Section 5

Gaussian Imagery

Imaging

Paraxial optics provides a simplified methodology to determine ray paths through optical systems. Using this method, the image location for a general system can be calculated relative to the Principal Planes of the system. This has also allowed the Focal Length of a general system to be defined in terms of the system's overall refractive properties.

However, the physical locations of the Principal Planes for a given optical configuration still need to be derived, so the analysis in not yet complete. The details of determining the system focal length of a combination of optical elements is also not clear.

In addition, the image size is as of yet unknown.

As will be demonstrated later, paraxial raytrace can be used to answer these questions.

Gaussian optics provides an alternative method of system analysis that treats imaging as a mapping from object space into image space.

Gaussian Optics

Gaussian optics treats imaging as a mapping from object space into image space. It is a special case of a collinear transformation applied to rotationally symmetric systems, and it maps points to points, lines to lines and planes to planes.

The corresponding object and image elements are called conjugate elements.

Assumptions:
- Axial Symmetry
- Meridional Plane
  - contains the axis of symmetry
  - usually the y-z plane

Sign Conventions and definitions of reference locations will be applied as usual.

The transverse magnification or lateral magnification is the ratio of the image point height from the axis to the conjugate object point height:

$$m \equiv \frac{h'}{h}$$

The fundamental principle of geometrical optics holds: any three-dimensional object scene is represented by a collection of independently radiating point sources. Each point source is independently imaged through the system to its conjugate image point (perfect imagery). The three-dimensional image is the superposition of all of the point images.

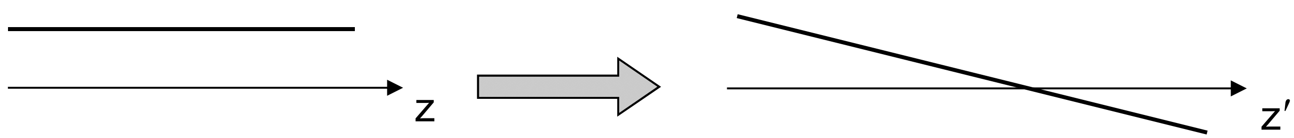
Gaussian Optics Theorems

Theorems (all derive from rotational symmetry of the system):

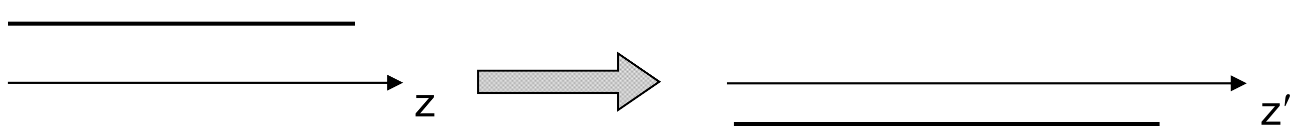
- Planes perpendicular to the axis in one space are mapped to planes perpendicular to the axis in the other space.

- Lines parallel to the axis in one space map to conjugate lines in the other space that either intersect the axis at a common point (focal system), or are also parallel to the axis (afocal system).

Focal System:



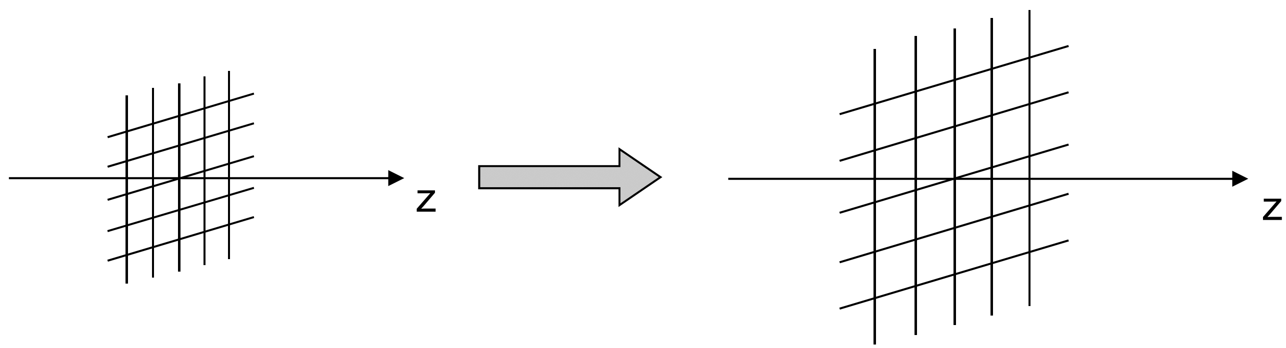
Afocal System:



In a Gaussian or collinear mapping, lines must map to lines and are conjugate elements.

Gaussian Optics Theorems - Continued

- The transverse magnification is constant in conjugate planes perpendicular to the axis.



If the magnification in a plane were not constant, the images of the grid lines would become curved or distorted.

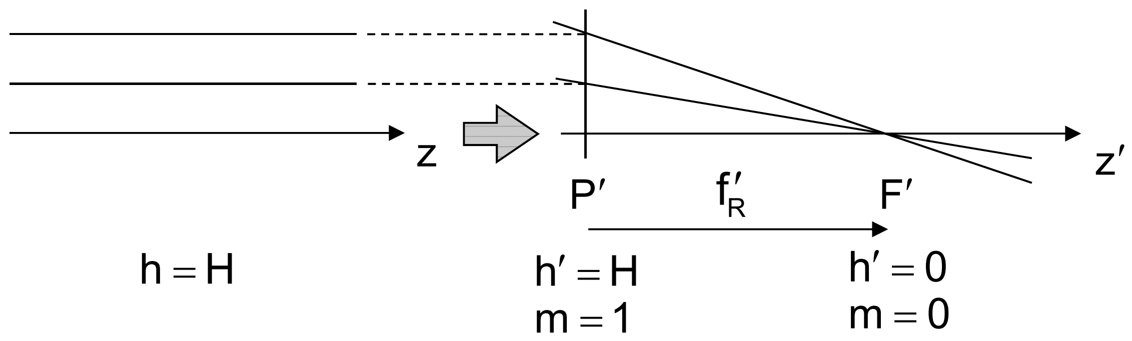
Cardinal Points and Planes

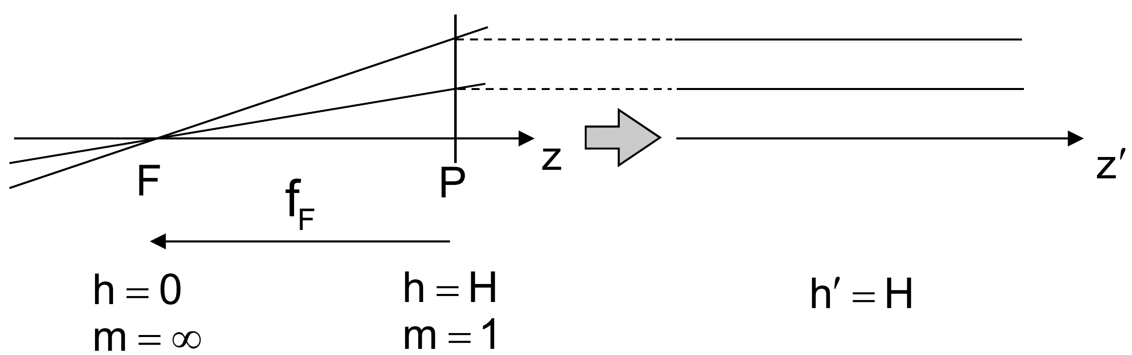
The cardinal points and planes completely describe the focal mapping. They are defined by specific magnifications:

F	Front focal point/plane	$m = \infty$
F'	Rear focal point/plane	$m = 0$
P	Front principal plane	$m = 1$
P'	Rear principal plane	$m = 1$

$$m \equiv \frac{h'}{h}$$

Consider conjugate line elements in object and image space:





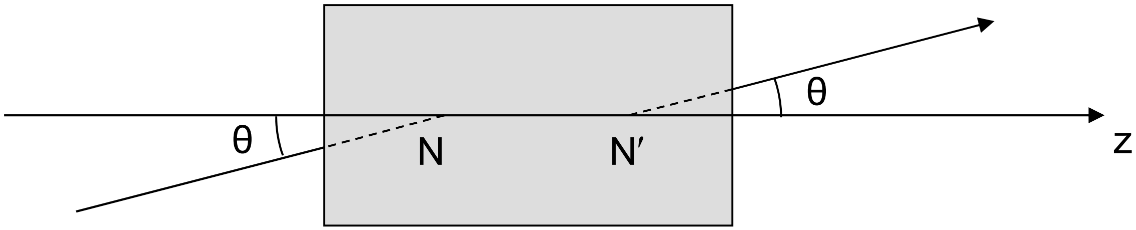
The front and rear focal lengths (f_F and f'_R) are defined as the directed distances from the front and rear principal planes to the respective focal points.

Cardinal Points and Planes

Unprimed variables are in object space.
Primed variables are in image space.

AP	Front antiprincipal plane/point	m = -1
AP'	Rear antiprincipal plane/point	m = -1
N	Front nodal point	Angular Mag = 1
N'	Rear nodal point	Angular Mag = 1

The two nodal points of a system are conjugate points.



For a focal imaging system, an object plane location is related to its conjugate image plane location through the transverse magnification associated with those planes.

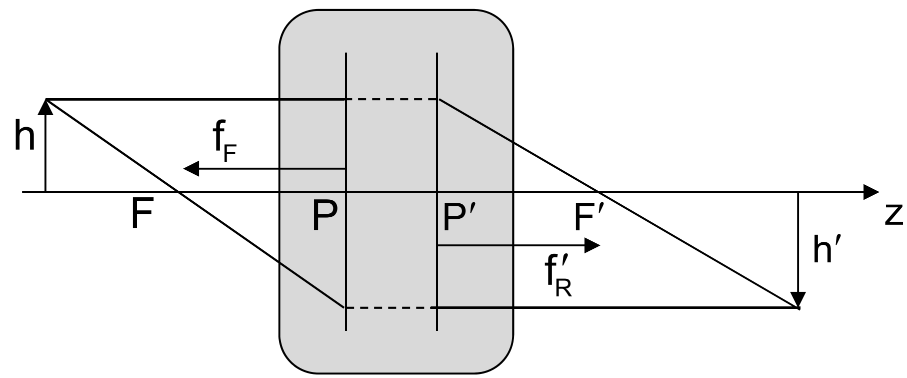
Newtonian equations measure object and image distances from the *focal planes*.

Gaussian equations measure object and image distances from the *principal planes*.

Locating an Image with the Cardinal Points

The optical system can be represented as a set of Principal Planes and a set of Focal Points (with the respective focal lengths).

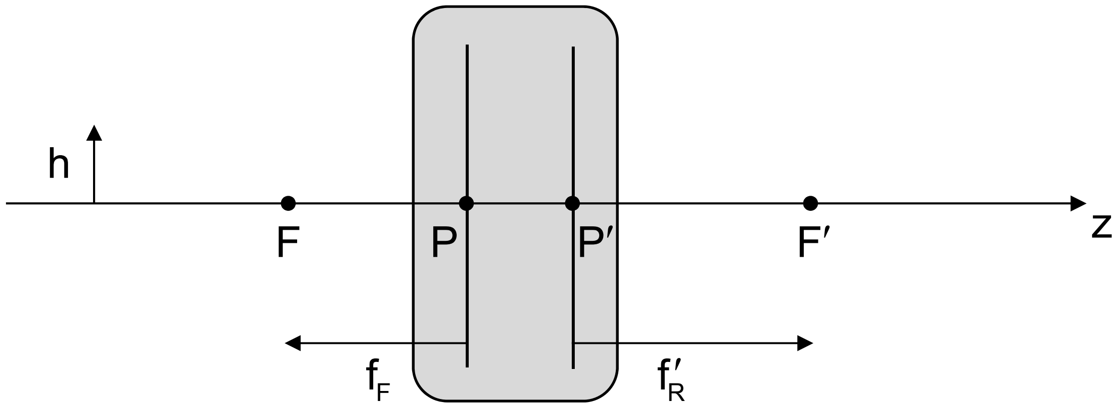
- A ray incident on the front principal plane will emerge from the rear principal plane at the same height.
- A ray parallel to the optical axis in object space passes through the rear focal point.
- A ray through the front focal point emerges parallel to the optical axis.
- The intersection of two rays defines an object or image point.

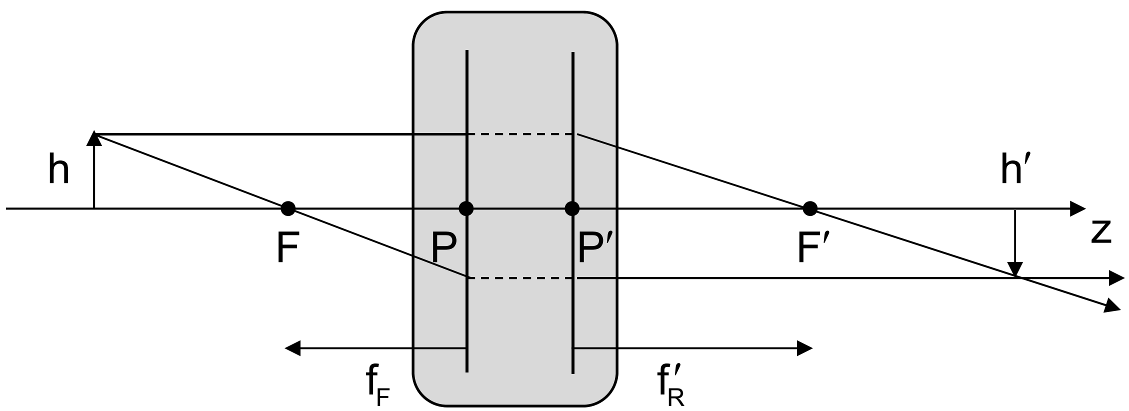


The Principal Planes serve as the planes of effective refraction between object space and image space.

Locating an Image with the Cardinal Points – Example 1

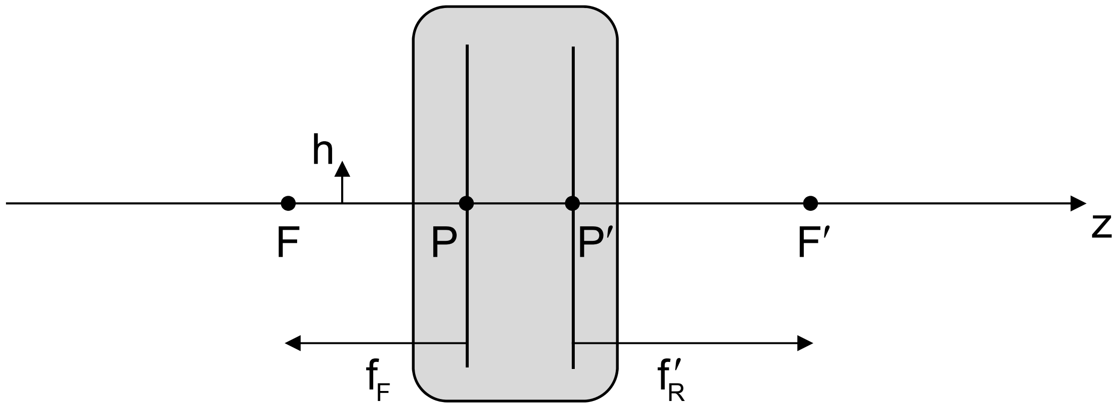
Positive System – Real object to the left of the front focal point F

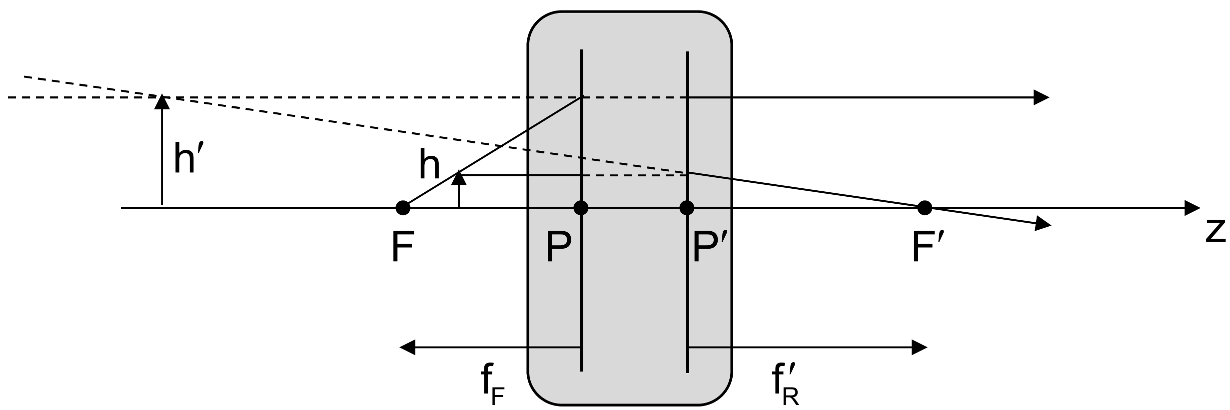




Locating an Image with the Cardinal Points – Example 2

Positive System – Real object between the front focal point F and the front principal plane

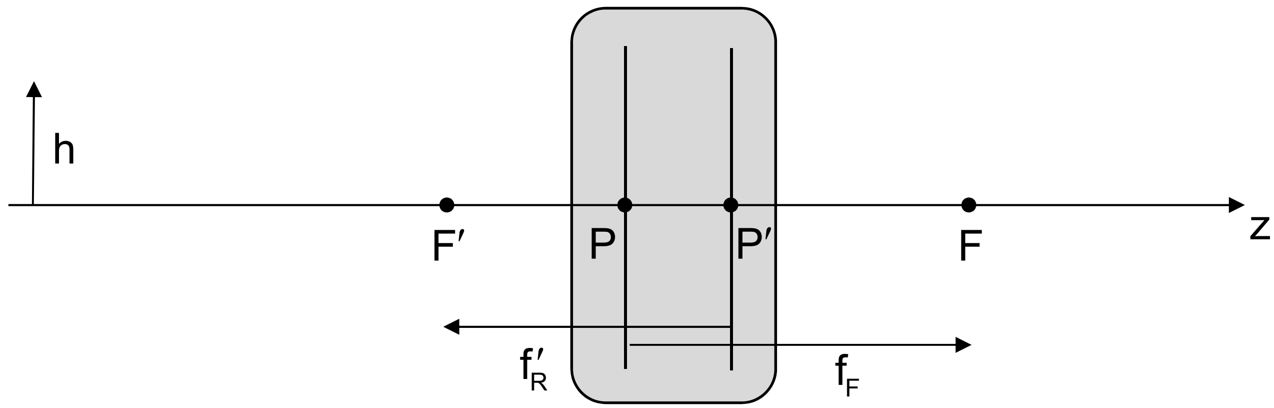


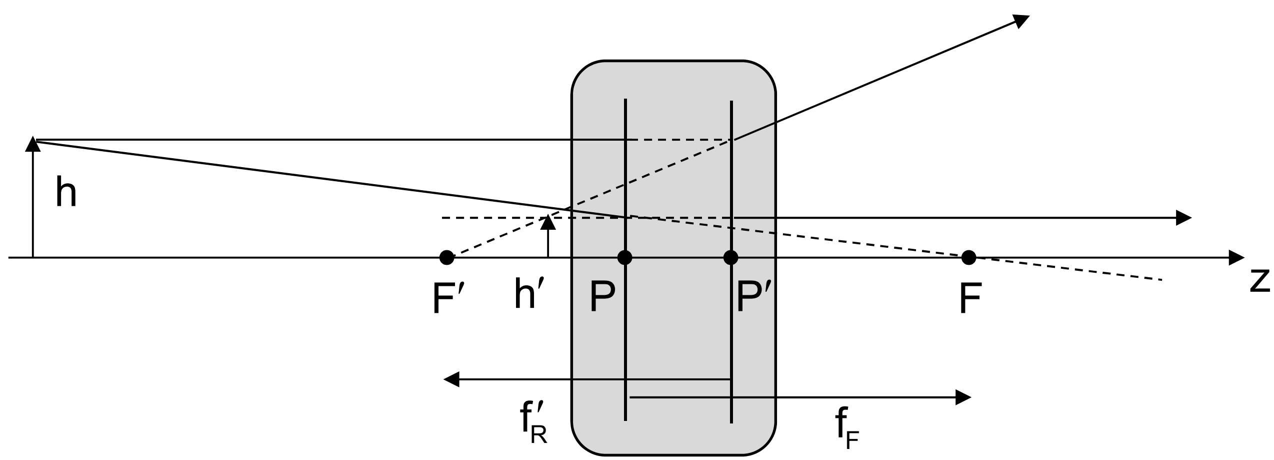


The two image space rays diverge and have a virtual crossing. An enlarged, erect virtual image is produced. The image is in image space.

Locating an Image with the Cardinal Points – Example 3

Negative System – Real object – Note the locations of the Focal Points. The Front Focal Point F and Front Principal Plane P are both in the system object space. Similarly, the Rear Focal Point F' and Rear Principal Plane P' are both in the system image space. The same image formation rules apply.

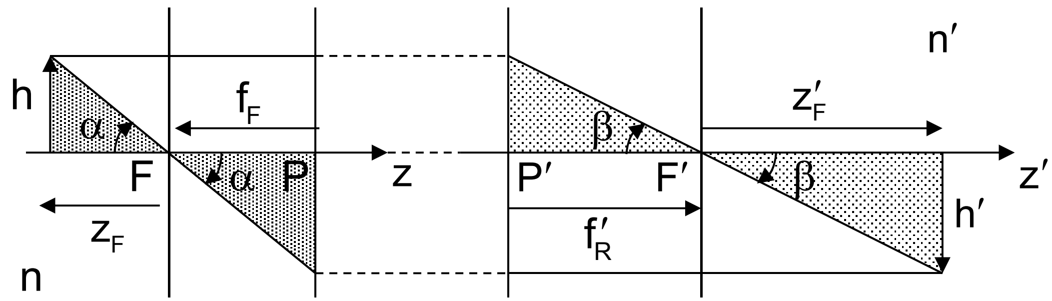




Once again, the two image space rays diverge and have a virtual crossing. A minified, erect virtual image is produced. The image is in image space.

Newtonian Equations

The Newtonian equations characterize this Gaussian mapping when the axial locations of the conjugate object and image planes are measured relative to the respective *Focal Points*. By definition, the front and rear focal lengths continue to be measured relative to the principal planes. Use similar triangles.



$\alpha < 0$

$\beta < 0$

Object Distance = z_F

Image Distance = z'_F

$$\frac{h}{z_F} = \tan\alpha = \frac{h'}{-f_F}$$

$$\frac{h}{-f'_R} = \tan\beta = \frac{h'}{z'_F}$$

$$m \equiv \frac{h'}{h} = -\frac{f_F}{z_F}$$

$$m \equiv \frac{h'}{h} = -\frac{z'_F}{f'_R}$$

$$\frac{z_F}{f_F} = -\frac{1}{m}$$

$$\frac{z'_F}{f'_R} = -m$$

$$\left(\frac{z_F}{f_F}\right)\left(\frac{z'_F}{f'_R}\right) = 1$$

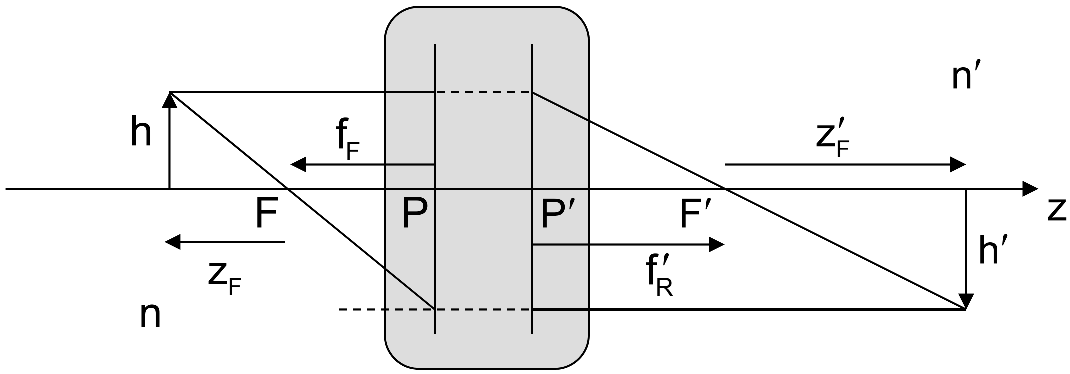
$$z_F \; z'_F = f_F \; f'_R$$

Magnification:

- Independent of the object height h
- Proportional to the image distance z'_F
- Inversely proportional to the object distance z_F

Newtonian Equations Applied to a System

The Newtonian equations characterize this Gaussian mapping when the axial locations of the conjugate object and image planes are measured relative to the respective *Focal Points*.



$$m \equiv \frac{h'}{h}$$

$$f'_R = \frac{n'}{\phi} = n' f_E$$

$$f_F = -\frac{n}{\phi} = -n f_E \qquad z_F = -\frac{f_F}{m} \qquad z'_F = -m f'_R \qquad z_F \, z'_F = f_F \, f'_R$$

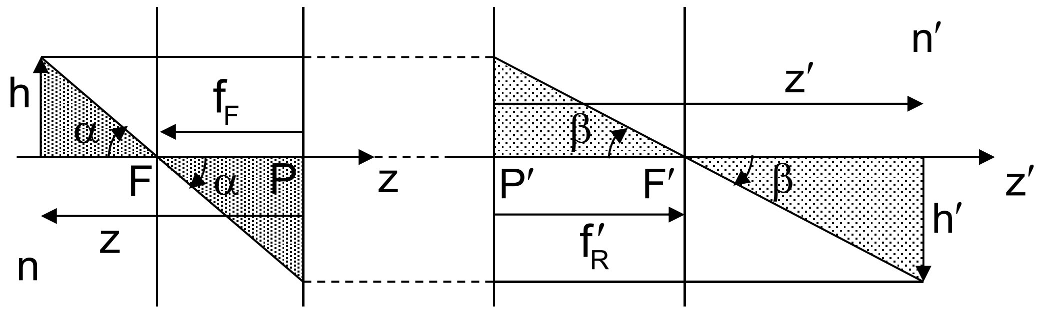
$$\frac{z_F}{n} = \frac{f_E}{m} \qquad \frac{z'_F}{n'} = -m f_E \qquad \left(\frac{z_F}{n}\right)\left(\frac{z'_F}{n'}\right) = -f_E^{\;2}$$

In air: $n = n' = 1$

$$z_F = \frac{f_E}{m} \qquad z'_F = -m f_E \qquad z_F \, z'_F = -f_E^{\;2}$$

Gaussian Equations

The Gaussian equations describe the focal mapping when the respective *Principal Planes* are the references for measuring the locations of the conjugate object and image planes. Use the same similar triangles.



$\alpha < 0$

$\beta < 0$

$$\frac{h}{z - f_F} = \tan\alpha = \frac{h'}{-f_F} \qquad \frac{h}{-f'_R} = \tan\beta = \frac{h'}{z' - f'_R}$$

Object Distance = z

Image Distance = z′

$$m \equiv \frac{h'}{h} = \frac{-f_F}{z - f_F} \qquad m \equiv \frac{h'}{h} = \frac{z' - f'_R}{-f'_R}$$

$$\frac{z}{f_F} = 1 - \left(\frac{1}{m}\right) = \frac{m-1}{m} \qquad \frac{z'}{f'_R} = 1 - m$$

Ratio: $$\frac{z'}{z} = \left(-\frac{f'_R}{f_F}\right) m$$

Add: $$\frac{f_F}{z} + \frac{f'_R}{z'} = \frac{m}{m-1} + \frac{1}{1-m} \qquad \frac{f_F}{z} + \frac{f'_R}{z'} = 1$$

Magnification:
- Independent of the object height h
- Proportional to the ratio of the image distance to the object distance z'/z

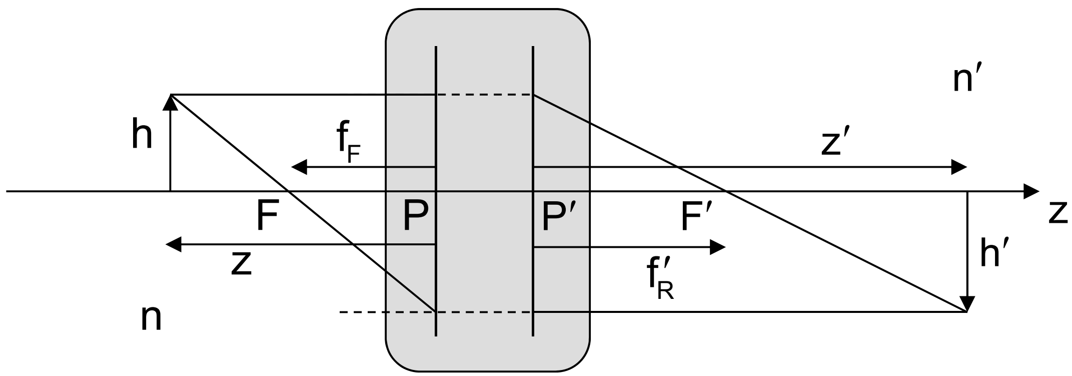
Gaussian Equations Applied to a System

The Gaussian equations describe the focal mapping when the respective *Principal Planes* are the references for measuring the locations of the conjugate object and image planes.

$$f'_R = \frac{n'}{\phi} = n' f_E$$

$$f_F = -\frac{n}{\phi} = -n f_E$$

$$-\frac{f'_R}{f_F} = \frac{n'}{n}$$



$$m \equiv \frac{h'}{h}$$

$$\frac{z}{f_F} = 1 - \frac{1}{m} \qquad \frac{z'}{f'_R} = 1 - m \qquad \frac{z'}{z} = \left(-\frac{f'_R}{f_F}\right) m \qquad \frac{f_F}{z} + \frac{f'_R}{z'} = 1$$

$$\frac{z}{n} = \frac{(1-m)}{m} f_E \qquad \frac{z'}{n'} = (1-m) f_E \qquad m = \frac{z'/n'}{z/n} \qquad \frac{n'}{z'} = \frac{n}{z} + \frac{1}{f_E}$$

In air:

$n = n' = 1$

$$z = \frac{(1-m)}{m} f_E \qquad z' = (1-m) f_E \qquad m = \frac{z'}{z} \qquad \frac{1}{z'} = \frac{1}{z} + \frac{1}{f_E}$$

Conjugate planes for m = 1, 0 and ∞

Use the Gaussian equations.

m = 1

$$\frac{z}{f_F} = 1 - \left(\frac{1}{m}\right) \qquad \frac{z'}{f'_R} = 1 - m$$

$$z = 0 \qquad z' = 0$$

Object plane located at P

Image plane located at P′

P and P' are conjugate.

m = 0

$$\frac{z}{f_F} = 1 - \left(\frac{1}{m}\right) \qquad \frac{z'}{f'_R} = 1 - m$$

$$z = \infty \qquad z' = f'_R$$

Object plane located at ∞

Image plane located at F'

∞ and F' are conjugate.

m = ∞

$$\frac{z}{f_F} = 1 - \left(\frac{1}{m}\right) \qquad \frac{z'}{f'_R} = 1 - m$$

$$z = f_F \qquad z' = \infty$$

Object plane located at F

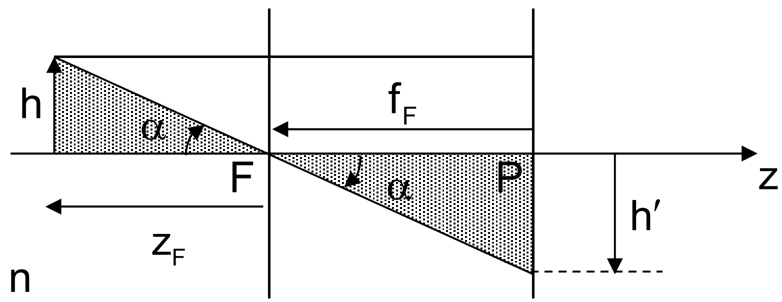
Image plane located at ∞

F and ∞ are conjugate.

Sign Conventions Revisited

Newtonian Equations Derivation

Original Configuration – Object to the left of F; z_F, h′ and f_F are negative.

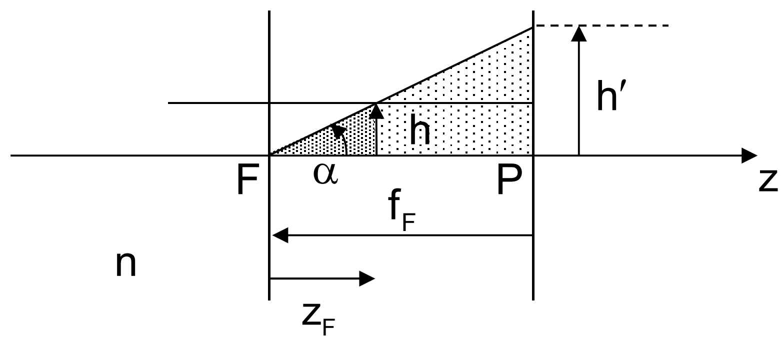


$$\alpha < 0$$

$$\frac{h}{z_F} = \tan\alpha = \frac{-h'}{f_F}$$

$$m \equiv \frac{h'}{h} = -\frac{f_F}{z_F}$$

New Configuration – Object to the right of F: now only f_F is negative.



$$\alpha > 0$$

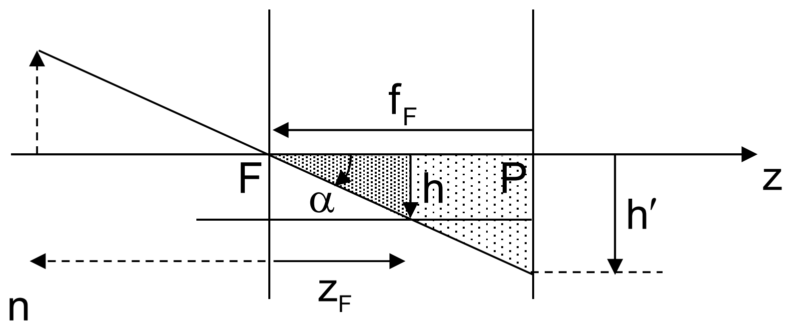
$$\frac{h}{z_F} = \tan\alpha = \frac{h'}{-f_F}$$

$$m \equiv \frac{h'}{h} = -\frac{f_F}{z_F}$$

Same result!

Sign Conventions Revisited

Original Figure – Object to the right of F; now, h, h′ and f_F are negative.



$$\alpha < 0$$

$$\frac{h}{z_F} = \tan\alpha = \frac{h'}{-f_F}$$

$$m \equiv \frac{h'}{h} = -\frac{f_F}{z_F}$$

Same result!

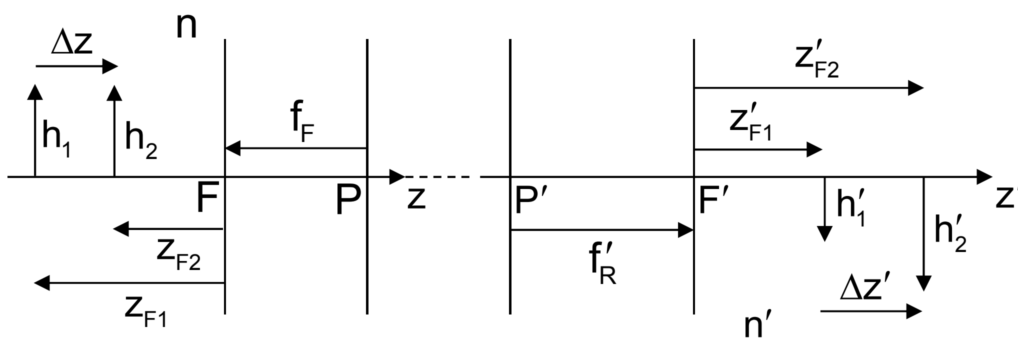
As the object distance z_F goes positive to place the object to the right of F, the object height h becomes negative to compensate in the equations. The net result is always the same. Note that in all of these figures, the reference location used to define the quantities has not changed.

Set the equations up to be consistent with the figure, the sign conventions will allow the equation to be valid for different configurations.

Distances Between Pairs of Conjugate Planes – Thickness Magnification

The thickness magnification relates the distances between pairs of conjugate planes.
Use Newtonian equations:

$$\frac{z_F}{f_F} = -\frac{1}{m} \qquad \frac{z'_F}{f'_R} = -m$$



$$\Delta z = z_{F2} - z_{F1} \qquad \Delta z' = z'_{F2} - z'_{F1}$$

$$\frac{\Delta z}{f_F} = \frac{z_{F2}}{f_F} - \frac{z_{F1}}{f_F} = \left(-\frac{1}{m_2}\right) - \left(-\frac{1}{m_1}\right) = \frac{1}{m_1} - \frac{1}{m_2} = \frac{m_2 - m_1}{m_1 m_2}$$

$$\frac{\Delta z'}{f'_R} = \frac{z'_{F2}}{f'_R} - \frac{z'_{F1}}{f'_R} = (-m_2) - (-m_1) = -(m_2 - m_1)$$

$$\frac{\Delta z' / f'_R}{\Delta z / f_F} = \frac{-(m_2 - m_1)}{(m_2 - m_1)/m_1 m_2} = -m_1 m_2$$

$$\frac{\Delta z'}{\Delta z} = \left(-\frac{f'_R}{f_F}\right) m_1 m_2$$

The thicknesses Δz and Δz′ are independent of the origins used.

Thickness Magnification and Longitudinal Magnification

$$\frac{\Delta z'}{\Delta z} = \left(-\frac{f_R'}{f_F} \right) m_1\, m_2$$

$$\frac{\Delta z'}{\Delta z} = \left(\frac{n'}{n} \right) m_1\, m_2$$

$$\frac{\Delta z' / n'}{\Delta z / n} = m_1\, m_2$$

$$f_R' = \frac{n'}{\phi} = n' f_E$$

$$f_F = -\frac{n}{\phi} = -n f_E$$

$$-\frac{f_R'}{f_F} = \frac{n'}{n}$$

The thickness magnification equations are valid for widely separated planes. Since it is a difference in position, the result is independent of the choice of origins.

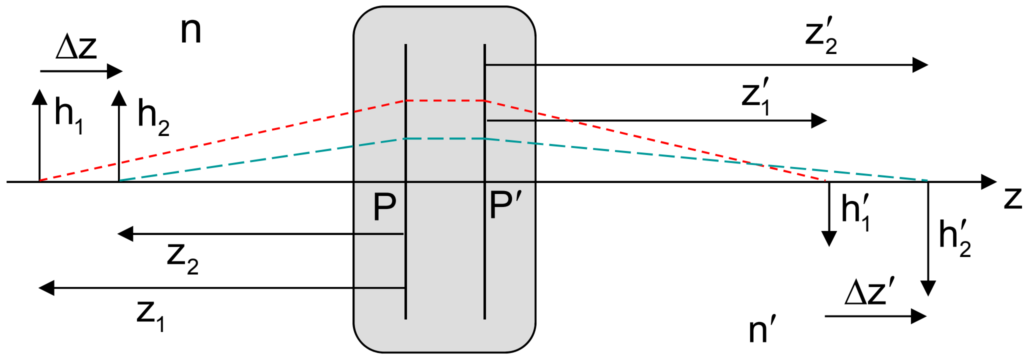
As the plane separation approaches zero, the local longitudinal or axial magnification is obtained.

$$\bar{m} = \lim_{\Delta z \to 0} \frac{\Delta z'}{\Delta z} = \left(\frac{n'}{n} \right) m^2 \qquad \frac{\Delta z' / n'}{\Delta z / n} = m^2$$

Since m varies with position, the longitudinal magnification and the thickness magnification are a function of z and z'.

Thickness Magnification and Longitudinal Magnification – System

The thickness magnification relates the distances between pairs of conjugate planes.



$$m_1 \equiv \frac{h_1'}{h_1}$$

$$m_2 \equiv \frac{h_2'}{h_2}$$

$$\Delta z = z_2 - z_1 \qquad \Delta z' = z_2' - z_1'$$

$$\frac{\Delta z'/n'}{\Delta z/n} = m_1\, m_2$$

In air: $$\frac{\Delta z'}{\Delta z} = m_1\, m_2$$

The thickness magnification equations are valid for widely separated planes. As the plane separation approaches zero, the local longitudinal or axial magnification is obtained:

$$\bar{m} = \lim_{\Delta z \to 0} \frac{\Delta z'}{\Delta z} = \frac{n'}{n} m^2$$

$$\frac{\Delta z'/n'}{\Delta z/n} = m^2$$

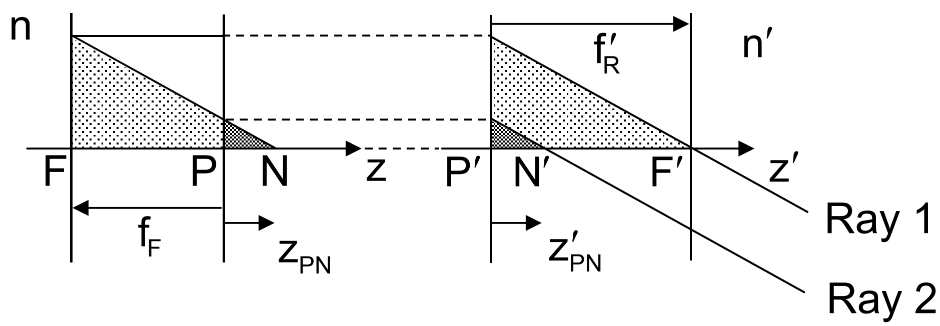
$$m_1 = m_2 = m$$

In air: $$\bar{m} = m^2$$

Since m varies with position, the longitudinal magnification and the thickness magnification are a function of z and z'.

Nodal Points

Two additional cardinal points are the front and rear nodal points (N and N') that define the location of unit angular magnification for a focal system. A ray passing through one nodal point of a system is mapped to a ray passing through the other nodal point having the same angle with respect to the optical axis. The nodal points are conjugate points.



Rays 1 and 2 must be parallel in image space, since their conjugate rays cross in the front focal plane. The indicated triangles are not only similar, but identical.

$$z'_{PN} = z_{PN} \qquad\qquad z_{PN} - f_F = f'_R$$

$$z'_{PN} = z_{PN} = f_F + f'_R$$

Using the thickness magnification (and the locations of the principal planes and nodal points):

$$\frac{z'_N - z'_P}{z_N - z_P} = \frac{z'_{PN}}{z_{PN}} = \frac{\Delta z'}{\Delta z} = \left(\frac{n'}{n}\right) m_P\, m_N \qquad\qquad m_P = 1$$

$$1 = \left(\frac{n'}{n}\right) m_N \qquad\qquad m_N = \frac{n}{n'}$$

Nodal Points of a System

$$z_{PN} = z'_{PN} = f_F + f'_R$$

$$m_N = \frac{n}{n'}$$

If the same index occurs in object space and image space:

$$n' = n$$

$$f_F = -f'_R$$

$$z_{PN} = z'_{PN} = 0$$

$$m_N = 1$$

The nodal points are located at the respective principal planes if the image space index of refraction equals the object space index of refraction.

Origins at the Nodal Points

If the object and image locations are measured relative to the Nodal points, an interesting and important result is obtained. Use the thickness magnification relationship:

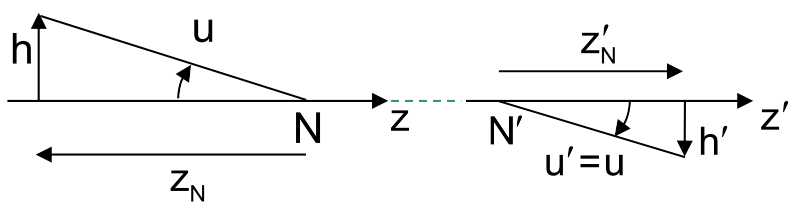
$$\frac{\Delta z'}{\Delta z} = \left(\frac{n'}{n}\right) m_1 m_2 \qquad \frac{z'_N}{z_N} = \frac{\tilde{z}' - \tilde{z}'_N}{\tilde{z} - \tilde{z}_N} = \left(\frac{n'}{n}\right) m_N\, m \qquad m_N = \frac{n}{n'}$$

$\tilde{z}$ and $\tilde{z}'$ are the object and image positions measured from some origins

$\tilde{z}_N$ and $\tilde{z}'_N$ are the nodal point positions measured from the same origins

$$\frac{z'_N}{z_N} = \left(\frac{n'}{n}\right)\left(\frac{n}{n'}\right) m = m$$

The angular subtense of an image as seen from the rear nodal point equals the angular subtense of the object as seen from the front nodal point.



$$u = \frac{h}{z_N} = \frac{h'}{z'_N} = u' \qquad m \equiv \frac{h'}{h} = \frac{z'_N}{z_N}$$

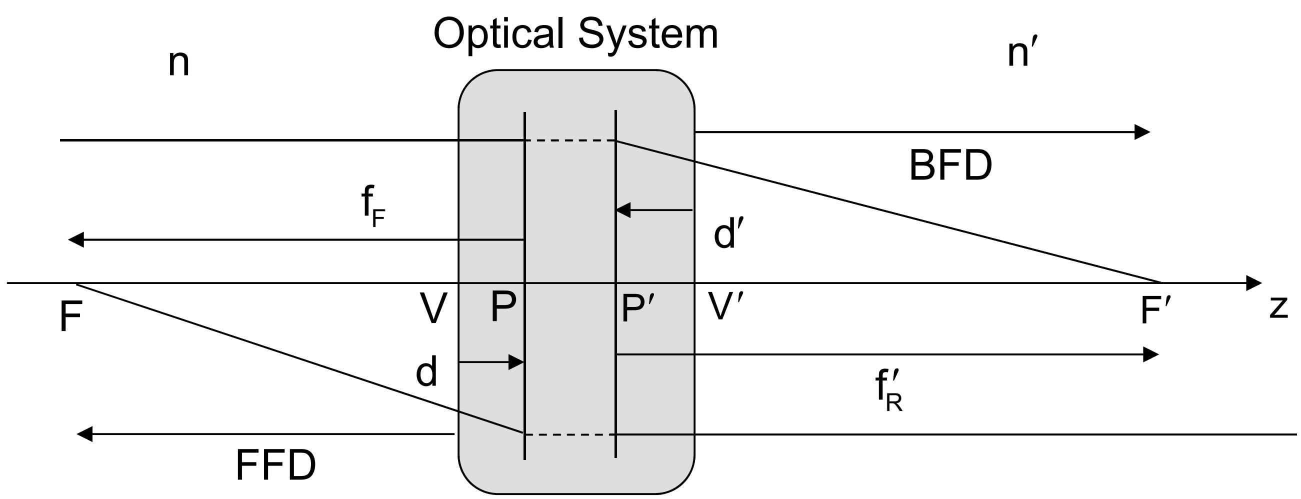
Cardinal Points and Planes of a System

The system focal length is f

The object space index of refraction is n

The image space index of refraction is n′

$$f = f_E = -\frac{f_F}{n} = \frac{f'_R}{n'}$$



The Front Focal Distance FFD is the distance from the system front vertex to the front focal point F

The Back Focal Distance BFD is the distance from the system back vertex to the back focal point F′

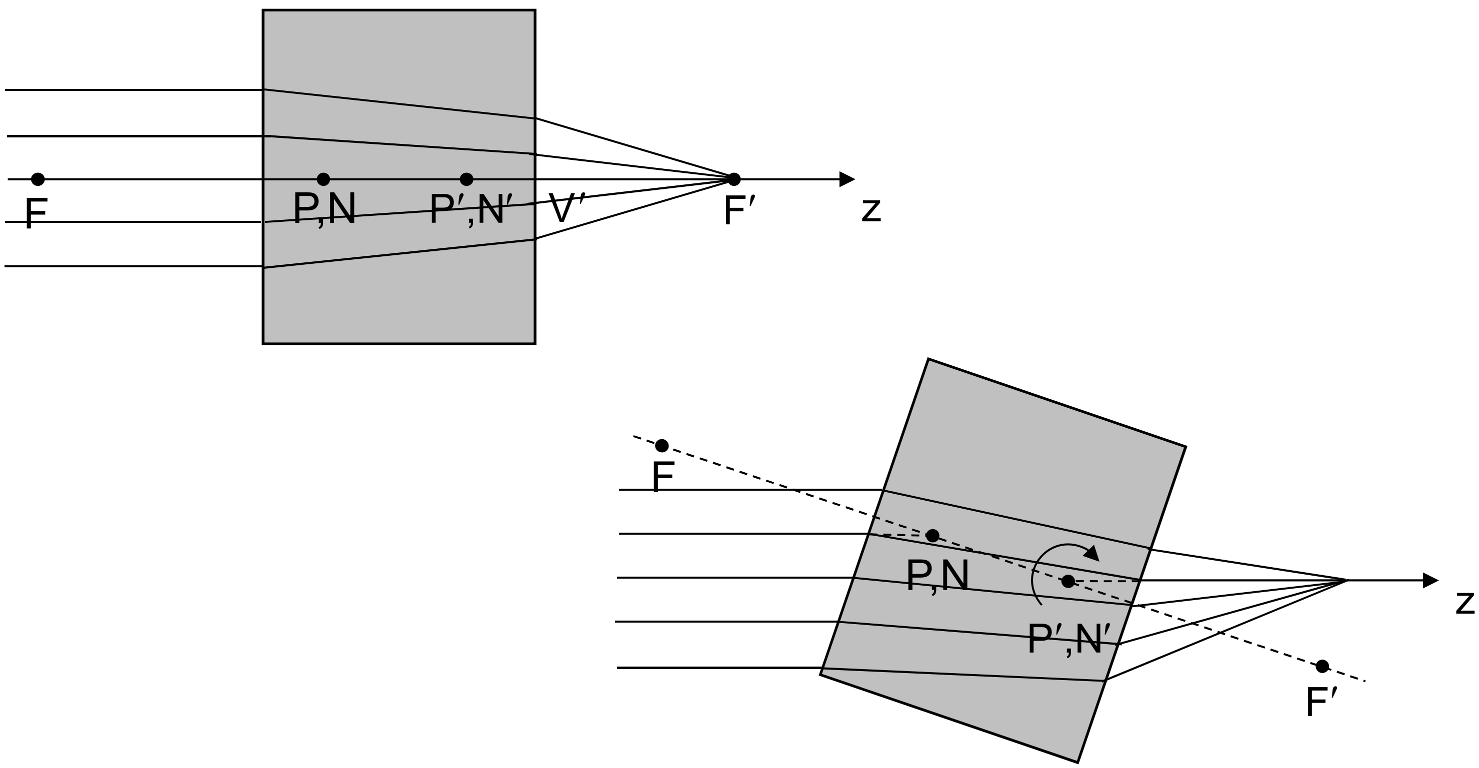
d is the shift of the system front Principal Plane P from the system front vertex V

d′ is the shift of the system rear Principal Plane P′ from the system back vertex V′

<u>Nodal Slide</u>

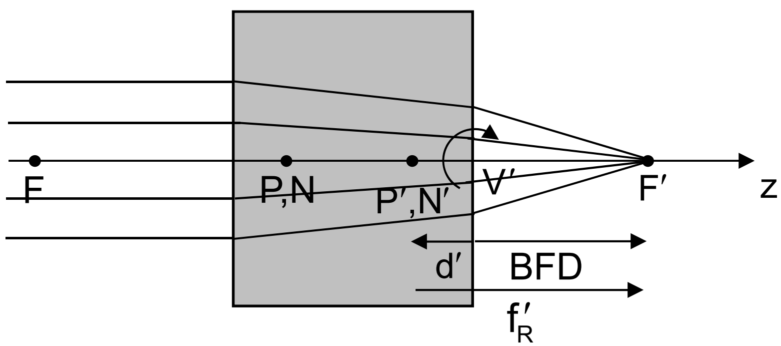
When a lens system is in air, the nodal points coincide with the principal points. The use of a nodal slide allows the principal planes and the focal length to be experimentally determined.

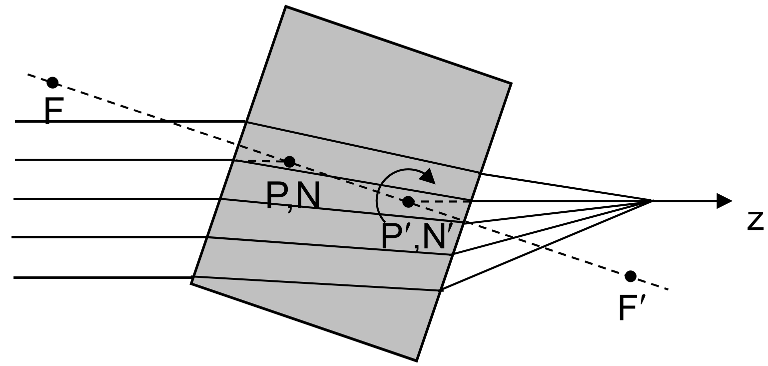
When the lens is rotated about its rear nodal point, the rays will converge to the same point. The image will not move even though the ray bundle forming the image is skewed, and F' is shifted to one side.



By inverting the lens, the front cardinal points (N, F, P) can also be located.

Nodal Slide – Procedure





- Mount the lens system on a translation stage which is on a rotation stage.
- Position the rear vertex over the rotation axis. When properly positioned, the vertex will not translate when the lens is rotated.
- Use collimated illumination.
- Use a microscope (with a micrometer) to measure the distance between the rear vertex V' and the focus (rear focal point F').
- This is the Back Focal Distance BFD – the distance from the rear vertex to the rear focal point.
- While observing the image, reposition the lens with the translation stage so that the image does not translate when the lens is rotated. The rear nodal point (and the rear principal point) are now over the rotation point.
- The amount the lens was moved is the separation d' between the rear vertex and the rear principal plane.
- The system focal length is found by

$$f = f_E = f'_R = BFD - d'$$

Gaussian Properties of a Single Refracting Surface

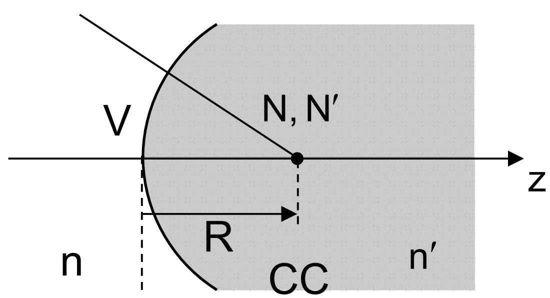
Principal Planes:

The front and rear principal planes are coincident and located at the surface vertex V.

A ray incident on the surface at some height will emerge from that surface at the same height. Unit magnification occurs at the surface.

Nodal Points:

Both nodal points are located at the center of the curvature CC of the optical surface.



A ray heading towards the center of curvature is normal to the refracting surface and is not refracted. The ray has the same angle in object space and image space (angular magnification = 1), and the ray's axial intercept at the CC defines both nodal points.

Surface Power and Focal Points:

$$\phi = (n' - n)C \qquad f'_R = \frac{n'}{\phi} = n' f_E \qquad f_F = -\frac{n}{\phi} = -n f_E$$

The front and rear focal lengths are measured from the surface vertex.

Nodal Points of a Single Refracting Surface

Verify the Nodal Point – Principal Plane separation:

$$f'_R = \frac{n'}{\phi} = n' f_E$$

$$f_F = -\frac{n}{\phi} = -n f_E$$

$$z_{PN} = z'_{PN} = f_F + f'_R$$

$$z_{PN} = z'_{PN} = -\frac{n}{\phi} + \frac{n'}{\phi} = \frac{n'-n}{\phi}$$

$$\phi = (n'-n)C$$

$$z_{PN} = z'_{PN} = \frac{1}{C} = R$$

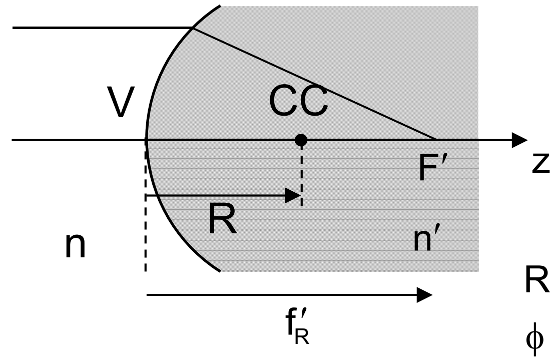
The principal planes are at the surface vertex

The nodal points are at the surface center of curvature.

Nodal point magnification for a single refracting surface: $m_N = \dfrac{n}{n'}$

Gaussian Properties of a Refracting Surface – Summary

Power: $\phi = (n' - n)C = \frac{(n' - n)}{R}$



R > 0
ϕ > 0
(n′ > n)

Focal Lengths: $f = f_E \equiv \frac{1}{\phi} = -\frac{f_F}{n} = \frac{f'_R}{n'}$

$$f_F = -\frac{n}{\phi} = -nf_E \qquad f'_R = \frac{n'}{\phi} = n'f_E$$

Principal planes/points – located at the surface vertex.
Nodal points – located at the center of curvature of the surface. $m_N = \frac{n}{n'}$

The front and rear focal lengths are measured from the surface vertex (principal planes).

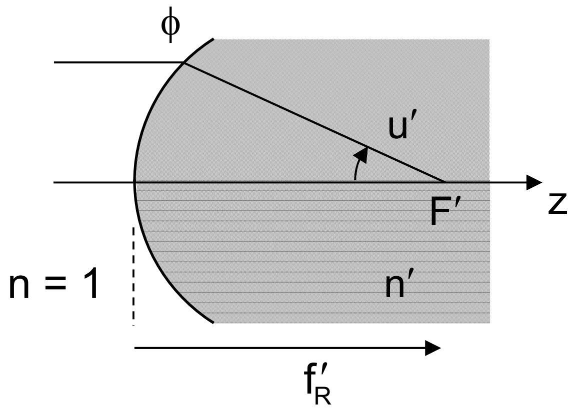
Principal plane - Nodal point separation: $z_{PN} = z'_{PN} = \frac{1}{C} = R$

Imaging: $\frac{n'}{z'} = \frac{n}{z} + \frac{1}{f_E} \qquad \frac{n'}{z'} = \frac{n}{z} + \phi$

Magnification: $m = \frac{z'/n'}{z/n}$

Reduced Distance Equivalence

Consider the example of a refracting surface and its thin lens equivalent. Both have the same power ϕ.

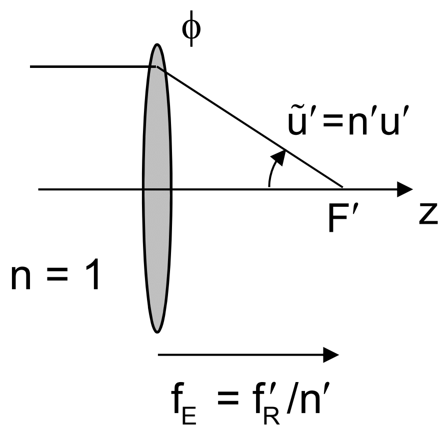


$$f_R' = n' f_E \qquad f_E = \frac{f_R'}{n'} \qquad f_E = \frac{1}{\phi}$$

The reduced focal length of the surface equals the focal length of the thin lens.

If the ray angle or slope for the refracting surface is u', then the ray angle is n'u' for the thin lens.

A ray angle multiplied by the refractive index of its optical space is called an optical angle: nu.



The paraxial raytrace equation is in terms of optical angles and the effective (or reduced) focal length of the system:

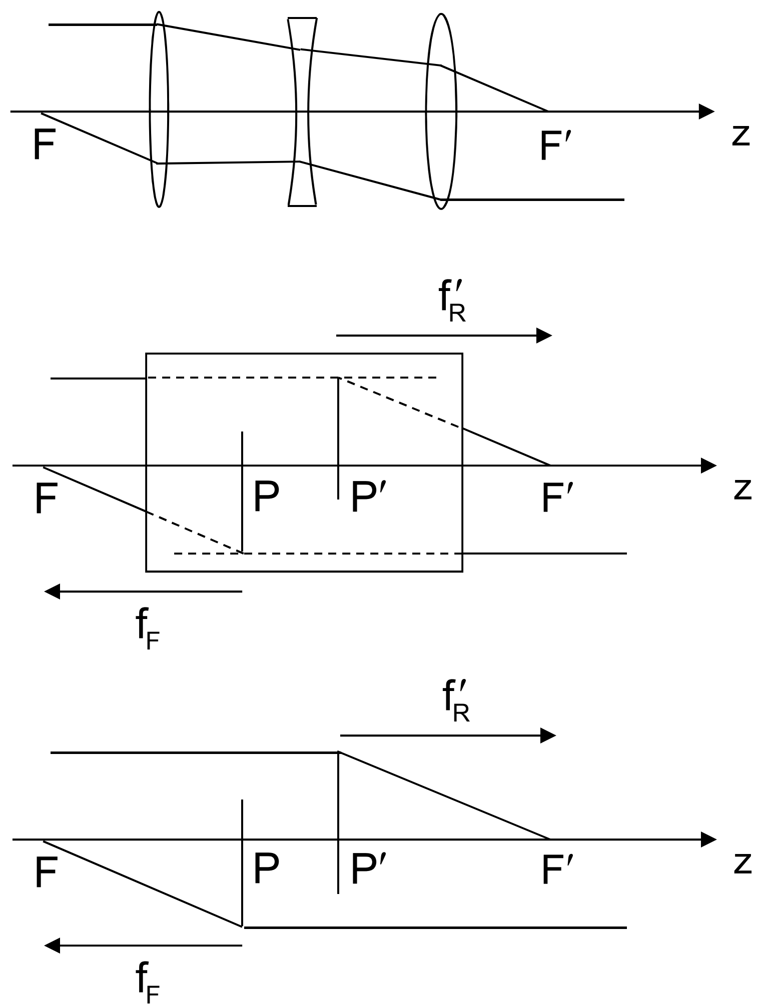
$$n'u' = nu - y\phi = nu - \frac{y}{f_E} = nu - \frac{y}{f_R' / n'}$$

The use of reduced distances and optical angles allows any system to be represented as an air-equivalent system comprised of thin lenses.

If the object is not at infinity, the image distance for the thin lens becomes the reduced image distance of the refracting surface.

Gaussian Imagery

The cardinal points, along with the associated focal lengths and power, completely specify the mapping from object space into image space for a focal system. Gaussian imagery aims to reduce any focal imaging system, regardless of the number of surfaces, to a single, unique set of cardinal points.



Imaging Example 1

Positive Focal System

$$f_E = 100\ mm$$
$$n = n' = 1.0$$

Object: 200 mm to left of F
h = 10 mm

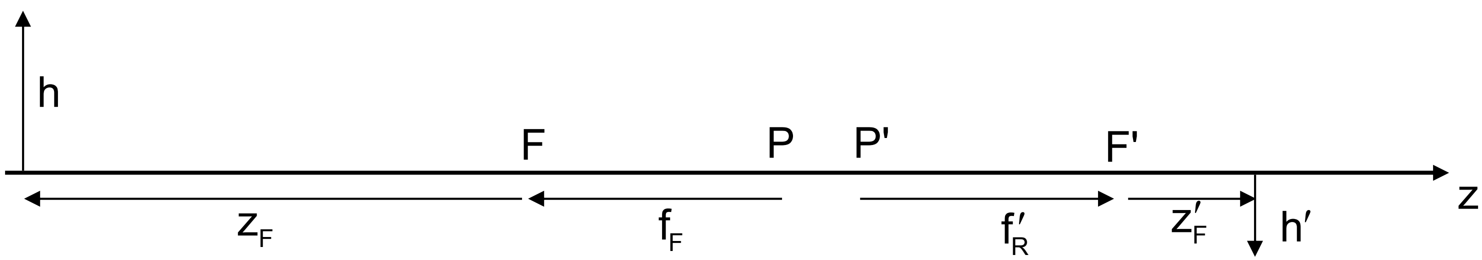
Use Newtonian equations:

$$z_F = -200\ mm$$
$$h = 10\ mm$$

$$\frac{z_F}{n} = \frac{f_E}{m} \qquad m = \frac{f_E}{z_F} = \frac{100}{-200} = -.50$$

$$h' = mh = -.50(10\,mm) = -5.0\,mm$$

$$\frac{z'_F}{n'} = -mf_E \qquad z'_F = -m\,f_E = -(-.50)(100\,mm) = 50\,mm$$



Note: The physical separation between P and P' is not known.

Imaging Example 1 – Gaussian Equations

Use Gaussian equations: Distances from Principal planes

$$f_E = 100\ mm$$

$$n = n' = 1.0$$

$$z = -200\,mm + f_F = -300\ mm$$

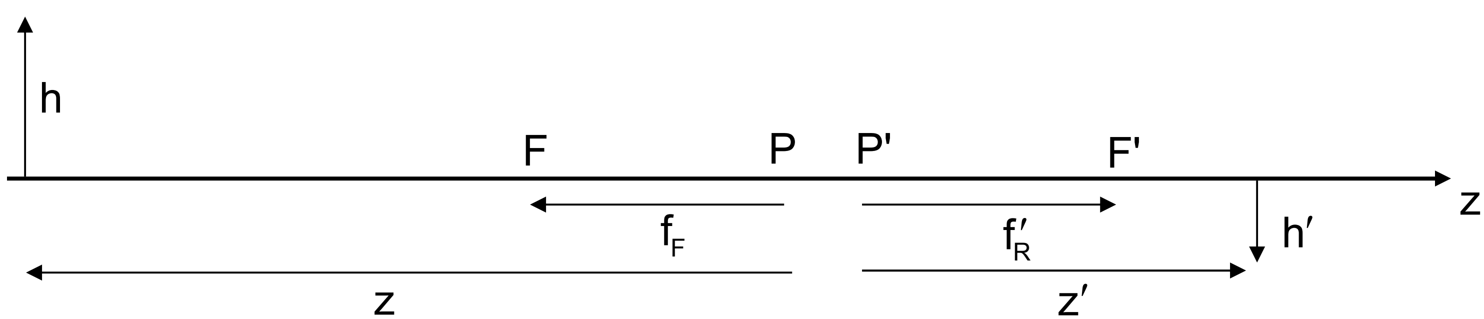
$$h = 10\,mm$$

$$\frac{z}{n} = \frac{(1-m)}{m} f_E \qquad m = \frac{f_E}{f_E + z} = \frac{100\,mm}{100\,mm - 300\,mm} = -.50$$

$$h' = m\,h = -.50\,(10\,mm) = -5.0\,mm$$

$$\frac{z'}{n'} = (1-m)\,f_E \qquad z' = f_E\,(1-m) = 100\,m(1.50) = 150\,mm$$

50 mm to the right of F'



Imaging Example 2

Same Positive Focal System

$$f_E = 100\ mm$$
$$n = n' = 1.0$$

Object: 40 mm to right of F
h = 10 mm

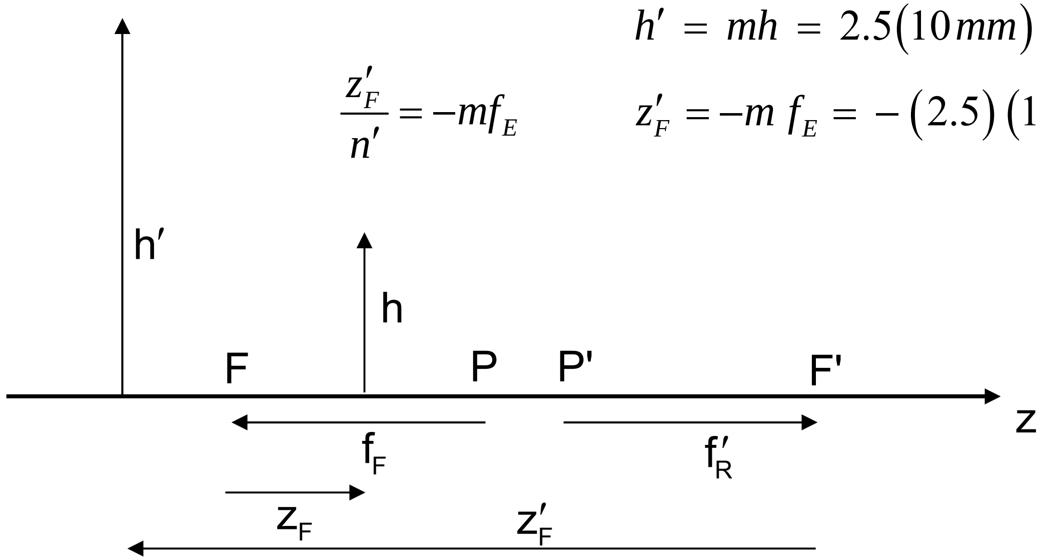
Use Newtonian equations:

$$z_F = 40\ mm$$
$$h = 10\ mm$$

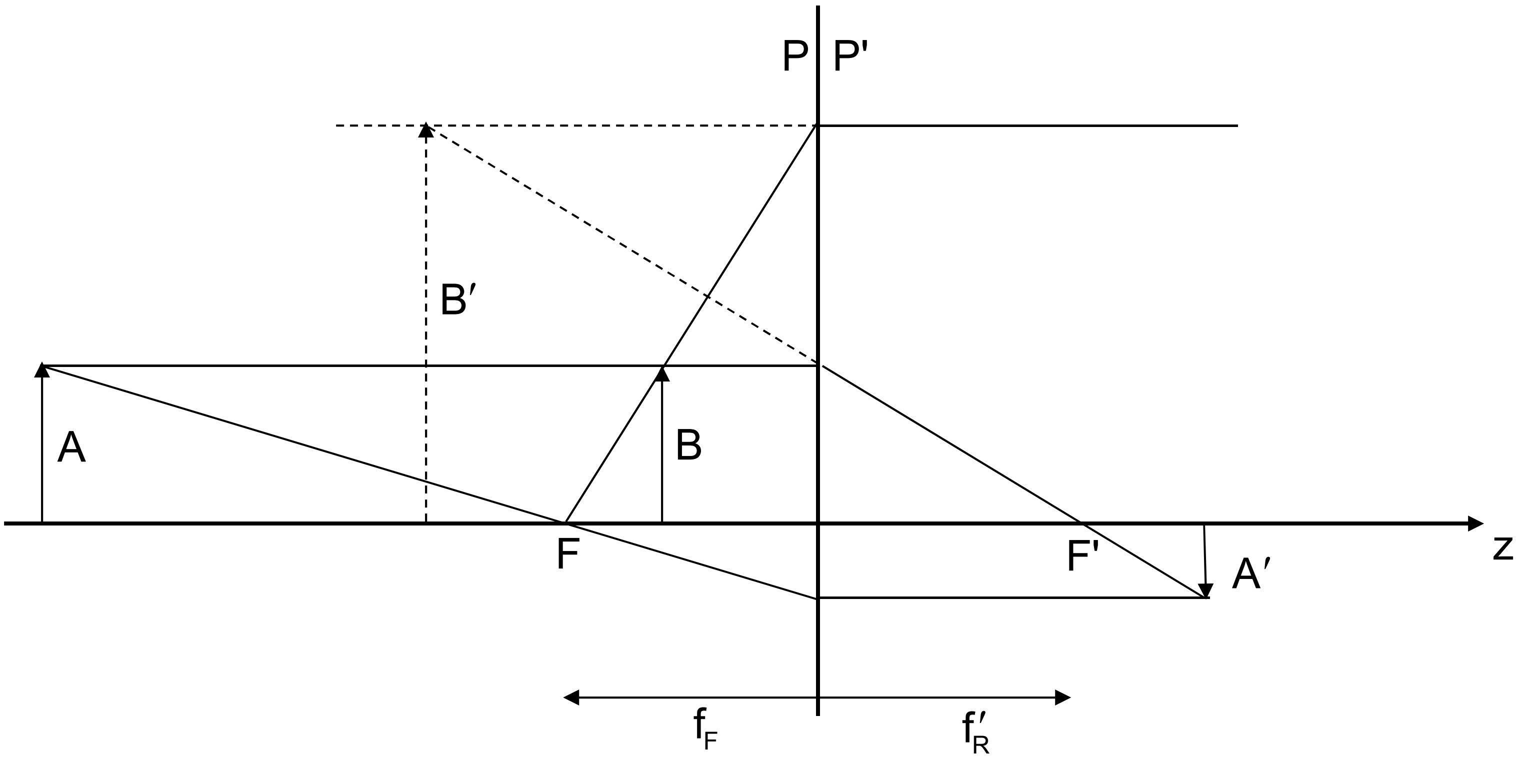
$$\frac{z_F}{n} = \frac{f_E}{m} \qquad m = \frac{f_E}{z_F} = \frac{100}{40} = 2.5$$

$$h' = mh = 2.5(10\,mm) = 25\,mm$$

$$\frac{z'_F}{n'} = -mf_E \qquad z'_F = -m\,f_E = -(2.5)(100\,mm) = -250\,mm$$



Example Summary



For convenience, the Principal Planes are shown as coincident.

Thickness and Longitudinal Magnification – Focal Systems

$$\frac{\Delta z'}{\Delta z} = \left(\frac{n'}{n}\right) m_1 m_2 \qquad \text{or} \qquad \frac{\Delta z'/n'}{\Delta z/n} = m_1 m_2$$

m_1 and m_2 are the lateral magnifications for the two planes

Gaussian Equations (distances measured from P, P')

$$m = \frac{z'/n'}{z/n}$$

$$\frac{\Delta z'}{\Delta z} = \left(\frac{n}{n'}\right) \frac{z_1' z_2'}{z_1 z_2} \qquad \frac{\Delta z'/n'}{\Delta z/n} = \frac{(z_1'/n')(z_2'/n')}{(z_1/n)(z_2/n)}$$

Newtonian Equations (distances measured from F, F')

$$m = -\frac{f_F}{z_F} \qquad \frac{\Delta z'/n'}{\Delta z/n} = \frac{f_F^2}{z_{F1} z_{F2}} \qquad \frac{\Delta z'/n'}{\Delta z/n} = \frac{f_E^2}{(z_{F1}/n)(z_{F2}/n)}$$

When Δz is small, the longitudinal magnification is obtained

$$m_1 \approx m_2 = m \qquad \text{Gaussian:} \quad \bar{m} = \left(\frac{n}{n'}\right) \frac{z'^2}{z^2}$$

$$\bar{m} = \left(\frac{n'}{n}\right) m^2 \qquad \text{Newtonian:} \quad \bar{m} = \left(\frac{n'}{n}\right) \frac{f_F^2}{z_F^2}$$

The image space spacing is inversely proportional to the Newtonian object distance squared.

Imaging Example 3 – Longitudinal Magnification

Same Positive Focal System

$$f_E = 100\ mm$$
$$n = n' = 1.0$$

Objects: 410 mm to left of F
400 mm to left of F

Use Newtonian equations:

$$z_{F1} = -410\,mm$$
$$z_{F2} = -400\ mm$$
$$\Delta z = 10\,mm \qquad \Delta z = z_{F2} - z_{F1}$$

$$\bar{m} = \left(\frac{n'}{n}\right)\frac{f_F^2}{z_F^2} = \frac{(-100\,mm)^2}{(-405\,mm)^2} = 0.061 \qquad z_F \approx -405\,mm$$

$$\Delta z' \approx \bar{m}\,\Delta z = 0.61\,mm$$

Exact thickness magnification:

$$\left(\frac{z_F}{n}\right)\left(\frac{z'_F}{n'}\right) = -f_E^{\ 2} \qquad z'_F = -\frac{f_E^2}{z_F}$$

$$z'_{F1} = 24.39\,mm$$
$$z'_{F2} = 25.0\,mm$$
$$\Delta z' = 0.61\,mm \qquad \Delta z' = z'_{F2} - z'_{F1}$$

$$\frac{\Delta z'}{\Delta z} = 0.061$$

Imaging Example 3A – Longitudinal Magnification

Same Positive Focal System

$$f_E = 100\ mm$$
$$n = n' = 1.0$$

Objects closer to the system: 50 mm to left of F
40 mm to left of F

Use Newtonian equations:

$$z_{F1} = -50\,mm$$
$$z_{F2} = -40\ mm$$
$$\Delta z = 10\,mm \qquad \Delta z = z_{F2} - z_{F1}$$

$$\bar{m} = \left(\frac{n'}{n}\right)\frac{f_F^2}{z_F^2} = \frac{(-100\,mm)^2}{(-45\,mm)^2} = 4.94 \qquad z_F \approx -45\,mm$$

$$\Delta z' \approx \bar{m}\,\Delta z = 49.4\,mm$$

Exact thickness magnification:

$$\left(\frac{z_F}{n}\right)\left(\frac{z'_F}{n'}\right) = -f_E^{\ 2} \qquad z'_F = -\frac{f_E^2}{z_F}$$

$$z'_{F1} = 200.0\,mm$$
$$z'_{F2} = 250.0\,mm$$
$$\Delta z' = 50.0\,mm \qquad \Delta z' = z'_2 - z'_1$$

$$\frac{\Delta z'}{\Delta z} = 5.00$$

Object Image Relationships

The general imagery relationships can be written either in terms of the front and rear focal lengths or in terms of the effective focal length and indices of refraction.

$$f = f_E \equiv \frac{1}{\phi} \qquad f_F = -\frac{n}{\phi} = -nf_E \qquad f_R' = \frac{n'}{\phi} = n'f_E$$

For clarity, the expressions will explicitly note the focal length as f_E. As the course progresses, the "E" subscript will be dropped, and the more common expression for the focal length f will be used.

Newtonian Equations (Origins at F, F'):

$$\frac{z_F}{f_F} = -\frac{1}{m} \qquad \Rightarrow \qquad \frac{z_F}{n} = \frac{f_E}{m}$$

$$\frac{z_F'}{f_R'} = -m \qquad \Rightarrow \qquad \frac{z_F'}{n'} = -mf_E$$

$$z_F\, z_F' = f_F\, f_R' \qquad \Rightarrow \qquad \left(\frac{z_F}{n}\right)\left(\frac{z_F'}{n'}\right) = -f_E^2$$

Object Image Relationships – Page 2

Gaussian Equations (Origins at P, P'):

$$\frac{z}{f_F} = 1 - \frac{1}{m} \quad\Rightarrow\quad \frac{z}{n} = \frac{(1-m)}{m} f_E$$

$$\frac{z'}{f'_R} = 1 - m \quad\Rightarrow\quad \frac{z'}{n'} = (1-m) f_E$$

$$\frac{z'}{z} = \left(-\frac{f'_R}{f_F} \right) m \quad\Rightarrow\quad m = \frac{z'/n'}{z/n}$$

$$\frac{f_F}{z} + \frac{f'_R}{z'} = 1 \quad\Rightarrow\quad \frac{n'}{z'} = \frac{n}{z} + \frac{1}{f_E}$$

Thickness and Longitudinal Magnification – Focal System:

$$\frac{\Delta z'}{\Delta z} = \left(-\frac{f'_R}{f_F} \right) m_1 m_2 = \left(\frac{n'}{n} \right) m_1 m_2 \quad\Rightarrow\quad \frac{\Delta z'/n'}{\Delta z/n} = m_1 m_2$$

$$\bar{m} = \left(-\frac{f'_R}{f_F} \right) m^2 \quad\Rightarrow\quad \bar{m} = \frac{n'}{n} m^2 \qquad \frac{\Delta z'/n'}{\Delta z/n} = m^2$$

Magnification of the Nodal Points:

$$m_N = -\frac{f_F}{f'_R} \quad\Rightarrow\quad m_N = \frac{n}{n'}$$

Reduced Distances and Optical Angles

When the Gaussian imagery equations are expressed in terms of the EFL or power (f_E or ϕ), all of the axial distances appear as a ratio of the physical distance to the index of refraction in the same optical space. This ratio is called a reduced distance and is usually denoted by a Greek letter. For example τ represents the reduced distance associated with the thickness *t*:

$$\tau = \frac{t}{n}$$

The EFL is the reduced focal length: it equals the reduced rear focal length or minus the reduced front focal length.

A ray angle multiplied by the refractive index of its optical space is called an optical angle:

$$\omega = nu$$

Imaging Equations in Air

$$n = n' = 1 \qquad f = f_E \equiv \frac{1}{\phi} = f'_R = -f_F$$

Newtonian Equations (Origins at F, F'):

$$z_F = \frac{f}{m} \qquad z'_F = -mf \qquad z_F z'_F = -f^2$$

Gaussian Equations (Origins at P, P'):

$$z = \frac{(1-m)}{m} f \qquad z' = (1-m) f \qquad m = \frac{z'}{z} \qquad \frac{1}{z'} = \frac{1}{z} + \frac{1}{f}$$

Thickness and Longitudinal Magnification – Focal System:

$$\frac{\Delta z'}{\Delta z} = m_1 m_2 \qquad \bar{m} = m^2$$

Magnification of the Nodal Points: $m_N = 1$

Afocal Systems:

$$m = -\frac{f_2}{f_1} \qquad \bar{m} = m^2 \qquad \frac{m}{\bar{m}} = \frac{1}{m}$$