Section 7
Gaussian Reduction

Paraxial Raytrace Equations

Refraction occurs at an interface between two optical spaces. The transfer distance $t'$ allows the ray height $y'$ to be determined at any plane within an optical space (including virtual segments).

\[
\omega = nu \\
\phi = (n' - n)C \\
\tau = \frac{t}{n}
\]

Refraction:
\[
n'u' = nu - y\phi
\]
\[
\omega' = \omega - y\phi
\]

Transfer:
\[
y' = y + t'u'
\]
\[
y' = y + n'u' \frac{t'}{n}
\]
\[
y' = y + \omega' \tau'
\]
Gaussian Reduction

Gaussian reduction is the process that combines multiple components two at a time into a single equivalent system. The Gaussian properties (power, focal lengths, and the location of the cardinal points) are determined.

Two component system – System Power:

Trace a ray parallel to the optical axis in object space. This ray must go through the rear focal point of the system.

Paraxial raytrace:

\[
\begin{align*}
\phi_1 & = \phi_2 \\
\phi' & = \phi_2' \\
\phi & = \phi_2 \\
\phi' & = \phi_2'
\end{align*}
\]

Transfer

\[
\begin{align*}
y' & = y + (n'u')(t'/n') \\
y' & = y + \phi' t'
\end{align*}
\]

Define the system power by applying the refraction equation to the system:

\[
\omega' = \omega_2' = \omega - y\phi \\
\omega = 0 \\
\omega' = -y_1\phi
\]

Two Component System – System Power

Trace the ray:

\[
\begin{align*}
\omega_2 & = \omega_2' = \omega_2 - y_2\phi_2 \\
y_2' & = y_2 + \phi_2 t \\
\omega' & = \omega_2' = \omega_2 - y_2\phi_2 \\
\omega' & = -y_2\phi_2 - (y_2 + \omega_2\phi_2) \phi_3 \\
\omega' & = -y_2\phi_2 - (y_2\phi_2 - \phi_2\phi_3) \phi_3 \\
\omega' & = -y_3\phi_3 \\
\omega' & = -y_3\phi_3 - (y_3\phi_3 - \phi_3\phi_3) \phi_5 \\
\omega' & = -y_3\phi_3 \\
\omega' & = -y_3(\phi_3 + \phi_5 - \phi_3\phi_5) t = -y_3\phi
\end{align*}
\]

System power:

\[
\phi = \phi_1 + \phi_2 - \phi_3\phi_5 t
\]
Two Component System – Rear Cardinal Points

The system rear principal plane is the plane of unit system magnification.

\[
\begin{align*}
u = u_1 &= 0 \\
n_1 &= n \\
y_1 &= y_1 \\
\phi_1 &= \phi_1 \\
\phi &= \phi \\
u' &= u'_1 = u'_2 = 0 \\
n_2 &= n'_1 \\
y'_1 &= y'_1 \\
\phi'_1 &= \phi'_1 \\
\phi' &= \phi' \\
\phi' &= \phi \\
u'' &= u''_1 = u''_2 = 0 \\
n_3 &= n'_2 = n'' \\
y''_1 &= y''_1 \\
\phi''_1 &= \phi''_1 \\
\phi'' &= \phi'' \\
\end{align*}
\]

\[
\begin{align*}
y_2' &= y_1 + \alpha \tau \\
\phi' &= -y_1 \phi \\
\end{align*}
\]

If \( P'_2 \) is the rear vertex of the system, then the distance \( P'_2 P \) is the BFD.

\[
\begin{align*}
\delta' &= \frac{y_1 - y_2}{u'} \\
\delta' &= \frac{\alpha \phi \tau}{u'} \\
\delta' &= \frac{\alpha \phi \tau}{n'} - \frac{\alpha \phi \tau}{n'} \\
\delta' &= \frac{\alpha \phi \tau}{n'} - \frac{\alpha \phi \tau}{n'} \\
\end{align*}
\]

Note that the shift \( \delta' \) of the system rear principal plane \( P' \) from the rear principal plane of the second element \( P'_2 \) occurs in the system image space \( n' \).

Two Component System – Front Properties

Repeat the process to determine the front cardinal points. Start with a ray at the system front focal point \( F \). It will emerge from the system parallel to the optical axis.

\[
\begin{align*}
u = u_1 &= 0 \\
n_1 &= n \\
y_1 &= y_1 \\
\phi_1 &= \phi_1 \\
\phi &= \phi \\
u' &= u'_1 = u'_2 = 0 \\
n_2 &= n'_1 \\
y'_1 &= y'_1 \\
\phi'_1 &= \phi'_1 \\
\phi' &= \phi' \\
u'' &= u''_1 = u''_2 = 0 \\
n_3 &= n'_2 = n'' \\
y''_1 &= y''_1 \\
\phi''_1 &= \phi''_1 \\
\phi'' &= \phi'' \\
\end{align*}
\]

\[
\begin{align*}
\alpha' &= \alpha_2 \phi \phi \\
\alpha' &= \alpha_2 \phi \\
y_2 &= y_1 + \alpha \tau \\
\phi' &= \phi \\
\end{align*}
\]

System power: same result as for forward ray.
Two Component System – Front Cardinal Points

If \( P_1 \) is the front vertex of the system, then the distance \( P_1F \) is the FFD.

\[
\begin{align*}
d &= \frac{y_2 - y_1}{u} \\
d' &= \frac{\phi_2 \tau}{\phi} \\
\delta &= \frac{d}{n} = \frac{\phi_2 \tau}{\phi} = \frac{\phi_2 r}{\phi} \\
\delta' &= \frac{\phi_2 \tau}{\phi} = \frac{\phi_2 r}{\phi} \\
f_r &= \frac{P_1F - d}{n} \\
f_r &= -\frac{1 - \phi_2 \tau}{\phi} = \frac{1 - \phi_2 r}{\phi} = -f_x \\
f_r &= \frac{k}{\phi}
\end{align*}
\]

Note that the shift \( d \) of the system front principal plane \( P \) from the front principal plane of the first element \( P_1 \) occurs in the system object space \( n \).

Gaussian Reduction - Summary

\[
\begin{align*}
\phi &= \phi_1 + \phi_2 - \phi_2 \phi \tau \\
\tau &= \frac{t}{n_2} \\
f_s' &= \frac{n'}{\phi} = n'f \\
f_r &= \frac{n}{\phi} = -nf \\
f_r &= \phi_2 \phi \tau = \phi_2 r = \phi_2 t \\
f_r &= \frac{1}{\phi}
\end{align*}
\]

- \( P \) and \( P' \) are the planes of unit system magnification (effective refraction for the system).
- \( d \) is the shift in object space of the front system principal plane \( P \) from the front principal plane of the first system \( P_1 \).
- \( d' \) is the shift in image space of the rear system principal plane \( P' \) from the rear principal plane of the second system \( P_2 \).
- \( t \) is the directed distance in the intermediate optical space from the rear principal plane of the first system \( P_1' \) to the front principal plane of the second system \( P_2 \). Both of these principal planes must be in the same optical space.
- Following reduction, the two original elements and the intermediate optical space \( n_2 \) are not needed or used.
Vertex Distances

The surface vertices are the mechanical datums in a system and are often the reference locations for the cardinal points.

Back focal distance BFD:

\[ BFD = f'_s + d' \]

Front focal distance FFD:

\[ FFD = f_r + d \]

Object and image vertex distances are determined using the Gaussian distances:

\[ s = z + d \]
\[ s' = z' + d' \]

Thick Lens in Air

A thick lens is the combination of two refracting surfaces.

\[ \phi_1 = (n_2 - n_1)C_1 \]
\[ \phi_2 = (n_1 - n_2)C_2 \]
\[ \phi_3 = (1 - n)C_3 = -(n - 1)C_3 \]

\[ \phi = \phi_1 + \phi_2 - \phi_3 \tau \]
\[ \tau = \frac{t}{n_2} = \frac{t}{n} \]
\[ d' = \frac{\phi_1}{\phi} \frac{t}{n} \]
\[ d = \frac{\phi_3}{\phi} \frac{t}{n} \]

\[ f' = \frac{n_1}{\phi} = \frac{1}{\phi} \quad f'_s = \frac{n_1}{\phi} = \frac{1}{\phi} \quad \frac{PP}{t - d + d'} = t - \frac{\phi_1}{\phi} \frac{t}{n} \]
\[ f = f'_s - f_r = -\frac{1}{\phi} \quad \frac{PP}{(n - 1)\tau} = -\frac{\phi_3}{\phi} \frac{t}{n} \]

The nodal points are located at the respective principal planes.
Thin Lens in Air

t \to 0 \quad \tau \to 0
\phi = \phi_1 + \phi_2 - \phi_0 \tau
\phi = \phi_1 + \phi_2
\phi_1 = (n-1)C_1
\phi_2 = -(n-1)C_2
\phi = (n-1)(C_1 - C_2)

f = f_x = f'_x = -f_0 = \frac{1}{\phi} \quad d = d' = 0 \quad BFD = f

The principal planes and nodal points are located at the lens.

Two Separated Thin Lenses in Air

n_2 = 1 \quad t = \tau
\phi = \phi_1 + \phi_2 - \phi_0 t
\phi = \phi_1 + \phi_2
f = f'_x = \frac{1}{\phi}

d' = -\frac{\phi_1}{\phi} t \quad d = \frac{\phi_2}{\phi} t

BFD = f'_x + d'
\overline{PP} = t - d + d' = \frac{\phi_1 \phi_2}{\phi} t^2

The nodal points are coincident with the principal planes.
Gaussian Reduction Example – Two Separated Thin Lenses in Air

Two 50 mm focal length lenses are separated by 25 mm.

\[ \phi = \phi_1 + \phi_2 - \phi_1 \phi_2 f \]

\[ d' = -\frac{\phi_1 f}{\phi} = -\frac{0.02 \text{mm}^{-1} \times 25 \text{mm}}{0.03 \text{mm}^{-1}} \]

\[ \phi = 0.02 \text{mm}^{-1} \times 0.02 \text{mm}^{-1} - (0.02 \text{mm}^{-1})^2 \times 25 \text{mm} \]

\[ d' = -16.667 \text{mm} \]

\[ \phi = 0.03 \text{mm}^{-1} \]

\[ f = f'_f = 33.333 \text{mm} \]

\[ BFD = f'_f + d' \]

\[ BFD = 16.667 \text{mm} \]

Diopters

Lens power is often quoted in diopters D.

Units are \( \text{m}^{-1} \)

\[ D = \phi \quad (\text{in m}^{-1}) \]

\[ D = \frac{1}{f_x} \quad (f_x \text{ in m}) \]

With closely spaced thin lenses, the total power is approximately the sum of the powers of the individual lenses. Focal lengths do not add.
Multi-Element Reduction

Multiple element systems are reduced two elements at a time.

A single system power and pair of principal planes results.

Given these quantities, the focal lengths and other cardinal points can be found.

There are several reduction strategies possible for multiple elements or surfaces.

\[ 1\ 2\ 3\ 4 \rightarrow (12)\ (34) \rightarrow (1234) \]

\[ 1\ 2\ 3\ 4 \rightarrow (12)\ 3\ 4 \rightarrow (123)\ 4 \rightarrow (1234) \]

The system principal planes are usually measured relative to the front and rear vertices of the systems:

- The system front principal plane is located relative to the front principal plane of the first surface or element.
- The system rear principal plane is located relative to the rear principal plane of the last surface or element.

\[ \begin{align*}
1 & \ 2 & \ 3 & \ 4 \\
\rightarrow & \ (12) & \ (34) & \ (1234)
\end{align*} \]
Reduction – One at a Time

\[ n = n_1 \quad n'_1 = n_2 \quad n'_2 = n_3 \quad n'_3 = n_4 \quad n'_4 = n' \]

\[ \phi_1 \quad \phi'_1 \quad \phi'_2 \quad \phi'_3 \quad \phi'_4 \]

\[ P_1 \quad P'_1 \quad P'_2 \quad P'_3 \quad P'_4 \]

\[ t'_1 \quad t'_2 \quad t'_3 \quad t'_4 \]

\[ d_{12} \quad d_{13} \quad d_{14} \]

\[ P_{12} \quad P_{13} \quad P_{14} \]

\[ n \quad n_3 \quad n_4 \quad n' \]

\[ n' \]

\[ \phi_{12} \quad \phi_{13} \quad \phi_{14} \]

\[ d_{123} \quad d_{124} \]

\[ P_{123} \quad P_{124} \]

\[ d' = d_{123} + d_{124} \]

\[ \phi'_{12} \quad \phi'_{13} \quad \phi'_{14} \]

\[ d'' = d'_{1234} \]

\[ \phi'_{123} \quad \phi'_{124} \quad \phi'_{134} \quad \phi'_{1234} \]

\[ P'_{12} \quad P'_{13} \quad P'_{14} \]

Gaussian Reduction – Example

Cemented Doublet

\[ n = n_1 = 1.0 \quad n_2 = 1.517 \quad n_3 = 1.649 \quad n' = n_4 = 1.0 \]

\[ C_1 \quad V \quad C_2 \quad C_3 \]

\[ t'_1 \quad t'_2 \quad t'_3 \]

\[ R_1 = 73.8950 \quad R_2 = -51.7840 \quad R_3 = -162.2252 \]

\[ C_1 = .0135327 \quad C_2 = -.0193110 \quad C_3 = -.00616427 \]

\[ \phi_1 = .00700 \quad \phi_2 = -.00255 \quad \phi_3 = .00400 \]

\[ t'_2 = 10.5 \quad t'_2 = 4.0 \]

\[ t'_3 = \frac{t'_2}{n_2} = 6.92 \quad t'_3 = \frac{t'_2}{n_3} = 2.43 \]
Gaussian Reduction – Example – Continued

First, reduce the first two surfaces:

\[ \phi_{12} = \phi_1 + \phi_2 - \phi_1 \phi_2 \phi_{12}' = 0.00457 \]
\[ \delta_{12} = \frac{\phi_1}{\phi_2} r_{12}' = -3.86 \quad d_{12} = \delta_{12} \]
\[ \delta_{12}'' = -\frac{d_{12}}{\phi_1} r_{12}' = -10.60 \quad d_{12}' = n_2 \delta_{12}'' = -17.48 \]

At this point, the first two surfaces are represented by \( \phi_{12} \) and the principal planes \( P_{12} \) and \( P_{12}' \).

\[ r_{12}' = r_{12}' - d_{12}' = 21.48 \]
\[ r_{12}' - \frac{d_{12}'}{n_1} r_{12}' - \delta_{12}' = 13.03 \]

Gaussian Reduction – Example – Continued

Add the third surface:

\[ \phi = \phi_1 + \phi_2 + \phi_3 \phi_1 \phi_{12}' = 0.00833 \]
\[ d_{133} = \delta_{13} = \frac{d_{13}}{\phi} r_{13}' = 6.26 \quad \text{Object space } n = 1 \]
\[ d_{133}' = \delta_{133}' = -\frac{\phi_3}{\phi_1} r_{13}' = -7.15 \quad \text{Image space } n' = 1 \]

\[ d = \delta = \delta_{12} + \delta_{13} = d_{12} + d_{13} = 2.40 \]
\[ d' = \delta' = \delta_{133}' = -7.15 \]
\[ \phi = 0.008333 \quad f_e = 120.0 \quad f'_e = -f_e = 120.0 \]
Gaussian Reduction – Example – Summary

\[ d = \delta_1 + \delta_{23} = d_{12} + d_{13}, \quad d' = \delta_1' + \delta_{23} = -7.15 \]

\[ \phi = 0.008333, \quad f'_e = 120.0, \quad f'_e = -f_e = 120.0 \]

\[ \overline{FF} = \overline{PP} = f_e + d = -117.6 \quad \text{Front Focal Distance} \quad \text{FFD} \]

\[ \overline{VF} = \overline{P'F'} = f'_e + d' = 112.85 \quad \text{Back Focal Distance} \quad \text{BFD} \]

Principal plane separation:

\[ \overline{PP'} = \overline{FF'} - d + d' = \ell'_1 + \ell'_2 - d + d' \]

\[ \overline{PP'} = 4.95 \]

Real Lens to Thin Lens Model
Consider a single reflecting surface with a radius of curvature of \( R \). The rays propagate in an index of refraction of \( n \).

The angles of incidence and refraction (\( \theta \) and \( \theta' \)) are measured with respect to the surface normal.

The ray angles \( U \) and \( U' \), as well as the elevation angle \( A \) of the surface normal at the ray intersection, are measured with respect to the optical axis. The usual sign conventions apply.

Relating the angles at the ray intersection with the surface:

\[
U' = A + I' \\
I = U - A \\
I' = U' - A
\]

Apply the Law of Reflection:

\[ I' = -I \]

\[ (U' - A) = -(U - A) \]
Single Reflecting Surface and the Law of Reflection

\[ I' = -I \]

\[ (U' - A) = -(U - A) \]

\[ \sin(U' - A) = -\sin(U - A) \]

\[ \sin(U'\cos A - \cos U'\sin A) = -[\sin U\cos A - \cos U\sin A] \]

\[ \frac{\sin(U' - \cos U'\sin A)}{\cos A} = -\frac{\sin U - \cos U\sin A}{\cos A} \]

\[ \frac{\sin U' - \cos U'\tan A}{\cos A} = -\sin U - \cos U\tan A \]

Approximation #1: \( \cos U \approx \cos U' \)

\[ |U'| = |U| \]

Approximation #1 implies that the magnitude of the ray angle is approximately constant.

Paraxial Angles

\[ \tan(U' - \tan A) = -\tan U - \tan A \]

\[ \tan U' = -\tan U + 2\tan A \]

Reformulate in terms of the paraxial angles or ray slopes:

\[ u = \tan U \quad u' = \tan U' \quad \alpha = \tan A \]

\[ u' = -u + 2\alpha \]

\[ \alpha = \tan A = -\frac{y}{(R - \text{Sag})} \]

Approximation #2: \( |\text{Sag}| << |R| \)

\[ \alpha = -\frac{y}{R} \]

\[ u' = -u - 2\frac{y}{R} \]

\[ u' = -u - 2yC \quad \text{This is the Paraxial Reflection Equation.} \]

Approximation #2 implies that the sag of the surface at the ray intersection is much less than the radius of curvature of the surface.
Reflection and Refraction

Refraction:
\[ n'u' = nu - y \phi = nu - y(u' - n)C \]
if \( n' = -n \)
\[ n'u' = -nu + y \phi = -nu - 2nyC \]
\[ u' = -u - 2yC \]

Reflection Equals Refraction with \( n' = -n \)
\[ \phi = (n' - n)C \]
\[ n'u' = nu - y \phi \]

Note that a reflector with a positive curvature has a negative power.

Reflection:
\[ f_r = -\frac{n}{\phi} = \frac{1}{2C} = \frac{R}{2} \]
\[ f'_r = f_r \quad n' = -n \]
\[ f''_r = \frac{n'}{\phi} = -\frac{n}{\phi} = f_r \]
\[ f_r = f'_r = -n f''_r = \frac{n}{\phi} = \frac{1}{2C} = \frac{R}{2} \]
\[ f_r = f'_r = \frac{1}{\phi} \]

The front and rear focal lengths are equal to half the radius of curvature.

Object and Image distances for a Single Reflecting Surface

The object and image distances (\( z \) and \( z' \)) are also both measured from the surface vertex.

Approximation #3: The object and image distances are much greater than the sag of the surface at the ray intersection.

\[ u = \tan U = -\frac{y}{(z - Sag)} \approx -\frac{y}{z} \]
\[ u' = \tan U' = -\frac{y}{(z' - Sag)} \approx -\frac{y}{z} \]
Surface Vertex Plane and Principal Planes

The same set of approximations hold for paraxial analysis of a reflecting surface as for a refraction surface:

- The surface sag is ignored and paraxial reflection occurs at the surface vertex.
- The ray bending at each surface is small.

By ignoring the surface sag in paraxial optics, the planes of effective refraction for the single reflecting surface are located at the surface vertex plane V. The Front and Rear Principal Planes (P and P') of the surface are both located at the surface.

The nodal points of a reflecting surface are located at its center of curvature as a ray perpendicular to the surface is reflected back on itself.

The Cosine Condition Applied to Reflection

\[ \cos U = \cos U' \quad \text{or} \quad \frac{\cos U'}{\cos U} = 1 \quad \text{or} \quad |U| = |U'| \]

Consider a reflecting surface perpendicular to the optical axis:

- \( n = 1.0 \)
- \( n' = -1.0 \)
- \( A = 0 \)
- \( I = U \)
- \( I' = -U \)

For a plane mirror perpendicular to the axis, there is no error associated with this approximation.
Cosine Condition – Curved Surfaces

With curved surfaces, this approximation is more difficult to interpret as it relates to the ray angle with respect to the optical axis, not with respect to the surface normal.

Consider a surface with a radius of curvature $R = 200 \text{ mm}$ with a ray height of 10 mm. The surface normal at the ray intersection is tilted at about 2.9 degrees.

A ray will be incident perpendicular to the surface when $U = A$.

The approximation error is approximately linear with the ray angle relative to the surface normal.

At normal incidence, there is no approximation error.

For rays incident within about 10 deg of the surface normal, the error is about 2%.

The approximation is in terms of the ray angle with respect to the optical axis, not the angle of incidence.
Transfer After Reflection

Transfer after reflection works exactly the same as for a refractive system, except that the distance to the next surface (to the left) is negative.

\[ t' < 0 \]
\[ u' = \frac{y' - y}{t'} \]
\[ t' u' = y' - y \]
\[ y' = y + t' u' \]
\[ y' = y + \tau' \omega' \]

The transfer equation is independent of the direction of transfer.

After reflection, the signs of \( \omega \) and \( \tau \) are opposite those of the corresponding \( u \) and \( t \). A drawing done in reduced distances and optical angles will unfold the mirror system and show a thin lens equivalent system.

Sign conventions and reflection:
- Use directed distances as defined by the usual sign convention. A distance to the left is negative, and a distance to the right is positive.
- The signs of all indices of refraction following a reflection are reversed.

Optical Surfaces

The front and rear principal planes of an optical surface are coincident and located at the surface vertex \( V \). Both nodal points of a single refractive or reflective surface are located at the center of curvature of the surface.

\[ \phi = (n' - n) C = \frac{(n' - n)}{R} \]
\[ C = \frac{1}{R} \]

\[ f = f_e = \frac{1}{\phi} \]
\[ f_r = -\frac{n}{\phi} = -n f_e \]
\[ f'_s = \frac{n'}{\phi} = n' f_e \]

A reflective surface is a special case with \( n' = -n \)

\[ \phi = -2n C = -\frac{2n}{R} \]
\[ f_s = f'_s = -\frac{n}{\phi} = -n f_e = \frac{R}{2} = \frac{1}{2C} \]
Refractive and Reflective Surfaces

Power of a refractive surface:  \( \phi = (n' - n)C \)

Assume \( n = 1 \) and \( n' = 1.5 \)

\( \phi = (0.5)C = C/2 \)

Power of a reflective surface:  \( \phi = (n' - n)C = -2nC \)

Assume \( n = 1 \)

\( \phi = -2C \)

For the same optical power, a reflective surface requires approximately one quarter of the curvature of a refractive surface. However the signs of the powers are opposite.

This is one advantage of using reflective surfaces.

Optical Surfaces – Cardinal points

\[ \phi = (n' - n)C = \frac{(n' - n)}{R} \]

\[ C = \frac{1}{R} \]

\[ f_x = -\frac{n}{\phi} = -nf_x \]

\[ f_y' = \frac{n'}{\phi} = nf'_y \]

P, P' at surface vertex

N, N' at center of curvature

Positive Refracting:

\( \phi > 0 \)

\( n' > n \)

\( C > 0 \)
Optical Surfaces – Cardinal points

$$\phi = (n' - n)C = \frac{(n' - n)}{R}$$
$$C = \frac{1}{R}$$

\( f_p = -\frac{n}{\phi} = -nf_e \) \( f_p = \frac{n'}{\phi} = n'f_e \)

Negative Refracting:

\( \phi < 0 \)
\( n' > n \)
\( C < 0 \)

\( \phi < 0 \)
\( n' < n \)
\( C > 0 \)

Positive Reflecting:

\( \phi > 0 \)
\( C < 0 \)
\( n > 0 \)

Negative Reflecting:

\( \phi < 0 \)
\( C > 0 \)
\( n > 0 \)

A concave mirror has a positive power and focal length, but negative front and rear focal lengths. A convex mirror has a negative power and focal length, but positive front and rear focal lengths.
Real and Virtual Images

Real images can be projected and made visible on a screen; virtual images cannot.

Real images— the actual rays in image space head towards the image.

Virtual – the actual rays in image space head away from the image. The rays must be projected backwards to find the image (virtual ray segments).

Shape Factor and Bending

Lenses having the same power can differ from each other in shape. The distribution of power between the two surfaces can vary.

Symmetrical Shape Factor: \( X = \frac{C_1 + C_2}{C_1 - C_2} \frac{\phi_1 - \phi_2}{\phi_1 + \phi_2} \)

For a thick lens in air of thickness \( t \) or \( \tau \) with a specified power \( \phi \) and shape factor \( X \):

\[
\phi_1 = \frac{1 - \sqrt{1 - (1 - X^2)\phi \tau}}{(1 - X)\tau}
\]

\[
\phi_2 = \frac{1 - \sqrt{1 - (1 - X^2)\phi \tau}}{(1 + X)\tau}
\]

For small \( t \) or \( \tau \):

\( \phi_1 \rightarrow \frac{1}{2}(1 + X)\phi \)

\( \phi_2 \rightarrow \frac{1}{2}(1 - X)\phi \)

\( d \rightarrow \frac{1}{2}(1 - X)\tau \)

\( d' \rightarrow \frac{1}{2}(1 + X)\tau \)

\( \frac{PP'}{(n - 1)\tau} \rightarrow \text{independent of } X \)

\( X = -1 \quad C_1 = 0 \quad \text{Plano – Spheric Lens} \)

\( X = +1 \quad C_2 = 0 \quad \text{Spheric – Plano Lens} \)

\( X = 0 \quad \text{Symmetric} \quad \text{Bi-Spheric Lens} \)

\( |X| > 1 \quad \text{Meniscus Lens} \)

Thin Lens – it is the difference in curvatures that determines the lens power:

\( \phi = (n - 1)(C_1 - C_2) \)
**Shape Factor**

Bending does not change the element power.

\[
X = \begin{array}{ccccc}
-3 & -1 & 0 & 1 & 3 \\
\end{array}
\]

\[
d' = 0 \\
d = 0
\]

**System Design Using Thin Lenses**

1) Obtain the thin lens solution to the problem:

2) Include the principal plane separations of real elements:

3) Locate the vertices of the real components:

The vertices and vertex-to-vertex separations are the mechanical datums for the system.
Gaussian Imagery and Gaussian Reduction

The utility of Gaussian optics and Gaussian reduction is that the imaging properties of any combination of optical elements can be represented by a system power or focal length, a pair of principal planes and a pair of focal points. In initial design, the P-P' separation is often ignored (i.e. a thin lens model).

Methods of System Analysis

- Raytrace
- Gaussian System
  - System Cardinal Points
  - Gaussian Reduction
  - Gaussian Imagery
- Imaging Properties
  - Oblique illumination
  - Oblique illumination
  - Oblique illumination
  - Oblique illumination

System Specification
- Radii, indices, thicknesses, spacings, element focal lengths, etc.
Thin Lens Design – Overall Object-to-Image Distance

\[ \frac{1}{z'} + \frac{1}{z} = \frac{1}{f_e} \quad m = \frac{h'}{h} \]

\[ z = \left(1 - \frac{m}{m'}\right)f_e \quad z' = (1 - m)f_e \]

\[ L = z' - z = (1 - m)f_e - \left(1 - \frac{m}{m'}\right)f_e \]

\[ L = \left(\frac{m - 1}{m}\right)f_e \]

Real object and real image: \( m < 0 \quad z < -f_e \quad z' > f_e \)

Minimum object to image distance: \( m = -1 \quad L = 4f_e \)

Reciprocal Magnifications

Overall object-to-image distance:

\[ L = \frac{(m - 1)^2}{m}f_e \]

For each \( L \), there are two possible magnifications and conjugates: Reciprocal magnifications.

\[ \frac{L}{f_e} = \frac{(m - 1)^2}{m} = \frac{(1/m - 1)^2}{1/m} \]
Positive Lens

\[ f_e > 0 \]

\[ z < 0 \]

\[ z = -f_e \]

\[ z = 0 \]

\[ z > 0 \]

\[ f_e \]

Object Location

Lens Position

Image Location

L

Real Object Virtual Image

Virtual Object Real Image

1/4

1/3

1/2

1

-1

-2

-3

-4

m

Position Relative To Object

Real Image

Virtual Object

z

\[ Z \]

Z'

Negative Lens

\[ f_e < 0 \]

z < 0

\[ z = -|f_e| \]

\[ z = |f_e| \]

z > 0

Virtual Object Virtual Image

Virtual Object

Real Image

1/4

1/3

1/2

1

-1

-2

-3

-4

m

Position Relative To Object

Real Image

Virtual Object

z

Z

Z'
Magnification Properties

The Gaussian Magnification may also be determined from the object and image ray angles.

\[ m = \frac{h'}{h} = \frac{z'/n'}{z/n} \]

\[ u = \frac{h_o}{z} \quad u' = \frac{h_o}{z'} \]

\[ m = \frac{\omega}{\omega'} = \frac{-h_o}{z/n} \] \( \frac{\omega}{\omega'} = \frac{-h_o}{z'/n'} \]

\[ z/n = \frac{h_o}{\omega} \quad z'/n' = \frac{h_o}{\omega'} \]

\[ m = \frac{h'}{h} = \frac{z'/n'}{z/n} = \frac{h_o/\omega'}{\omega} \]

\[ m = \frac{\omega}{\omega'} = \frac{mu}{n'u'} \]

This angle relationship holds for all rays passing through on-axis conjugate points.

Thin Lens Design – Two Lenses at Fixed Separation

Given: \( t_1, t_2, t_3, m \)

Determine: \( \phi_1, \phi_2 \)

\( n = 1 \quad u = \omega \)

Let: \( \omega_1 = u_1 = 1 \)

\[ \phi_1 = \frac{\omega_1 - \omega_2}{h_1} = \frac{1}{t_1} = \frac{t_1 + t_2 + t_3}{m} \]

\[ h_1 = \omega_o t_1 = t_1 \]

\[ \omega_1 = \omega_o/m = 1/m \]

\[ h_2 = -\omega_o t_2 = -t_2/m \]

\[ \phi_2 = \frac{\omega_2 - \omega_3}{h_2} = \frac{-t_2 + t_3 + t_4}{m} = \frac{m t_1 + t_2 + t_3}{t_3} \]

\[ \omega_2 = \frac{h_2 - h_3}{t_2} = -\frac{t_2 + t_3 + t_4}{t_2} \]

\[ \phi = \phi_1 + \phi_2 - \phi_2 \phi_2 = -\frac{m t_1 + t_2 + t_3}{t_3} \]
Thin Lens Design – Two Given Lenses

Given: \( L, m, \phi_1, \phi_2 \) (or \( f_1, f_2 \))

Determine: \( t_1, t_2, t_3, \phi \)

Define: \( M = \frac{(1-m)^2}{m} \)

\[ F_i = \frac{f_i}{L} \quad F_z = \frac{f_z}{L} \]

\[ L = \frac{M}{\phi} + \frac{\phi t_1^2}{\phi} = \frac{\phi t_2^2}{\phi} \]

\[ -\phi L = -M + \phi \phi t_1^2 = -L(\phi + \phi t_1) \]

\[ -M f_1 f_2 + t_2^2 = -L(f_1 + f_2) + L t_2 \]

\[ t_2^2 - L t_2 + L(f_1 + f_2) - M f_1 f_2 = 0 \]

\[ \frac{1}{L} t_2^2 - t_2 + L(F_i + F_z) - L M F_i F_z = 0 \]

\[ t_2 = \frac{L}{2} \left[ 1 \pm \sqrt{1 - 4(\frac{L F_i + F_z}{M F_i F_z})} \right] \]

Cardinal Points Example

The power and the relative locations of the cardinal points of a system completely define the imaging mapping.

Different combinations of elements can produce interesting situations.

As an example consider this true 1:1 imaging system consisting of cascaded 2f-2f systems. An inverted intermediate image is formed.

Because the object and image planes are planes of unit magnification, the system front and rear principal planes are coincident with the object and image planes.

Where are the focal points?
Cardinal Points Example - Continued

To find the rear focal point of the system, launch a ray parallel to the axis:

![Diagram showing ray path and focal points]

The system rear focal point is to the left of the system rear principal plane, and the system power is negative! This infinity ray diverges from the rear principal plane.

\[
\phi = \phi_1 = \frac{1}{f} \quad t = 4f
\]

\[
\phi = \phi_1 + \phi_2 = \frac{2}{f} - \frac{4f}{f^2} = \frac{2}{f}
\]

\[
f_{\text{system}} = f'_{\text{R}} = -0.5f
\]

\[
d' = -\frac{\phi}{\phi} t = \frac{1/f}{-2/f} 4f = 2f
\]

In a simplified Gaussian model that ignores the P-P' separation, this system looks just like a negative thin lens:

By symmetry, the front focal point of the system is to the right of the system front principal plane.

Mini Quiz

Two 100 mm focal length thin lenses are separated by 50 mm. What is the focal length of this combination of lenses?

- [ ] a. 66.67 mm
- [ ] b. 200 mm
- [ ] c. 50 mm
- [ ] d. It is an afocal system
Mini Quiz – Solution

Two 100 mm focal length thin lenses are separated by 50 mm. What is the focal length of this combination of lenses?

[X] a. 66.67 mm  
[ ] b. 200 mm  
[ ] c. 50 mm  
[ ] d. It is an afocal system

\[
f_1 = f_2 = 100 \text{mm} \quad \phi = \frac{1}{f} \quad \phi_1 = \phi_2 = 0.01 \text{mm}^{-1}
\]

\[
t = 50 \text{mm}
\]

\[
\phi = \phi_1 + \phi_2 - \phi \phi t
\]

\[
\phi = 0.015 \text{mm}^{-1}
\]

\[
f = 66.67 \text{mm}
\]