Section 22

Radiative Transfer

Radiometry

Radiometry characterizes the propagation of radiant energy through an optical system. Radiometry deals with the measurement of light of any wavelength; the basic unit is the Watt (W). The spectral characteristics of the optical system (source spectrum, transmission and detector responsivity) must be considered in radiometric calculations.

Radiometric terminology and units:

Energy	Q	Joules (J)	
Flux	Φ	W	Power
Intensity	I	W/sr	Power per unit solid angle
Irradiance	E	W/m^2	Power per unit area – incident
Exitance	M	W/m^2	Power per unit area – exiting
Radiance	L	W/m^2sr	Power per unit area per unit solid angle incident or exiting

The basic assumptions of radiometry:

- The source is incoherent. Any scene is a collection of independently radiating point sources. There is no interference.
- Geometric optics applies and light propagates along rays. There is no diffraction.

In this simplified discussion, objects and images are assumed to be on-axis and perpendicular to the optical axis. With this assumption, the projected area equals the area.

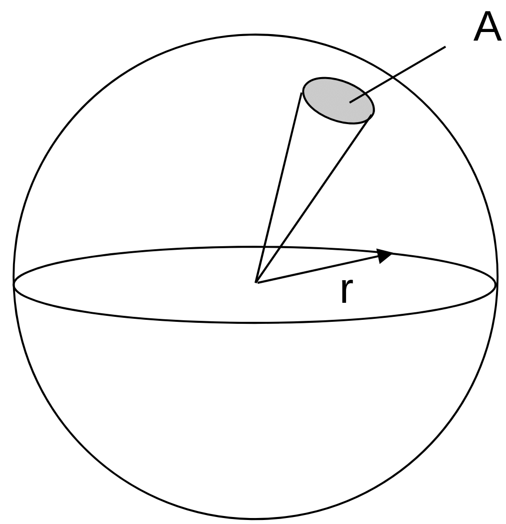
Solid Angle

The solid angle Ω equals the surface area of the unit sphere that is subtended by a surface relative to a point at the center of the unit sphere. There are 4π steradians in a full sphere.

The solid angle of an area A at a distance from a reference point is

$$d\Omega = \frac{dA}{r^2}$$

$$\Omega = \frac{A}{r^2}$$



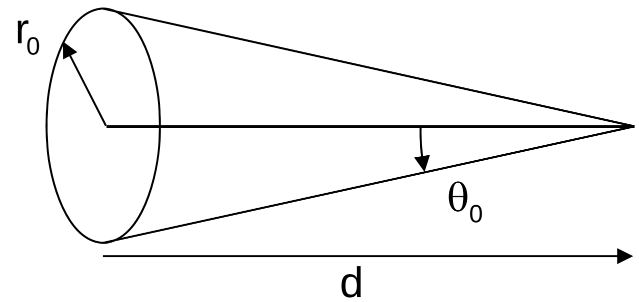
In polar coordinates:

$$d\Omega = \sin\theta d\theta d\phi$$

Note that the solid angle is a function of θ and ϕ, so that only the boundaries of that area are important in determination of the solid angle.

Right Circular Cone

The solid angle of a right circular cone:



$$\Omega \approx \frac{A}{d^2} = \frac{\pi r_0^2}{d^2}$$

$$\frac{r_0}{d} \approx \sin\theta_0 \approx \theta_0$$

$$\Omega \approx \pi\theta_0^2$$

Exact: The area of a spherical end cap must be used.

$$\Omega = 2\pi\left(1-\cos\theta_0\right) = 4\pi\sin^2\left(\theta_0/2\right)$$

Hemisphere:

$$\Omega = 2\pi \; sr$$

Radiative Transfer

Geometrical optics aims to determine the image size location and quality. Radiative transfer uses first-order geometrical principles to determine the amount of light from an object that reaches an image or a detector. It models the propagation of radiant energy through an optical system.

The Problem: An object is imaged with a lens of a particular f/#. Given a certain amount of radiant power per unit area (irradiance or E) falling on the object, what is the power per unit area (irradiance) in the image?

Irradiance:

The scene is illuminated by a certain irradiance E (W/m^2).

For solar illumination, the mean solar constant can be used:

$E = 1368\ W/m^2$	outside the atmosphere
$E = 1000\ W/m^2$	on the surface of the earth

Radiative Transfer – Reflectance and Exitance

Exitance M is the amount of light leaving the surface per unit area. Exitance and irradiance are related by the reflectance of the surface ρ. Note that this is the power reflectance (not the electric field amplitude reflectivity).

$$M = \rho E$$

Photographic research has shown that $\rho = 18\%$ for the average scene. Exposures are often set using this value, and 18% gray cards are available to provide a reflectance reference. This value is important as a simple photographic printer will expose the print to that average, so that the print reflectance ends up being 18% to match the average scene. As a result, scenes that do not conform to this standard (snow, for example) are printed incorrectly. 18% gray snow results.

More advanced printers analyze the scene and vary the print reflectance to get better results.

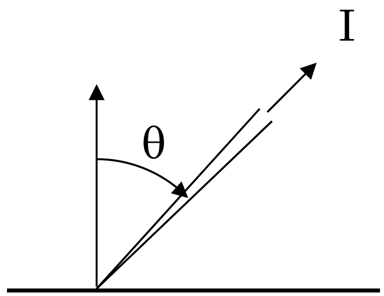
Radiative Transfer – Radiance and Lambertian Sources

The exitance M gives the power per unit area, but it contains no information about the directionality or angular distribution of the light leaving the scene. This information is contained in the radiance L.

The most common assumption for diffuse scenes is that the radiance is constant or independent of angle. This is a Lambertian source.

$$L(\theta,\phi) = \text{Constant}$$

A Lambertian source is considered to be perfectly diffuse. The intensity falls off with the apparent source size or its projected area. This result comes from the fact that the source size appears to decrease as the extended source is viewed obliquely. This is Lambert's law:



$$I = I_0 \cos\theta$$

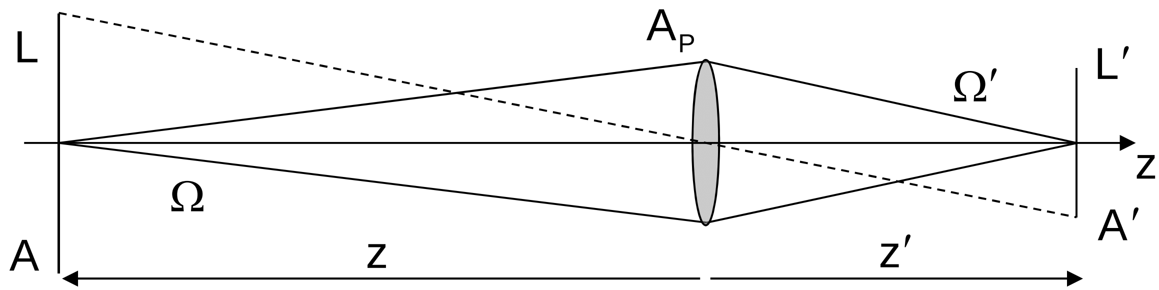
The exitance M of a Lambertian source or scene is related to its radiance L by π.

$$M = \pi L \qquad \pi L = \rho E \qquad L = \rho E / \pi$$

This relationship is π (instead of the expected 2π for a hemisphere) because of the falloff of the projected area with θ.

Radiative Transfer – the Optical System

The next step is to determine the amount of optical power Φ from an area A on the source that is captured by the optical system.



$$\Phi = L\left(\text{Object area}\right)\left(\text{Solid angle subtended by the lens}\right) = LA\Omega$$

The object distance is z, and the area of the pupil is A_P. A thin lens equivalent system is assumed (both the EP and XP are at the lens and of equal size).

$$\Omega = \frac{A_P}{z^2} = \frac{\pi D_P^2}{4z^2} \qquad \Phi = LA\frac{\pi D_P^2}{4z^2}$$

All of this power is now transferred to the image with a magnification of m:

$$A' = m^2 A$$

$$\Phi = \frac{\pi L A' D_P^2}{4m^2 z^2}$$

Radiative Transfer – The Optical System - Continued

The object and image distances are related by the Gaussian equations. Assume a thin lens in air:

$$z = \frac{(1-m)}{m} f$$

$$\Phi = \frac{\pi L A' D_P^2}{4(1-m)^2 f^2} = \frac{\pi L A'}{4(1-m)^2 (f/\#)^2}$$

The image plane irradiance can be found by dividing by the image area:

$$E' = \frac{\Phi}{A'} = \frac{\pi L}{4(1-m)^2 (f/\#)^2} = \frac{\pi L}{4(f/\#_W)^2} = \pi L (NA)^2$$

Which can be simplified for distant objects:

$$E' = \frac{\Phi}{A'} = \frac{\pi L}{4(f/\#)^2} = \pi L (NA)^2 \qquad L = \frac{\rho E_0}{\pi}$$

This result is known as the Camera Equation, and it relates the image irradiance to the scene radiance.

Camera Equation

$$E' = \frac{\pi L}{4(1-m)^2 (f/\#)^2} = \frac{\pi L}{4(f/\#_W)^2} = \pi L (NA)^2$$

Spectral dependence can also be added, starting with the scene irradiance (now in units of power per unit area per unit wavelength, or for example, W/m^2nm).

$$E_0(\lambda) = \text{Object Irradiance}$$

$$\rho(\lambda) = \text{Object Reflectance}$$

$$L(\lambda) = \text{Object Radiance}$$

$$L(\lambda) = \frac{M(\lambda)}{\pi} = \frac{\rho(\lambda) E_0(\lambda)}{\pi}$$

$$E'(\lambda) = \frac{\rho(\lambda) E_0(\lambda)}{4(1-m)^2 (f/\#)^2}$$

Which can be integrated for the total (non-spectral) irradiance:

$$E' = \frac{1}{4(1-m)^2 (f/\#)^2} \int_{\lambda_1}^{\lambda_2} \rho(\lambda) E_0(\lambda) d\lambda$$

Exposure

For the camera equation, an on-axis Lambertian object and small angles are assumed. The object and image planes are perpendicular to the optical axis. Including obliquity factors associated with off-axis objects leads to the cosine fourth law. The image irradiance falls off as the $\cos^4$ of the field angle.

Most detectors respond to energy per unit area rather than power per unit area. Multiplying the image irradiance by the exposure time gives the exposure (J/m^2):

$$H = E'\Delta t$$

Photometry

Photometry is the subset of radiometry that deals with visual measurements, and luminous power is measured in lumens lm.

The lumen is a Watt weighted to the visual photopic response. The peak response occurs at 555 nm, where the conversion is 683 lm/W. The dark-adapted or scotopic response peaks at 507 nm with 1700 lm/W.

Photometric terminology and units:

Luminous power	Φ_V	lm
Luminous intensity	I_V	lm/sr
Illuminance	E_V	lm/m^2
Luminous exitance	M_V	lm/m^2
Luminance	L_V	lm/m^{2}sr
Exposure	H_V	lm s/m^2

All of the rules and results of radiometry and radiative transfer apply.

Luminous Photopic Sensitivity

λ (nm)	lm/W
400	0.3
420	2.7
440	15.7
460	41.0
480	95.0
500	221
520	485
540	652
560	680
580	594
600	425
620	260
640	120
660	41.7
680	11.6
700	2.8
720	0.7

More Photometric Units

Other common photometric units and conversions include:

I_V:	candela (cd)	= lm/sr
E_V:	lux (lx)	= lm/m^2
	foot-candle (fc)	= lm/ft^2
		1 fc = 10.76 lx
L_V:	foot-lambert (fL)	= $1/\pi$ cd/ft^2
	nit (nt)	= cd/m^2
		1 fL = 3.426 nt
H_V:	lux-second (lx s)	= lm s/m^2

The unit meter-candle-second (mcs) is an obsolete unit of exposure equal to the lux-second.

Typical illuminance levels:

Sunny day:	10^5 lux	Moonlit night:	10^{-1} lux
Overcast day:	10^3 lux	Starry night:	10^{-3} lux
Interior:	10^2 lux	Desk lighting:	10^3 lux

Remember, photometric quantities work just like radiometric quantities.

Candela

The candela (cd) is a fundamental SI unit for luminous intensity (lm/sr). It is the connection between photometric and radiometric units.

"The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz (555 nm) and that has a radiant intensity in that direction of 1/683 watt per steradian."

At 555 nm:

$$1 \text{ cd} = 1 \text{ lumen per steradian} = 1/683 \text{ Watt per steradian}$$

AΩ Product

The flux through a system is given by

$$\Phi = L\left(\text{Object area}\right)\left(\text{Solid angle subtended by the lens}\right) = LA\Omega$$

The assumptions implicit in this result are

- Small angles and on axis scene
- Lambertian Source
- Object, image and pupils are perpendicular to the optical axis (no obliquity effects or projected areas are included).

The AΩ product appears to be the geometric portion of the above relationship, while L would be related to the source characteristics.

$$\Omega = \pi\theta^2$$

$$\theta = \text{half angle of cone at area } A$$

$$A\Omega = \pi A\theta^2$$

In an object or an image plane:

$$A = \pi\bar{y}^2 \qquad \theta = u$$

$$A\Omega = \pi^2\bar{y}^2u^2 = \pi^2Ж^2/n^2 \qquad Ж = n\bar{y}u$$

AΩ Product - Continued

In a pupil plane:

$$A = \pi y^2 \qquad \theta = \bar{u}$$

$$A\Omega = \pi^2 y^2 \bar{u}^2 = \pi^2 Ж^2 / n^2 \qquad Ж = ny\bar{u}$$

In both cases:

$$n^2 A\Omega = \pi^2 Ж^2$$

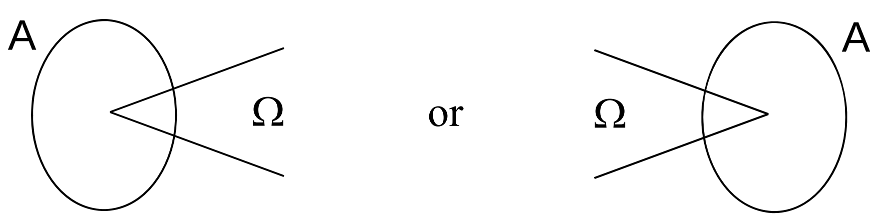
The $n^2A\Omega$ product is proportional to the square of the Lagrange invariant. The square is due to the fact that the AΩ product involves areas, and the Lagrange invariant is a linear measure. This quantity is known as the Basic Throughput and is invariant.

In air, the AΩ product is called the Throughput of the system, and it is also invariant.

$$A\Omega = \pi^2 Ж^2 \qquad n = 1$$

$$A\Omega = A'\Omega' \qquad \Phi = LA\Omega = L'A'\Omega'$$

Reviewing the areas and solid angles under discussion:



Conservation of Basic Throughput – Alternate Derivation

In conjugate planes:

$$\frac{\bar{y}'}{\bar{y}} = \frac{h'}{h} = m = \frac{\omega}{\omega'}$$

$$\bar{y}'\omega' = \bar{y}\omega$$

$$\bar{y}'n'u' = \bar{y}nu$$

$$\bar{y}'^2 n'^2 u'^2 = \bar{y}^2 n^2 u^2$$

$$\pi\bar{y}'^2 \pi n'^2 u'^2 = \pi\bar{y}^2 \pi n^2 u^2$$

$$A = \pi\bar{y}^2 \qquad \Omega = \pi u^2$$

$$n'^2 A'\Omega' = n^2 A\Omega \qquad\qquad n^2 A\Omega = \pi^2 Ж^2$$

Basic Throughput is conserved.

Conservation of Radiance

For a lossless optical system (no reflection or absorption losses), the flux Φ through the system is constant. An index n = 1 is assumed.

$$\Phi = L_1 A_1 \Omega_1 = L_2 A_2 \Omega_2 = L_3 A_3 \Omega_3 = \cdots$$

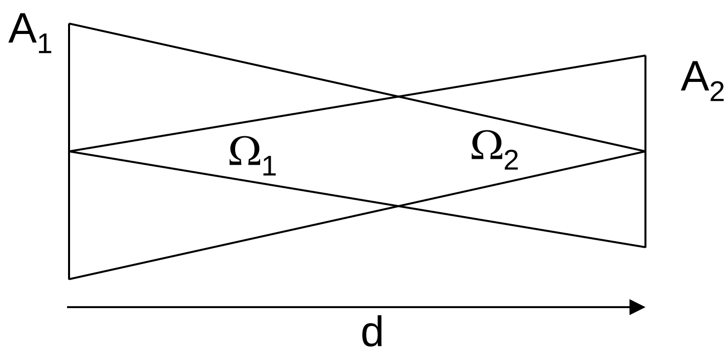
Since the AΩ product is also a constant, the radiance L must also be constant throughout the system.

$$L = \text{Constant}$$

This is one of the basic laws of radiative transfer, and is very useful for system analysis.

$$\Phi = L A_i \Omega_i$$

$$\Phi = \frac{L A_1 A_2}{d^2}$$



Note that radiance can be evaluated at any point along a ray. Radiance can therefore be associated with images, pupils, etc.

Camera Equation - Revisited

The conservation of radiance can be used to derive the camera equation:

$$L_{IMAGE} = L_{SOURCE} \qquad L' = L$$

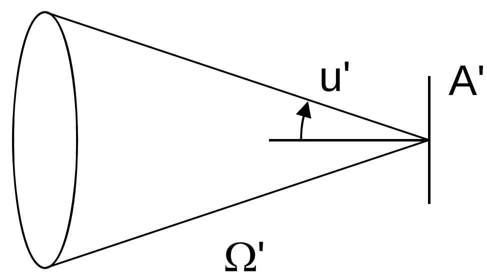
$$\Phi = LA\Omega$$

$$E' = \frac{\Phi}{A'} = L'\Omega'$$

$$\Omega' \approx \pi u'^2$$

$$E' = L\pi u'^2 = \pi L(NA)^2$$

$$E' = \pi L(NA)^2 = \frac{\pi L}{4(f/\#_W)^2}$$

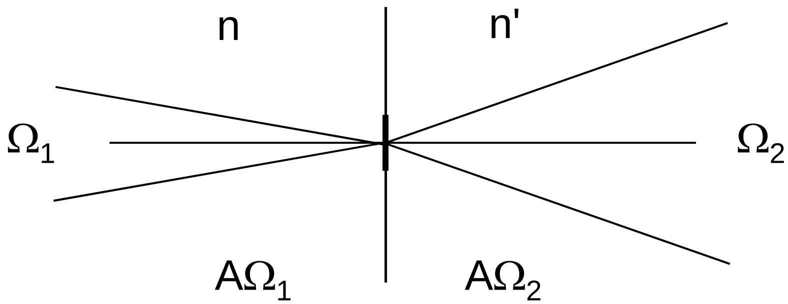


$$NA \approx n|u| \qquad n = 1$$

Basic Radiance

If the index of refraction is not unity and changes, the radiance is not conserved.

When crossing a refractive boundary, the radiance will change across that boundary because the solid angle associated with a ray bundle changes.



$$A\Omega_1 \neq A\Omega_2$$

$$\Phi = L_1 A\Omega_1 = L_2 A\Omega_2$$

$$\therefore L_1 \neq L_2$$

The flux through the system is still constant and found by:

$$\Phi = L A\Omega = \text{Constant}$$

Rewriting in terms of the invariant basic throughput:

$$\Phi = \left(L / n^2\right)\left(n^2 A\Omega\right) = \text{Constant}$$

$$n^2 A\Omega = \text{Constant}$$

$$L/n^2 = \text{Constant}$$

This is the Basic Radiance of the system and is invariant.