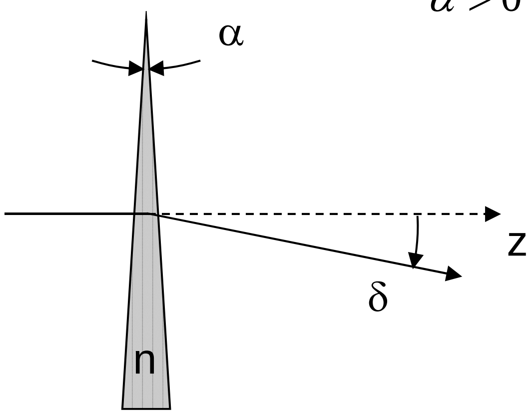
Section 20

Thin Prisms

Thin Prism Deviation

Thin prisms introduce small angular beam deviations and are useful as alignment devices. The beam deviation δ is approximately independent of the incident angle:

$$\delta \approx -(n-1)\alpha$$

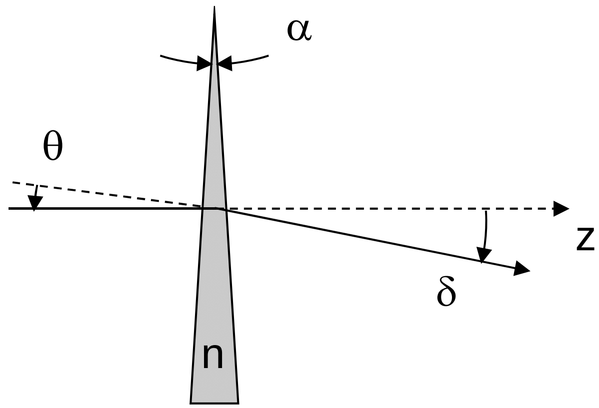


The beam deviation is parallel to a principal section of the prism and towards the thick end of the prism.

Thin prisms are used for optometric correction of strabismus (a misalignment of the axes of the eyes). The deviation is measured in prism diopters. A prism of 1 diopter deviates a beam by 1 cm at 1 m.

For small deviations: $D \approx \delta$ (δ in 0.01 radians)

Thin Prism Deviation - Derivation



$$\delta = \alpha - \sin^{-1}\left[\sqrt{n^2 - \sin^2\theta}\sin\alpha - \cos\alpha\sin\theta\right] - \theta$$

$$\sin\alpha \approx \alpha \qquad \sin\theta \approx \theta$$

Neglect terms in α^2, θ^3 and higher:

$$\delta = \alpha - \sin^{-1}\left[\alpha n\sqrt{1-\theta^2/n^2} - \theta\right] - \theta$$

$$\sqrt{1-\theta^2/n^2} \approx 1 - \theta^2/2n^2$$

$$\delta = \alpha - \sin^{-1}\left[\alpha n - \alpha\theta^2/2n - \theta\right] - \theta$$

$$\sin^{-1} x \approx x + x^3/6 + \ldots$$

$$\sin^{-1}\left[\alpha n - \alpha\theta^2/2n - \theta\right] \approx \alpha n - \alpha\theta^2/2n - \theta + \left(3\alpha n\theta^2\right)/6$$

$$\delta \approx \alpha - \alpha n + \alpha\theta^2/2n + \theta - \alpha n\theta^2/2 - \theta$$

$$\delta \approx -\alpha\left[\left(n-1\right) + \frac{\theta^2}{2}\left(n - \frac{1}{n}\right)\right]$$

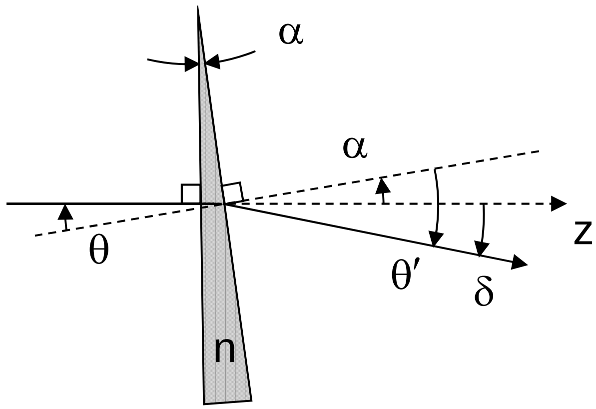
$$\delta \approx -\alpha\left(n-1\right)\left[1 + \frac{\theta^2\left(n+1\right)}{2n}\right]$$

For θ small:

$$\delta \approx -\left(n-1\right)\alpha$$

Thin Prism Deviation – Alternate Derivations

The thin prism is always used at minimum deviation. Tip the prism so that the front face is perpendicular to the input ray:



$$\theta = -\alpha \qquad \theta' = \delta - \alpha$$

Snell's Law: $n\theta = \theta'$

$$-n\alpha = \delta - \alpha$$

$$\delta \approx -(n-1)\alpha$$

At minimum deviation:

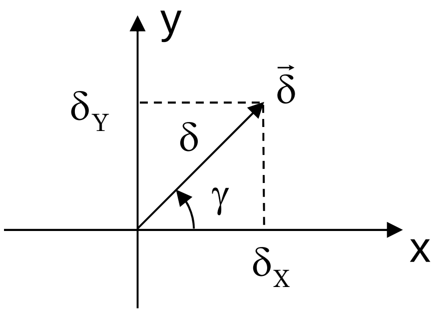
$$n = \frac{\sin\left[(\alpha - \delta_{MIN})/2\right]}{\sin(\alpha/2)}$$

$$n \approx \frac{\alpha - \delta}{\alpha}$$

$$\delta \approx -(n-1)\alpha$$

<u>Combinations of Thin Prisms</u>

The beam deviation is parallel to a principal section of the prism and towards the thick end of the prism. The magnitude and direction of this deviation defines a vector perpendicular to the optical axis (in the x-y plane).

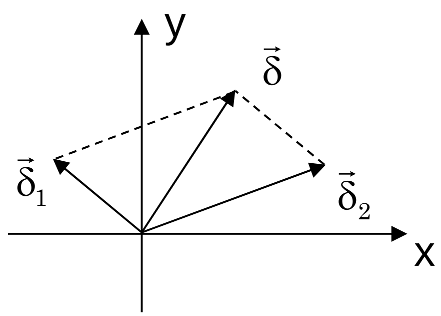


$$\delta_X = \delta \cos\gamma$$

$$\delta_Y = \delta \sin\gamma$$

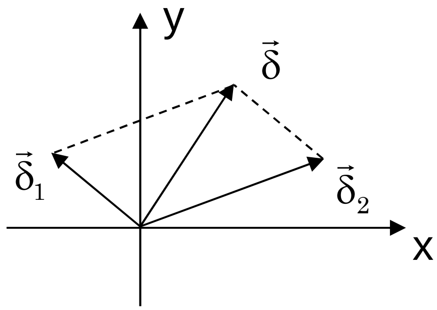
The net deviation vector for a series of thin prisms is the sum of the component vectors.

$$\vec{\delta} = \vec{\delta}_1 + \vec{\delta}_2$$



Two Thin Prisms

Net deviation and direction:



$$\delta = \left(\delta_1^{\,2} + \delta_2^{\,2} + 2\delta_1\delta_2\cos(\gamma_1 - \gamma_2)\right)^{1/2}$$

$$\tan\gamma = \frac{\delta_Y}{\delta_X} = \frac{\delta_1\sin\gamma_1 + \delta_2\sin\gamma_2}{\delta_1\cos\gamma_1 + \delta_2\cos\gamma_2}$$

Adding:

$$\gamma_1 = \gamma_2$$

$$\delta = \delta_1 + \delta_2$$

$$\gamma = \gamma_1 = \gamma_2$$

The net deviation is the sum of the individual deviations.

Opposing:

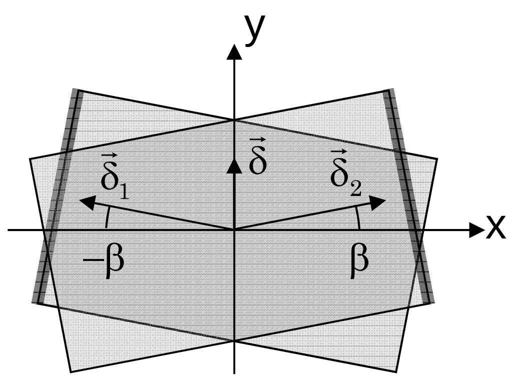
$$\gamma_1 - \gamma_2 = 180°$$

$$\delta = \delta_1 - \delta_2$$

The net deviation is the difference of the individual deviations.

Risley Prism

A Risley prism consists of a pair of identical, but opposing, thin prisms. The prisms are counter-rotated by +/-β to obtain a variable net deviation in a fixed direction (shown with the net deviation in the y-direction).



$$\delta_0 = \left|\vec{\delta}_1\right| = \left|\vec{\delta}_2\right|$$

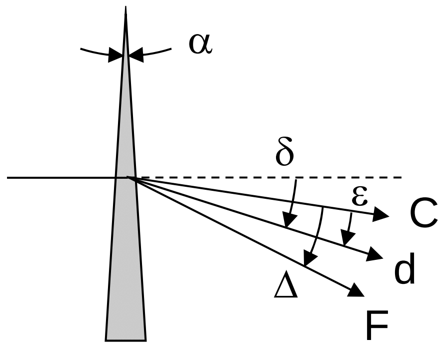
$$\delta = 2\delta_0 \sin(\beta)$$

The Risley prism allows the fine angular alignment of an optical system by adjusting the prism orientations β.

Thin Prism Dispersion

The dispersion of a thin prism Δ measures the total angular spread for C to F light, and the secondary dispersion ε gives the spread from the C to d wavelengths. The results depend on the index n_d, Abbe number ν and partial dispersion ratio P of the glass.

Deviation: $\delta = -(n_d - 1)\alpha$



Dispersion:

$$\Delta = (n_F - 1)(-\alpha) - (n_C - 1)(-\alpha) = -(n_F - n_C)\alpha$$

Secondary Dispersion:

$$\varepsilon = (n_d - 1)(-\alpha) - (n_C - 1)(-\alpha) = -(n_d - n_C)\alpha$$

$$\frac{\delta}{\Delta} = \frac{n_d - 1}{n_F - n_C} = \nu$$ Abbe Number

$$\Delta = \frac{\delta}{\nu}$$

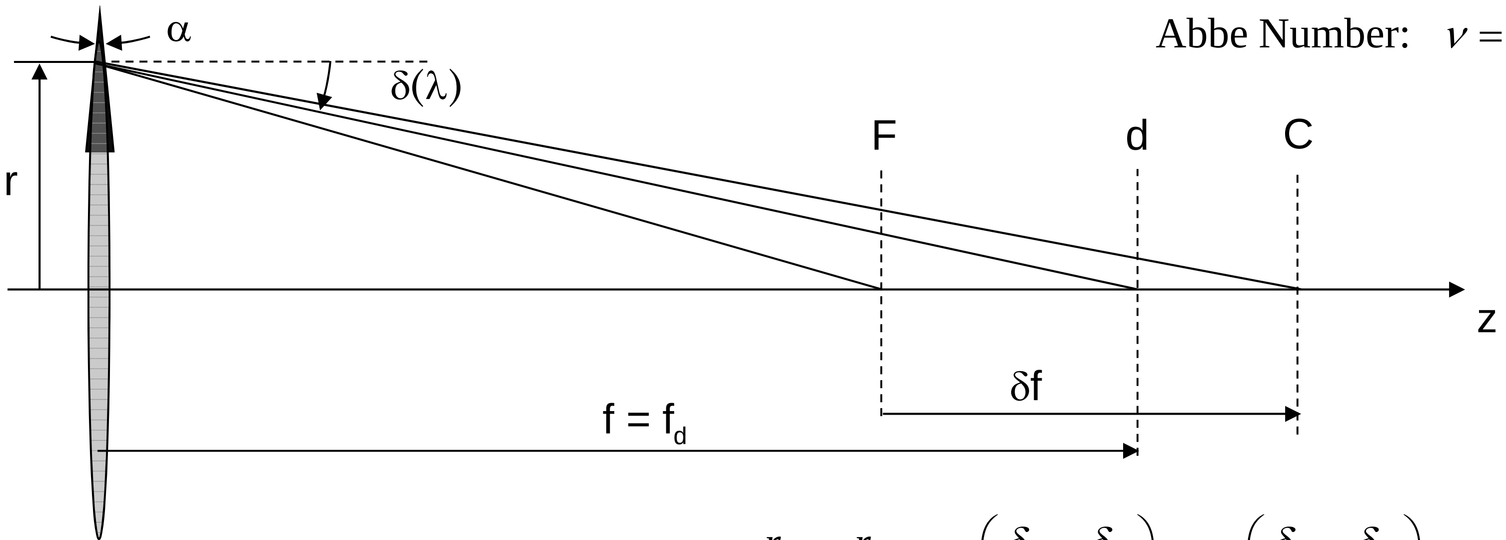
$$\frac{\varepsilon}{\Delta} = \frac{n_d - n_C}{n_F - n_C} = P$$ Partial Dispersion Ratio

$$\varepsilon = P\frac{\delta}{\nu}$$

Thin Prism Dispersion and the Chromatic Aberration of a Thin Lens

The edge of a thin lens can be considered to be a thin prism. The prism angle α will depend on the power and diameter of the lens. Each wavelength will be deviated by a different angle. The focal length of the lens is defined for d light.

Abbe Number: $\nu = \dfrac{n_d - 1}{n_F - n_C}$



$$\delta(\lambda) = -(n_\lambda - 1)\alpha$$

$$f(\lambda) = -\frac{r}{\delta(\lambda)}$$

$$f_d = -\frac{r}{\delta_d}$$

$$\delta f = f_C - f_F = -\frac{r}{\delta_C} + \frac{r}{\delta_F} = -r\left(\frac{\delta_F - \delta_C}{\delta_F \delta_C}\right) \approx -r\left(\frac{\delta_F - \delta_C}{\delta_d^2}\right)$$

$$\delta_C \delta_F \approx \delta_d^{\,2}$$

$$\frac{\delta f}{f_d} = \frac{f_C - f_F}{f_d} = \frac{-r\left(\dfrac{\delta_F - \delta_C}{\delta_d^2}\right)}{-\dfrac{r}{\delta_d}} = \left(\frac{\delta_F - \delta_C}{\delta_d^2}\right)\delta_d = \left(\frac{\delta_F - \delta_C}{\delta_d}\right)$$

$$\frac{\delta f}{f_d} = \frac{-(n_F - 1)\alpha + (n_C - 1)\alpha}{-(n_d - 1)\alpha} = \frac{-n_F + n_C}{-(n_d - 1)} = \frac{n_F - n_C}{n_d - 1} = \frac{1}{\nu}$$

$$\boxed{\frac{\delta f}{f_d} = \frac{1}{\nu}}$$

Systems of Thin Prisms

Deviations and dispersions add.

An inverted prism (with its vertex at the bottom) deviates a ray up and has a negative vertex angle α.

Deviation: $\delta = \sum_i \delta_i$

Dispersion: $\Delta = \sum_i \Delta_i$

Secondary Dispersion: $\varepsilon = \sum_i \varepsilon_i$

Achromatic Thin Prism

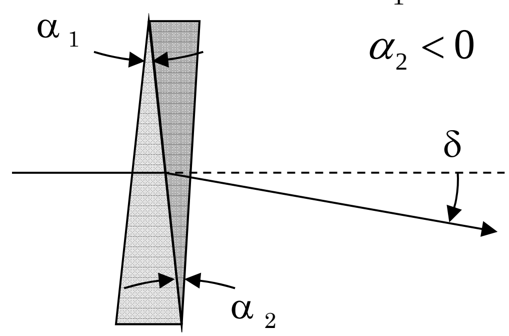
An achromatic thin prism or achromatic wedge provides deviation without dispersion. Opposing prisms made from two different glasses (n_{d1}, ν_1, P_1 and n_{d2}, ν_2, P_2) are combined to force the dispersion between the F and C wavelengths to be zero.

$$\Delta = \Delta_1 + \Delta_2 = 0 \qquad \Delta_1 = \frac{\delta_1}{\nu_1} \qquad \Delta_2 = \frac{\delta_2}{\nu_2}$$

$$\delta_2 = -\frac{\nu_2}{\nu_1}\delta_1$$

$$\alpha_1 > 0$$

$$\alpha_2 < 0$$



A deviation of δ is maintained for d light:

$$\delta = \delta_1 + \delta_2 = \delta_1 - \frac{\nu_2}{\nu_1}\delta_1 = (\nu_1 - \nu_2)\frac{\delta_1}{\nu_1}$$

$$\delta_1 = -(n_{d1} - 1)\alpha_1$$

$$\delta = -(\nu_1 - \nu_2)\frac{(n_{d1} - 1)\alpha_1}{\nu_1}$$

$$\frac{\alpha_1}{\delta} = \left(\frac{1}{\nu_2 - \nu_1}\right)\left(\frac{\nu_1}{n_{d1} - 1}\right)$$

$$\frac{\alpha_2}{\delta} = -\left(\frac{1}{\nu_2 - \nu_1}\right)\left(\frac{\nu_2}{n_{d2} - 1}\right)$$

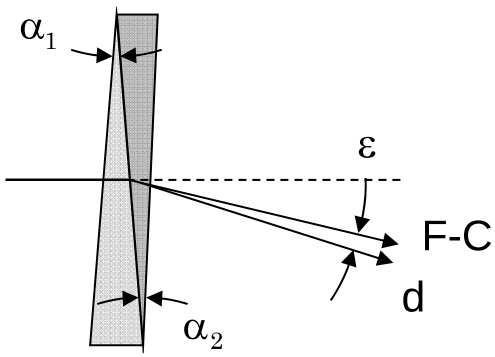
The high dispersion prism is inverted to obtain an opposing deviation.

Achromatic Thin Prism – Secondary Dispersion

The achromatic thin prism design forces F and C wavelengths to have the same deviation, but it does not require that the d wavelength have this same deviation. The residual dispersion is the secondary dispersion ε.

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad \varepsilon_1 = P_1 \frac{\delta_1}{\nu_1} \qquad \varepsilon_2 = P_2 \frac{\delta_2}{\nu_2}$$

$$\varepsilon = P_1 \frac{\delta_1}{\nu_1} + P_2 \frac{\delta_2}{\nu_2}$$



From the achromatic condition:

$$\nu_1 \delta_2 = -\nu_2 \delta_1 \qquad \delta_2 = \delta - \delta_1$$

$$\nu_1(\delta - \delta_1) = -\nu_2 \delta_1$$

$$\nu_1 \delta = (\nu_1 - \nu_2)\delta_1$$

$$\frac{\delta_1}{\nu_1} = \frac{\delta}{\nu_1 - \nu_2} \qquad \frac{\delta_2}{\nu_2} = \frac{-\delta}{\nu_1 - \nu_2}$$

$$\varepsilon = \frac{P_1 \delta}{\nu_1 - \nu_2} - \frac{P_2 \delta}{\nu_1 - \nu_2}$$

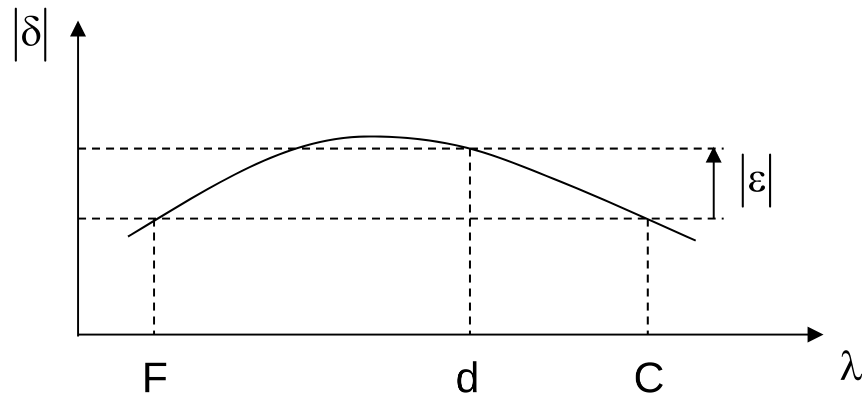
$$\frac{\varepsilon}{\delta} = \frac{P_1 - P_2}{\nu_1 - \nu_2} = \frac{\Delta P}{\Delta \nu}$$

While the F and C wavelengths are corrected, a residual secondary dispersion remains. For most glass pairs, d light will be bent more than the F and C wavelengths.

Secondary Dispersion

For most glasses:

$$\frac{\varepsilon}{\delta} = \frac{\Delta P}{\Delta \nu} > 0$$



The shape of this curve depends on the details of the two dispersion curves. The maximum dispersion does not occur for d-light.

Simple thin prism: $\frac{\Delta}{\delta} = \frac{1}{\nu}$ $\nu \approx 30 - 70$

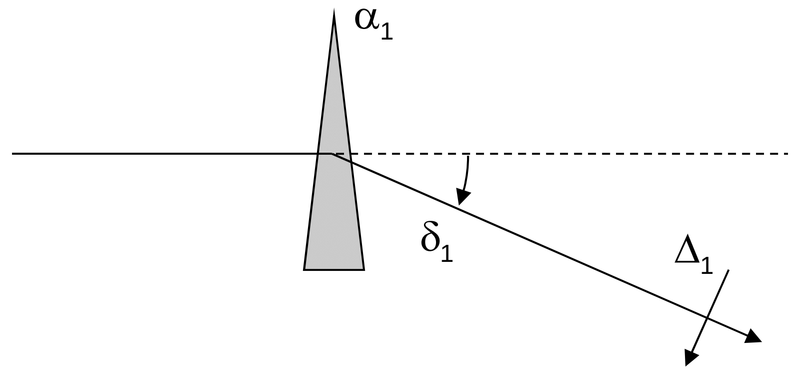
The dispersion is about 1.5-3% of the deviation.

Achromatic thin prism: $\frac{\varepsilon}{\delta} = \frac{\Delta P}{\Delta \nu}$ $\frac{\Delta P}{\Delta \nu} \approx 0.00045$

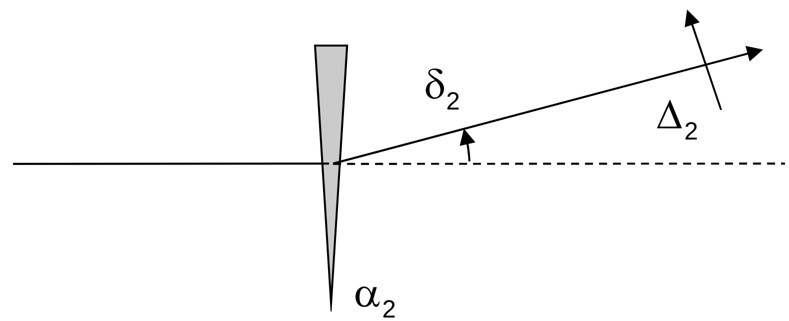
The residual or secondary dispersion is about 0.05% of the deviation.
The dispersion is reduced by a factor of about 40 over the simple thin prism.

Achromatic Prism – How Does It Work?

Low dispersion prism:



High dispersion prism:



$$\Delta_1 = -\Delta_2$$

$$\delta_1 \neq \delta_2$$

$$\delta = \delta_1 + \delta_2$$

Direct Vision Prism

A direct vision prism uses opposing prisms to provide dispersion without deviation of the d light. An object viewed through this prism is smeared by the local spectrum of the object.

For a desired dispersion Δ:

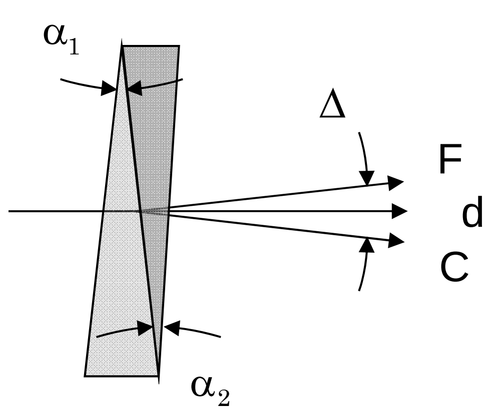
$$\delta = \delta_1 + \delta_2 = 0$$

$$\Delta = \Delta_1 + \Delta_2 = \frac{\delta_1}{\nu_1} + \frac{\delta_2}{\nu_2}$$

$$\Delta = -\left(\frac{\nu_1 - \nu_2}{\nu_1 \nu_2}\right)\delta_1$$

$$\frac{\alpha_1}{\Delta} = \left(\frac{\nu_1 \nu_2}{\nu_1 - \nu_2}\right)\left(\frac{1}{n_{d1} - 1}\right)$$

$$\frac{\alpha_2}{\Delta} = -\left(\frac{\nu_1 \nu_2}{\nu_1 - \nu_2}\right)\left(\frac{1}{n_{d2} - 1}\right)$$



In order to get sufficient dispersion, it is often necessary to use more than two prisms.