\frac{f_{12}}{f_{11}} \right) \left( -\frac{f'_2}{f'_1} \right) m_1 m_2 = \left( \frac{n}{n} \right) m_1 m_2
\]

Object-Image Zones

Newtonian Equations (Origins at F, F’)
Case A: \( z_F < 0 \) Object to the Left of F
\[
\frac{z_r}{f_r} = \frac{1}{m}
\]
\[
z_r = -\frac{f_r}{m} < 0
\]
\[
f_r > 0
\]
\[
\frac{z_r'}{f'_r} = -m > 0
\]
\[
f'_r > 0, \quad m > 0
\]
\[
f'_r > 0, \quad m > 0
\]
\[
\frac{z_r'}{f'_r} = -m > 0
\]
\[
f'_r > 0, \quad m > 0
\]
\[
\frac{z'_r}{z'_r} < 0, \quad z'_r > 0
\]
\[
z'_r > 0, \quad z'_r < 0
\]
\[
\left( -\frac{f'_r}{f'_r} \right) > 0, \quad \left( -\frac{f'_r}{f'_r} \right) = 0, \quad \left( -\frac{f'_r}{f'_r} \right) < 0
\]
\[
\frac{\Delta z'}{\Delta z} = \left( -\frac{f'_r}{f'_r} \right) m_1 m_2
\]
\[
\frac{\Delta z'}{\Delta z} < 0, \quad \frac{\Delta z'}{\Delta z} > 0, \quad \frac{\Delta z'}{\Delta z} > 0, \quad \frac{\Delta z'}{\Delta z} < 0
\]
Positive Focal System

Object to the Left of $F$
Real Object – Real Image

$z_r < 0$  $f_r < 0$  $m < 0$  $\frac{\Delta z'}{\Delta z} > 0$

$z' > 0$  $\frac{\Delta z'}{\Delta z} > 0$

(Newtonian distances)

This representation hides the fact that both the object and image spaces are separate and extend from minus infinity to plus infinity.

In addition, the physical relationship between the locations of the Front and Rear Principal Planes on the system configuration (number and type of elements – including reflective, spacings and thicknesses, refractive indices, etc.). Physically, $P'$ can be to the right of $P$, to the left of $P$ or coincident with $P$. It depends on the system configuration.

Images are inverted
Objects and images are in the same order
Object-Image Zones

Newtonian Equations (Origins at F, F')

Case B: \( z_F > 0 \) Object to the Right of F

\[
\frac{z_F}{m} = \frac{1}{f_F}, \quad \frac{z_F}{m} = \frac{f_F}{m} > 0
\]

\[
f_F < 0 \quad m > 0 \quad f_F > 0 \quad m < 0
\]

\[
\frac{z_F^{'}}{f_F^{'}} = -m < 0 \quad \frac{z_F^{'}}{f_F^{'}} = -m > 0
\]

\[
f_F^{'}, z_F^{'}, \left( \frac{f_F^{'}}{f_F} \right) > 0
\]

\[
\Delta z^{'}, \Delta m \quad m_1, m_2
\]

\[
\frac{\Delta z^{'}}{\Delta z} < 0 \quad \frac{\Delta z^{'}}{\Delta z} > 0
\]

Positive Focal System

\( 0 < z_F < -f_F \quad f_F < 0 \quad m > 0 \quad (m > 1) \quad \Delta z^{'}, \Delta z^{'}, \Delta z^{'}, \Delta z < 0 \)

Object between F and P
Real Object – Virtual Image

Images are erect and magnified
Objects and images are in the same order
Positive Focal System

$f_f < 0 \quad m > 0 \quad (0 < m < 1) \quad \frac{\Delta z}{\Delta s} > 0$

Object to the Right of P
Virtual Object – Real Image

(Newtonian distances)

Images are erect and minified
Objects and images are in the same order

Positive Focal System – Zones

$f_f < 0 \quad f_f' > 0$

Object Space

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>P</th>
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<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
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Image Space

<table>
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<tr>
<th></th>
<th>F'</th>
<th>P'</th>
</tr>
</thead>
<tbody>
<tr>
<td>B'</td>
<td>C'</td>
<td>A'</td>
</tr>
</tbody>
</table>

$m > 1 \quad 1 > m > 0 \quad m < 0$
Negative Focal System

Object to the Left of P
Real Object – Virtual Image

$$z_p < -f_p \quad f_p' > 0 \quad m > 0 \quad (0 < m < 1) \quad \frac{\Delta z'}{\Delta z} > 0$$

Images are erect and minified
Objects and images are in the same order

Object Between P and F
Virtual Object – Real Image

$$-f_p < z_p < 0 \quad f_p > 0 \quad m > 0 \quad (m > 1) \quad \frac{\Delta z'}{\Delta z} > 0$$

Images are erect and magnified
Objects and images are in the same order
Negative Focal System

Object to the Right of F
Virtual Object – Virtual Image

Images are inverted
Objects and images are in the same order

Negative Focal System – Zones

Object Space

Image Space

m < 0 0 < m < 1 m > 1
Positive Focal System – Reflective

Object to the Left of F
Real Object – Real Image

\[ z_r < 0 \quad f_r < 0 \quad m < 0 \quad \Delta z' < 0 \]

(Newtonian distances)

Images are inverted
Objects and images are in the opposite order

Positive Focal System – Reflective

Object between F and P
Real Object – Virtual Image

\[ 0 < z_r < -f_r \quad f'_r < 0 \quad m > 0 \quad (m > 1) \quad \Delta z' < 0 \]

(Newtonian distances)

Images are erect and magnified
Objects and images are in the opposite order
Positive Focal System – Reflective

\[ z_p > -f_r \quad f_r' < 0 \quad m > 0 \quad (0 < m < 1) \quad \Delta z' < 0 \]

Object to the Right of \( P \)

Virtual Object – Real Image

Images are erect and minified

Objects and images are in the opposite order
Object-Image Zones Summary

Positive Focal System

<table>
<thead>
<tr>
<th>Zone</th>
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<th>B</th>
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<tbody>
<tr>
<td>m</td>
<td>&lt; 0</td>
<td>1</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>f_c</td>
<td>&lt; 0</td>
<td>f'_c</td>
<td>&gt; 0</td>
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Positive Focal System – Reflective

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Negative Focal System

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Negative Focal System - Reflective

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The object-image zones show the general image properties as a function of the object location relative to the cardinal points.

An object in Zone A will map to an image in Zone A', etc. All optical spaces extend from \(-\infty\) to \(+\infty\).

A net reflective system (an odd number of reflections) inverts image space about \(P\).

Object Space/Image Space Mapping – Focal Systems

\[
\Delta z' = \left( -\frac{f'_c}{f_c} \right) m_1 m_2
\]

\(m_1\) and \(m_2\) are the lateral magnifications for the two planes.

Newtonian Equations (distances measured from \(F, F'\))

\[
z = -\frac{1}{m} \quad \frac{z'_r}{f'_s} = -m
\]

The magnification is proportional to the Newtonian image distance and inversely proportional to the Newtonian object distance.

\[
m = -\frac{f_r}{z_r} \quad m = -\frac{z'_r}{f'_s}
\]

When \(\Delta z\) is small, the longitudinal magnification is obtained

\[
m_1 = m_2 = m \quad \bar{m} = \left( -\frac{f'_c}{f_c} \right) m_2
\]

The image space spacing is proportional to the Newtonian image distance squared and inversely proportional to the Newtonian object distance squared.
Object Space/Image Space Mapping – Positive Focal System

\[ f_r < 0 \]
\[ f'_r > 0 \]

Object Space/Image Space Mapping – Negative Focal System

\[ f_r > 0 \]
\[ f'_r < 0 \]
Collinear Transformation

A collinear transformation maps points to points, lines to lines, and planes to planes. The general mapping equations associated with a collinear transformation are:

\[ x' = \frac{a_x y + b_y x + c_z + d_3}{a_x y + b_y x + c_z + d_3} \]
\[ y' = \frac{a_x y + b_y x + c_z + d_3}{a_x y + b_y x + c_z + d_3} \]
\[ z' = \frac{a_x y + b_y x + c_z + d_3}{a_x y + b_y x + c_z + d_3} \]

By applying the symmetries associated with a rotationally-symmetric system and the definitions of the magnification and the cardinal points, all of the relationships of Gaussian imagery can be derived (for both focal and afocal systems) from these general mapping equations.

These derivations have been prepared by Prof. Roland Shack and are included as Appendix A to these notes.