21 COATING OF OPTICAL SURFACES

21.1 INTRODUCTION

21.1.1 Uses. Thin films of dielectric, metallic or even semi-conducting materials are most often applied to optical components such as lenses, plates and reflectors for the purpose of altering their energy reflectances or transmittances. A great variety of distributions of spectral reflectances and transmittances can be achieved over the ultraviolet, visible and infra-red regions. However, the number of materials having suitable optical, mechanical and chemical properties for use in the ultraviolet region is severely limited. Occasionally, thin films are used for modifying, especially at oblique incidence, phase changes as well as amplitude changes upon reflection. Thin films can also serve as protective coatings for surfaces of soft materials such as aluminum or silver. In another type of application, films are deposited with non-uniform thickness in order to achieve a slight degree of aspherization of the coated surface or in order to produce wedges that transmit non-uniformly in a specified manner. In still another broad class of films, a combination of optical properties such as transmittance, and of electrical properties such as conductance is provided. A diversity of specialized films consisting of combinations of two or more materials in a multilayer is required to meet an increasing list of modern applications.

21.1.2 Properties of thin films. We shall be concerned herein with the physical principles governing the optical properties of thin films. Fortunately, thin films have been found to behave to a good first approximation as homogeneous, plane-parallel layers that can be regarded as infinite in lateral dimensions. This idealized model of single films or multilayers can be analyzed without further approximation as a boundary problem involving Maxwell's equations. Several related forms of this useful theory will be treated. Actually, thin films are not homogeneous either laterally or along the thickness direction. Small departures from the predictions of the idealized model are therefore likely to occur. Theories dealing with inhomogeneity along the thickness direction are under active investigation; but it must be expected that these theories will be of greatest value in designing films whose inhomogeneities are increased deliberately.

21.2 DEFINITIONS AND PRINCIPLES

21.2.1 The optical constants. The optical constants, n and K, of an homogeneous, isotropic film are defined in the following manner. We take the solutions for the electric vector, E, and the magnetic vector, H, in the form

$$E = U e^{-i\omega t} ; \quad H = V e^{-i\omega t} ; \tag{1}$$

in which U and V are vectors; $U = (U_x, U_y, U_z)$ and $V = (V_x, V_y, V_z)$. Maxwell's curl relations become

$$\text{Curl } V + i \frac{\omega}{c} m^2 U = 0; \tag{2}$$

$$\text{Curl } U - i \frac{\omega}{c} \mu V = 0; \tag{3}$$

and the wave equations for U and V become

$$\nabla^2 U + \frac{\omega^2}{c^2} \mu m^2 U = 0; \tag{4}$$

$$\nabla^2 V + \frac{\omega^2}{c^2} \mu m^2 V = 0; \tag{5}$$

wherein

$$m^2 = \epsilon + i\, 4\pi \sigma/\omega \qquad \text{(defining m);} \tag{6}$$

$$m = n(1 + iK) \qquad \text{(defining n and K).} \tag{7}$$

The magnetic permeability, μ, and the dielectric constant, ϵ, are defined so that they are unity for vacuum. σ is the electric conductivity; c is velocity in vacuum and $\omega = 2\pi/T$, where T is the period of vibration of the wave. $i = \sqrt{-1}$.

$$\omega/c = 2\pi/\lambda_o = k \qquad (8)$$

where λ_o denotes wavelength in vacuum. As so defined, n and K are the optical constants that are usually listed in handbooks. In most optical problems one will take $\mu = 1$. We shall regard n and K as the fundamental properties that are measured from experimental observation on the wave motion. Under this view, ϵ and σ are derived from knowledge of n and K.

21.2.2 Physical significance. So much confusion exists about the physical significance of the optical constants, n and K, that further discussion is warranted. It is often believed that n is the refractive index such that the phase velocity v is always given by c/n. Suppose that a plane wave is propagated along Z with its electric vector polarized to vibrate along X. Then $U = (U_x, 0, 0)$. Since the wave is plane, U_x is, by assumption, independent of x and y. Hence the wave equation reduces to

$$\frac{\partial^2 U_x}{\partial z^2} + \frac{\omega^2}{c^2}\mu m^2 U_x = 0. \qquad (9)$$

It is easily verified by substitution into (9) that a solution is

$$U_x = E_o\, e^{i\frac{\omega}{c}\mu^{1/2} mz} = E_o\, e^{-k\mu^{1/2} nKz}\, e^{i\frac{\omega}{c}\mu^{1/2} nz} \qquad (10)$$

One concludes from equations (1) and (10) that

$$E = (1, 0, 0)\, E_o\, e^{-k\mu^{1/2} nKz}\, e^{-i\omega(t - \frac{z}{v})} \qquad (11)$$

wherein

$$v = c/\mu^{1/2} n \qquad (12)$$

is the phase velocity of the wave in the medium, and $k\mu^{1/2} n K$ is an absorption or extinction coefficient that determines the rate of attenuation of the amplitude with increasing z. We observe from equation (11) that the parallel planes, z = constant, are planes of equiphase (wavefronts) and of equiamplitude. When the planes of equiphase and equiamplitude are parallel, the wave is said to be homogeneous. If, then, $\mu = 1$ and the wave is homogeneous, the optical constant, n, is in fact the refractive index. When a homogeneous wave is incident normally upon any system of plane parallel layers, the wave remains homogeneous. But when a homogeneous wave is incident obliquely upon a system of plane parallel layers, the wave becomes inhomogeneous in the absorbing layers, i.e. planes of equiphase and equiamplitude do not remain strictly parallel when $K \neq 0$. With inhomogeneous waves, the phase velocity is not given by equation (12) and rate of attenuation is not governed by the product $k\mu^{1/2} n K$. A generalized form of Snell's law of refraction applies, but the actual refractive index is not the optical constant n even when $\mu = 1$. It will not be an objective of this text to dwell upon the effects exhibited by inhomogeneous waves in absorbing media; but it is worth noting that the following systems of equations will include the effects produced by inhomogeneous waves in the absorbing layers. When the system is free of absorption, the waves remain homogeneous even at oblique incidence.

21.2.3 Fresnel's coefficients for normal incidence.

21.2.3.1 Fresnel's coefficients of reflection and transmission with respect to a plane interface between two homogeneous media can be derived with a high degree of rigor. Derivations based upon Maxwell's equations and realistic boundary conditions will be found in almost all textbooks dealing with physical optics or electromagnetic theory. The following Fresnel coefficients apply to normal incidence upon the interface as illustrated in Figure 21.1. The Z-direction is chosen normal to the interface, and the point $z = 0$ shall fall at the interface. To date, the permeabilities, μ, have invariably been unity in the optical applications of thin films. In view of the unlikelihood of cases $\mu \neq 1$, the following considerations will be restricted to cases $\mu = 1$.

21.2.3.2 Let τ_o be a complex number that specifies the amplitude and phase of the incident E-vector at the left hand boundary, $z = 0$. Let r_o be a complex number that specifies the amplitude and phase of the reflected E-vector at the left hand boundary, $z = 0$. Similarly, let τ_1 specify the amplitude and phase of the transmitted E-vector at the right hand boundary, $z = 0$. It can be shown that

$$\frac{r_o}{\tau_o} = \frac{m_o - m_1}{m_o + m_1} \equiv W_1\,, \qquad (13)$$

and that

$$\frac{\tau_1}{\tau_o} = \frac{2m_o}{m_o + m_1}\,, \qquad (14)$$

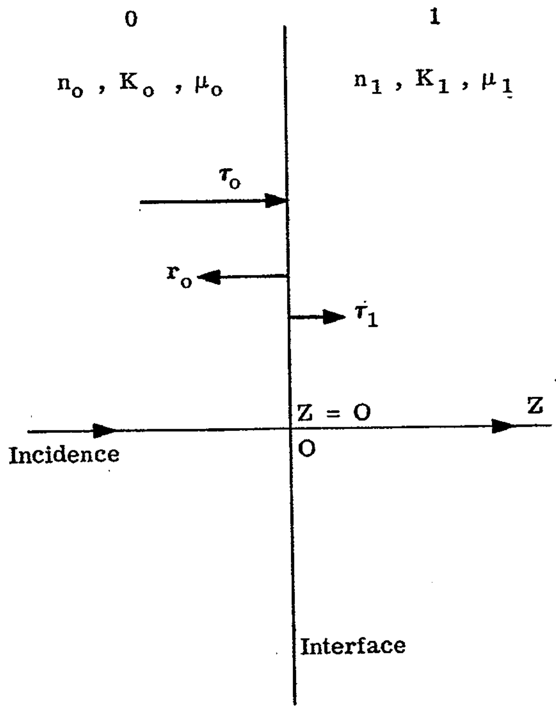

Figure 21.1- Notation with respect to Fresnel's coefficients for normal incidence.

in which

$$m_j = n_j (1 + i K_j) ; \quad j = 0 \text{ and } 1. \tag{15}$$

The ratios r_o/τ_o and τ_1/τ_o are, respectively, Fresnel's coefficient of reflection and transmission at normal incidence. One may take τ_o as unity without any essential loss of generality.

21.2.3.3 If neither medium is absorbing, $K_o = K_1 = 0$. Hence for interfaces between non-absorbing media the Fresnel coefficients reduce to the better known results

$$r_o/\tau_o = \frac{n_o - n_1}{n_o + n_1} , \tag{13a}$$

and

$$\tau_1/\tau_o = \frac{2n_o}{n_o + n_1} , \tag{14a}$$

in which n_o and n_1 are physically the refractive indices of the two media. When $n_1 > n_o$, one may write

$$r_o/\tau_o = \frac{|n_o - n_1|}{n_o + n_1} \quad e^{\pm i\pi} . \tag{13b}$$

Thus one concludes that, with respect to the electric vector, the phase change on reflection is $\pm \pi$ radians when $n_1 > n_o$. The phase change on transmission across the interface is always zero when the two media are non-absorbing, for then the ratio τ_1/τ_o is necessarily real and positive.

21.2.3.4 As a second example, consider incidence from a non-absorbing medium upon an absorbing medium. Since $m_o = n_o$ and $m_1 = n_1 + i n_1 K_1$, one obtains from equation (13)

$$\frac{r_o}{\tau_o} = \frac{n_o - n_1 - i n_1 K_1}{n_o + n_1 + i n_1 K_1} = \frac{n_o^2 - n_1^2 (1 + K_1^2) - i\, 2 n_o n_1 K_1}{(n_o + n_1)^2 + n_1^2 K_1^2} . \tag{16}$$

For reflection from air to metals, $n_1^2 (1 + K_1^2)$ invariably exceeds n_o^2 . Hence the real and imaginary parts of the Fresnel reflection coefficient r_o/τ_o will usually be negative and the phase angle on reflection will fall in the third quadrant. Thus with $r_o/\tau_o = |r_o/\tau_o|\, e^{i\theta}$, $180^\circ < \theta < 270^\circ$. From equation (14),

$$\frac{\tau_1}{\tau_o} = \frac{2n_o}{n_o + n_1 + in_1K_1} = 2\,\frac{n_o(n_o+n_1) - i n_o n_1 K_1}{(n_o+n_1)^2 + n_1^2 K_1^2} \ . \qquad (17)$$

The phase angle introduced by transmission through the interface will fall in the fourth quadrant.

21.2.4 <u>Fresnel's coefficients for oblique incidence.</u>

21.2.4.1 The direction, Z, has been taken along the normal to the interface. It is convenient to choose the X-direction in the plane of incidence as illustrated in Figure 21.2. X, Z is then the plane of incidence. Introduce for brevity,

$$p_o = \sin i_o; \quad q_o = \cos i_o; \qquad (18)$$

where i_o is the angle of incidence. Let

$$M_\nu = (m_\nu^2 - m_o^2 p_o^2)^{1/2} \qquad (19)$$

wherein the suffix $\nu = 0, 1$ and refers to the media of Figure 21.2, and wherein m_ν is defined by equation (15). M_ν is complex imaginary whenever m_o or m_1 is complex imaginary. It can happen, as in total internal reflection, that $(m_\nu^2 - m_o^2 p_o^2)$ is real and less than zero. In such cases

$$M_\nu = i\,|m_\nu^2 - m_o^2 p_o^2|^{1/2}; \quad i = \sqrt{-1}\ . \qquad (19a)$$

21.2.4.2 We need to introduce two more quantities W_ν and F_ν. These are

$$W_\nu = \frac{M_{\nu-1} - M_\nu}{M_{\nu-1} + M_\nu}, \qquad (20)$$

and

$$F_\nu = \frac{m_\nu^2 M_{\nu-1} - m_{\nu-1}^2 M_\nu}{m_\nu^2 M_{\nu-1} + m_{\nu-1}^2 M_\nu}\ . \qquad (21)$$

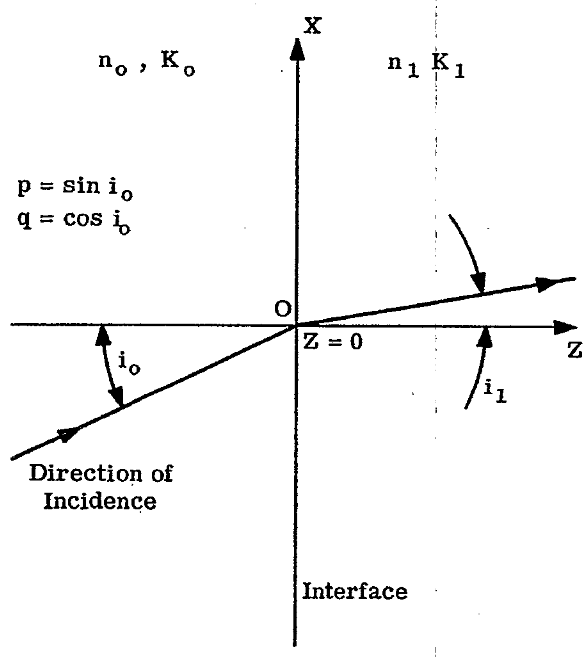

Figure 21.2 - Notation with respect to Fresnel's coefficients for oblique incidence. Plane XZ is chosen as the plane of incidence. i_o is the angle of incidence.

The laws of reflection depend upon the state of polarization of the incident wave. It suffices to consider the Fresnel coefficients of reflection and transmission for two states of polarization. In one of these states, the electric vector is perpendicular to the plane of incidence so that $E = (0, E_y, 0)$. In the second state of polarization, the magnetic vector is perpendicular to the plane of incidence so that $H = (0, H_y, 0)$. A minimum number of four Fresnel coefficients becomes necessary.

21.2.4.3 Consider first the state of polarization in which the electric vector is perpendicular to the plane of incidence. Let τ_o and r_o be complex numbers that specify the amplitude and phase of the incident and reflected electric vector at the origin, O, at left hand boundary of the interface, $z = 0$, Figure 21.2. Let, similarly, τ_1 specify the amplitude and phase of the transmitted electric vector at the point $x = z = 0$ at the right hand boundary of the interface. The ratios r_o/τ_o and τ_1/τ_o are now Fresnel's coefficients of reflection and transmission, respectively, for the electric vector. These ratios are given by

$$\frac{r_o}{\tau_o} = W_1 = \frac{M_o - M_1}{M_o + M_1} , \tag{22}$$

and

$$\frac{\tau_1}{\tau_o} = \frac{2M_o}{M_o + M_1} . \tag{23}$$

The similarity of equations (13) and (22), and of equations (14) and (23) should be noted. These results become alike when $p_o = 0$ (normal incidence).

21.2.4.4 Consider next the state of polarization in which the magnetic vector is perpendicular to the plane of incidence. Let T_o and R_o be complex numbers that specify the amplitude and phase of the incident and reflected H-vectors, respectively, at point $x = 0$ at the left hand boundary of the interface at $z = 0$. Let T_1 specify the amplitude and phase of the transmitted H-vector at point $x = 0$ at the right hand boundary of the interface. The ratios R_o/T_o and T_1/T_o define Fresnel's coefficient of reflection and transmission, respectively, for the perpendicular component of the magnetic vector. These ratios are given by

$$\frac{R_o}{T_o} = F_1 = \frac{m_1^2 M_o - m_o^2 M_1}{m_1^2 M_o + m_o^2 M_1} , \tag{24}$$

and

$$\frac{T_1}{T_o} = \frac{2m_1^2 M_o}{m_1^2 M_o + m_o^2 M_1} . \tag{25}$$

21.2.4.5 As an application or test of equations (22) and (24), consider total internal reflection. This phenomenon occurs when neither medium absorbs, so that $m_o = n_o$ and $m_1 = n_1$, and when $n_o > n_1$. Total internal reflection occurs when the angle of incidence $i_o \geq \sin^{-1} \frac{n_1}{n_o}$, i.e. when $n_o^2 p_o^2 \geq n_1^2$. Therefore, from equation (19a) it follows that M_1 is a pure imaginary number for angles i_o beyond the critical angle for total internal reflection. Since m_o, m_1 and M_o are real numbers, the numerators of equations (22) and (24) are complex conjugates with respect to the denominators. Hence it follows at once that both $|r_o/\tau_o|^2$ and $|R_o/T_o|^2$ are unity. The energy reflectance is therefore total, as required by experiment. One can verify that the numerator of equation (24) is zero at Brewster's angle. This means that the H-vector will not be reflected when it is perpendicular to the plane of incidence, or, equivalently, that the E-vector will not be reflected at Brewster's angle when it vibrates in the plane of incidence.

21.2.4.6 Whereas $|r_o/\tau_o|^2$ and $|R_o/T_o|^2$ can always be interpreted as energy reflectances for the states of polarization to which they apply, the square of the absolute values of Fresnel's coefficients of transmission τ_1/τ_o and T_1/T_o do not necessarily equal the actual energy transmittances. This matter will be treated in some detail since it has been the source of much confusion. With respect to thin films or interfaces, one invariably wishes to compute energy transmittance for cases in which the initial and last media are non-absorbing. Accordingly, emphasis will be placed upon non-absorbing initial and final media.

21.2.5 <u>The electromagnetic field when the electric vector is perpendicular to the plane of incidence.</u>

21.2.5.1 Let us suppose that an homogeneous incident wave has been generated in the initial medium such that the incident electric vector is the known vector,

$$E_{incident} = (0, 1, 0)\, \tau_o\, e^{-i\omega t}\, e^{ikm_o (p_o x + q_o z)} \tag{26}$$

From equations (1) and (26),

$$U_{incident} = (0, 1, 0)\ \tau_o\ e^{ikm_o(p_o x + q_o z)} \tag{26a}$$

Curl relation, equation (3), serves to determine V from U. For plane waves incident in the X, Z plane, vectors U and V are independent of y. Consequently equation (3) yields directly the result,

$$V_x = \frac{i}{k\mu}\frac{\partial U_y}{\partial z}, \quad V_y = 0, \quad V_z = \frac{-i}{k\mu}\frac{\partial U_y}{\partial x}; \tag{27}$$

a result that holds in any medium. Since U_y is given from equation (26a), one can compute vector, V, from equation (27) and then write the incident H-vector from equation (1). One obtains in a straightforward manner for the case $\mu = 1$,

$$H_{incident} = (-q_o, 0, p_o)\ m_o \tau_o\ e^{-i\omega t}\ e^{ikm_o(p_o x + q_o z)}, \tag{26b}$$

The incident electromagnetic field becomes known when τ_o is assigned. The <u>time averaged Poynting vector</u>, S, is given except for an unimportant factor by the vector product

$$S = E \times \bar{H} + \bar{E} \times H. \tag{28}$$

From equations (26), (26b) and (28),

$$S_{incident} = (p_o, 0, q_o)\ 2n_o |\tau_o|^2\ e^{-2kn_o K_o(p_o x + q_o z)}, \tag{26c}$$

since $m_o + \bar{m}_o = 2n_o$. This energy flux is along the direction of propagation of the incident wave.

21.2.5.2 The incident E-vector is reflected with the Fresnel reflection coefficient r_o/τ_o. Consequently,

$$E_{reflected} = (0, 1, 0)\ \frac{r_o}{\tau_o}\ \tau_o\ e^{-i\omega t}\ e^{ikm_o(p_o x - q_o z)}. \tag{29}$$

Just as equation (27) served for determining $H_{incident}$ from $E_{incident}$, it serves again for determining $H_{reflected}$ from $E_{reflected}$. One obtains straightforwardly,

$$H_{reflected} = (q_o, 0, p_o)\ m_o\ \frac{r_o}{\tau_o}\ \tau_o\ e^{-i\omega t}\ e^{ikm_o(p_o x - q_o z)}. \tag{29a}$$

The Fresnel coefficient r_o/τ_o of equation (22) determines the reflected electromagnetic field. Upon evaluating the Poynting vector, S, from equations (28), (29) and (29a), one obtains

$$S_{reflected} = (p_o, 0, -q_o)\ 2n_o \left|\frac{r_o}{\tau_o}\right|^2 |\tau_o|^2\ e^{-2kn_o K_o(p_o x - q_o z)}, \tag{29b}$$

an energy flux along the direction of the reflected wave.

21.2.5.3 The electric vector transmitted across the interface $z = 0$, Figure 21.2, has the form*

$$E_{transmitted} = (0, 1, 0)\left(\frac{\tau_1}{\tau_o}\right)\tau_o\ e^{-i\omega t}\ e^{ik(m_o p_o x + M_1 z)}. \tag{30}$$

Correspondingly, equation (27) yields

$$H_{transmitted} = (-M_1, 0, m_o p_o)\left(\frac{\tau_1}{\tau_o}\right)\tau_o\ e^{-i\omega t}\ e^{ik(m_o p_o x + M_1 z)}. \tag{30a}$$

The wave described by equations (30) and (30a) is in general inhomogeneous,** It is more convenient for many

*It can be verified easily by substitution, that vector U defined by equations (1) and (30) satisfies wave equation (4) in medium number one, provided that M_1 obeys equation (19).

**Suppose, for example, that $m_o = n_o$ but that m_1 is complex. The first medium is then non-absorbing and the second medium is absorbing. M_1 is now complex imaginary. Write $M_1 = R_e(M_1) + i\,I_m(M_1)$. Since

$$e^{ik(m_o p_o x + M_1 z)} = e^{-kI_m(M_1)z}\ e^{ik[n_o p_o x + R_e(M_1) z]}$$

the planes of equiamplitude are parallel to the interface $z = 0$ whereas the planes of equiphase are the planes $n_o p_o x + R_e(M_1) z$ = constant.

purposes to write the second exponential in equations (30) and (30a) in the expanded form,

$$e^{ik(m_o p_o x + M_1 z)} = e^{-k[n_o K_o p_o x + I_m(M_1) z]} \; e^{ik[n_o p_o x + R_e(M_1) z]} , \quad (30b)$$

wherein $R_e(M_1)$ and $I_m(M_1)$ denote, respectively, the real and imaginary parts of M_1. The Poynting vector, S, can now be found in a straightforward manner by applying equation (28) to equations (30), (30a) and (30b). The most general result is

$$S_{transmitted} = \left[n_o p_o,\ 0,\ R_e(M_1)\right] \ 2 \left|\frac{\tau_1}{\tau_o}\right|^2 \left|\tau_o\right|^2 \ e^{-2k[n_o K_o p_o x + I_m(M_1) z]} \quad (30c)$$

By comparing the three components $n_o p_o$, 0 and $R_e(M_1)$ with the arguments $n_o p_o x + R_e(M_1) z$ of the second exponential of equation (30b), one finds that $S_{transmitted}$ is an energy flow along the direction of propagation of the equiphase surfaces (wavefronts) in the second medium when the electric vector is perpendicular to the plane of incidence. Equation (30c) becomes most significant and useful when the initial medium has negligible absorption so that $K_o = 0$; for then the Poynting vector is constant in planes z = constant. Explicitly,

$$S_{transmitted} = \left[n_o p_o,\ 0,\ R_e(M_1)\right] \ 2 \left|\frac{\tau_1}{\tau_o}\right|^2 \left|\tau_o\right|^2 \ e^{-2k\, I_m(M_1) z} . \quad (30d)$$

If, also, the second medium is non-absorbing, $m_1 = n_1$ and M_1 is real. Attenuation with z does not occur because $I_m(M_1)$ is zero.

21.2.5.4 Since the x - and z - components of $S_{transmitted}$ are proportional to $n_o p_o$ and $R_e(M_1)$, respectively, the angle, i_1, between $S_{transmitted}$ and the Z-axis is given by

$$\tan i_1 = n_o p_o / R_e(M_1) , \quad (30e)$$

or

$$\sin i_1 = n_o p_o / \left[n_o^2 p_o^2 + R_e^2(M_1)\right]^{1/2} . \quad (30f)$$

As stated, this direction of the Poynting vector, S, is parallel to the direction of propagation of the wavefronts. Let

$$(n_a)_1 = \left[n_o^2 p_o^2 + R_e^2(M_1)\right]^{1/2} . \quad (31)$$

Equation (30f) now states that the law of refraction is

$$(n_a)_1 \ \sin i_1 = n_o p_o = n_o \sin i_o \quad (32)$$

in which $(n_a)_1$ is in fact the actual refractive index of medium number one, Figure 21.2. When $m_o = n_o$ and $m_1 = n_1$, $(n_a)_1 = n_1$ so that the more general law of equation (32) degenerates into the usual form known as Snell's law. Equation (32) serves to determine the direction of the Poynting vector and the "rays" in medium number one.

21.2.5.5 The manner in which the actual refractive index, $(n_a)_1$, depends upon the angle of incidence, i_o, and upon $n_1 K_1$, is described by the table in Table 21.1 for the case in which $n_o = 1$, $K_o = 0$, and $n_1 = 1.75$. $(n_a)_1$ increases with the angle of incidence and with the product $n_1 K_1$. Because nK is less than 0.02 in the usual lenses, plates, etc., the more general law of equation (32) is not of great importance to geometrical optics.

21.2.5.6 The absolute value of the vector $n_o p_o$, 0, $R_e(M_1)$ in equation (30d) is $n_o^2 p_o^2 + R_e^2(M_1)^{1/2}$. Hence equation (30d) can be written in the more useful and significant form

$$\left|S_{transmitted}\right| = 2\,(n_a)_1 \left|\frac{\tau_1}{\tau_o}\right|^2 \left|\tau_o\right|^2 \ e^{-2k\, I_m(M_1) z} . \quad (33)$$

The relations between the Fresnel coefficients, r_o/τ_o and τ_1/τ_o, and the energy reflectance and energy transmittance, respectively, can now be clarified unambiguously. First, we note from equations (26c) and (29b) that at $z = 0$,

$$\frac{\left|S_{reflected}\right|}{\left|S_{incident}\right|} = \left|\frac{r_o}{\tau_o}\right|^2 \quad \text{(energy reflectance)} , \quad (34)$$

a result that holds whether or not m_o is complex. Secondly, we note from equations (33) and (26c), that

$n_1 K_1$ \ i_o	0^o	10^o	20^o	40^o	60^o	80^o
0.01	1.750000	1.750000	1.750001	1.7500045	1.750009	1.750013
0.02	1.750000	1.750001	1.7500045	1.750018	1.750037	1.750053
0.04	1.750000	1.7500045	1.750015	1.750071	1.750148	1.750212
0.06	1.750000	1.750010	1.750041	1.750160	1.750333	1.750493
0.08	1.750000	1.7500205	1.750072	1.750284	1.750591	1.750843
0.10	1.750000	1.750028	1.750113	1.750444	1.750921	1.751314
0.50	1.750000	1.750656	1.752605	1.760028	1.770224	1.778142
1.00	1.750000	1.752131	1.758389	1.781126	1.809765	1.830155
2.00	1.750000	1.754882	1.768962	1.817250	1.872569	1.908867
4.00	1.750000	1.757218	1.7778585	1.846750	1.922468	1.970524

Table 21.1- Table of values of the actual refractive index $(n_a)_1$ of the second medium as a function of the angle of incidence, i_o, and of $n_1 K_1$ for the case in which the first medium does not absorb and has the optical constant $n_o = 1$. The optical constant $n_1 = 1.750000$.

at $z = 0$,

$$\frac{|S_{\text{transmitted}}|}{|S_{\text{incident}}|} = \frac{(n_a)_1}{n_o}\left|\frac{\tau_1}{\tau_o}\right|^2 \tag{35}$$

a result that holds when $m_o = n_o$. Consider next the conservation of energy flow across any element of area, ΔA, of the interface $z = 0$, Figure 21.3. Energy is conserved provided that

$$|S_{\text{reflected}}|\ \Delta A_r + |S_{\text{transmitted}}|\ \Delta A_t = |S_{\text{incident}}|\ \Delta A_i, \tag{36}$$

in which the elements of area are interrelated as indicated in Figure 21.3. Division of equation (36) by the right hand member produces the following important result,

$$\frac{|S_{\text{reflected}}|}{|S_{\text{incident}}|}\frac{\Delta A_r}{\Delta A_i} + \frac{|S_{\text{transmitted}}|}{|S_{\text{incident}}|}\frac{\Delta A_t}{\Delta A_i} = 1. \tag{37}$$

The first left hand member is energy reflectance from the element of area ΔA. Because $\Delta A_r = \Delta A_i$, comparison of the first left hand member of equations (37) and (34), shows that the ratio $|r_o/\tau_o|^2$ is in fact energy reflectance. The second left hand member of equation (37) is energy transmittance of the element of area ΔA. Since $\Delta A_t/\Delta A_i = \cos i_1/\cos i_o$, equations (37) and (35) show that the energy transmittance of any element ΔA of the interface, $z = 0$, is given by

$$\text{Energy transmittance} = \frac{(n_a)_1}{n_o}\left|\frac{\tau_1}{\tau_o}\right|^2 \frac{\cos i_1}{\cos i_o}, \tag{38}$$

in which $(n_a)_1$ is the actual refractive index of the last medium, and τ_1/τ_o is Fresnel's transmission coefficient for the case in which the E-vector is perpendicular to the plane of incidence.

21.2.5.7 We shall verify that equations (22) and (23) for the Fresnel coefficients are consistent with the law of conservation of energy. Upon introducing equations (34) and (35) into equation (37), one obtains the condition for conservation of energy at the interface $z = 0$ in the form

$$\left|\frac{r_o}{\tau_o}\right|^2 + \frac{(n_a)_1 \cos i_1}{n_o \cos i_o}\left|\frac{\tau_1}{\tau_o}\right|^2 = 1. \tag{39}$$

Hence one should obtain

$$\frac{|M_o - M_1|^2}{|M_o + M_1|^2} + \frac{(n_a)_1 \cos i_1}{n_o \cos i_o}\ \frac{4|M_o|^2}{|M_o + M_1|^2} = 1. \tag{40}$$

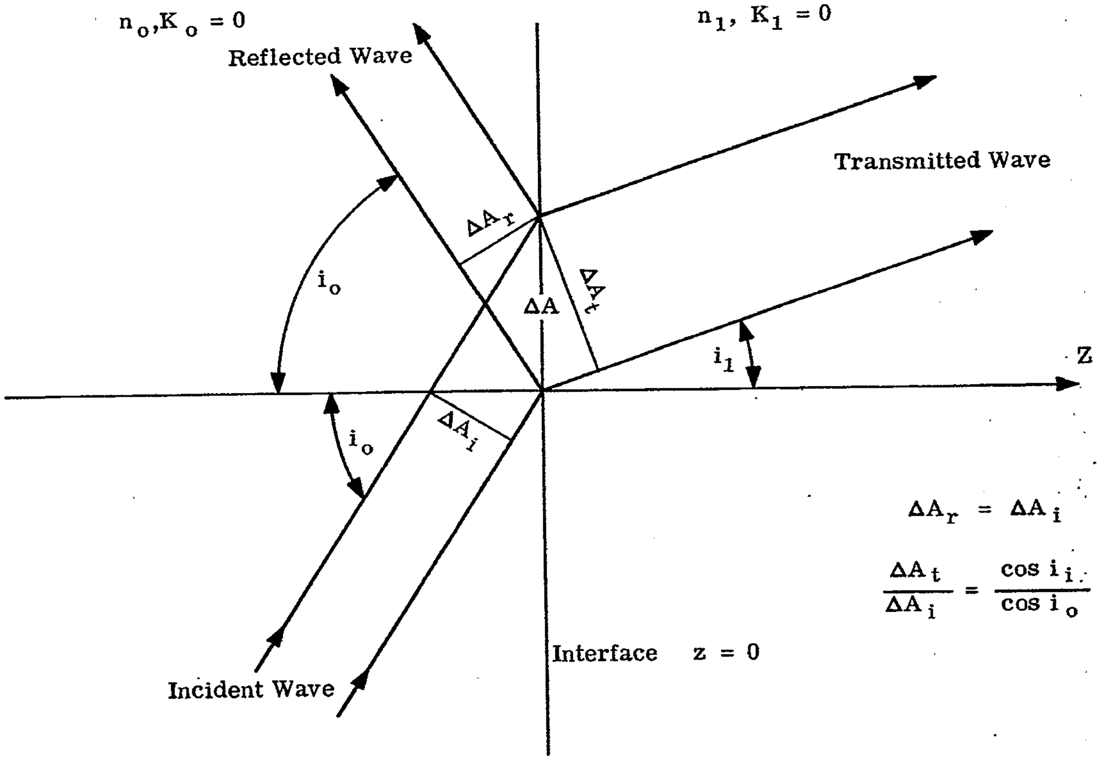

Figure 21. 3- Notation with respect to the flow of energy flux in the Poynting vectors for cases in which absorption is negligible in the initial and final media.

To avoid difficulties, let us test the case $m_o = n_o$ and $m_1 = n_1$. In this case $n_o \cos i_o = M_o$ and $(n_a)_1 \cos i_1 = M_1$. Hence equation (40) becomes the required identity.

21. 2. 6 <u>The electromagnetic field when the magnetic vector is perpendicular to the plane of incidence.</u>

21. 2. 6. 1 When the y-component of the H-vector is given, the curl relation, equation (2), determines the vector U. In the present case

$$U_x = \frac{-i}{km^2}\frac{\partial V_y}{\partial z} \quad ; \quad U_y = 0 \; ; \quad U_z = \frac{i}{km^2}\frac{\partial V_y}{\partial x} \; . \tag{41}$$

Consistent with equation (26), we suppose that the incident magnetic vector is given as the homogeneous wave

$$H_{\text{incident}} = (0, 1, 0)\; T_o \; e^{ikm_o(p_o x + q_o z)} \; . \tag{42}$$

Then from equations (41) and (42),

$$E_{\text{incident}} = (q_o , 0, -p_o)\; \frac{T_o}{m_o} \; e^{ikm_o(p_o x + q_o z)} . \tag{42a}$$

Therefore,

$$S_{\text{incident}} = (p_o , 0, q_o)\; \frac{2\, n_o}{|m_o|^2}\; |T_o|^2 \; e^{-2kn_o K_o (p_o x + q_o z)} . \tag{42b}$$

Since the Fresnel reflection coefficient is now R_o/T_o , equation (24),

$$H_{\text{reflected}} = (0, 1, 0)\left(\frac{R_o}{T_o}\right) T_o \; e^{ikm_o(p_o x - q_o z)} , \tag{43}$$

$$E_{\text{reflected}} = (-q_o, 0, -p_o)\; \frac{T_o}{m_o}\left(\frac{R_o}{T_o}\right) e^{ikm_o(p_o x - q_o z)} , \tag{43a}$$

and

$$S_{\text{reflected}} = (p_o , 0, -q_o)\; \frac{2\, n_o}{|m_o|^2}\; (T_o)^2 \left|\frac{R_o}{T_o}\right|^2 e^{-2kn_o K_o (p_o x + q_o z)} . \tag{43b}$$

The incident and reflected Poynting vectors point along the direction of propagation of the corresponding waves.

21.2.6.2 As in equation (30),

$$H_{transmitted} = (0,\ 1,\ 0) \left(\frac{T_1}{T_o}\right) T_o\ e^{ik(m_o p_o x + M_1 z)}, \tag{44}$$

therefore ,

$$E_{transmitted} = (M_1,\ 0,\ -m_o p_o)\ \frac{1}{m_1^2} \left(\frac{T_1}{T_o}\right) T_o\ e^{ik(m_o p_o x + M_1 z)}. \tag{44a}$$

When $m_o = n_o$,

$$S_{transmitted} = \left(n_o p_o\left[\frac{1}{m_1^2} + \frac{1}{\overline{m_1^2}}\right],\ 0,\ \frac{M_1}{m_1^2} + \frac{\overline{M}_1}{\overline{m_1^2}}\right) \left|\frac{T_1}{T_o}\right|^2 |T_o|^2\ e^{-2k\, I_m (M_1) z}. \tag{44b}$$

Since

$$\frac{1}{m_1^2} + \frac{1}{\overline{m_1^2}} = \frac{m_1^2 + \overline{m_1^2}}{|m_1|^4} = \frac{2 R_e (m_1^2)}{|m_1|^4}, \tag{44c}$$

the x-component of S would change sign with $R_e (m_1^2)$. The possibility $R_e (m_1^2) < 0$ would violate also equation (6) because a negative dielectric constant, ϵ_1, has no physically acceptable meaning. Thus, measured values of n and K can be valid (1a) only when

$$R_e (m^2) = n^2 (1 - K^2) = \epsilon > 0. \tag{44d}$$

With respect to the listed values of n and K for metals, one generally finds that $K^2 > 1$ so that $n^2 (1 - K^2) < 0$. Application of the theory to cases in which the listed K-values exceed unity is to be regarded, therefore, with skepticism.(1) When $R_e (m_1^2) > 0$, one finds quite directly from equation (44b) that

$$S_{transmitted} = \left(n_o p_o,\ 0,\ R_e (M_1)\right) 2\ \frac{R_e (m_1^2)}{|m_1|^4} \left|\frac{T_1}{T_o}\right|^2 |T_o|^2\ e^{-2k\, I_m (M_1) z}$$
$$+ (0,\ 0,\ 1)\ 2\ I_m (M_1)\ \frac{I_m (m_1^2)}{|m_1|^4} \left|\frac{T_1}{T_o}\right| |T_o|^2\ e^{-2k\, I_m (M_1) z}. \tag{44e}$$

The first right hand vector is parallel to the normals to the equiphase surfaces (wavefronts) in the second medium. The second right hand vector is normal to the interface and vanishes as the imaginary part of m approaches zero. As $n_1 K_1$ is increased from zero, the transmitted Poynting vector, S, departs from the direction of the wave normals and tends toward perpendicularity with the interface when the incident H-vector is perpendicular to the plane of incidence.

21.2.6.3 We shall restrict our considerations of equation (44e) to cases in which K_1 is so small that the vector with components (0, 0, 1) is negligible. The transmitted Poynting vector, S, and the wave normals are then practically parallel. As in 21.2.5, we obtain

$$|S_{transmitted}| = \frac{2 (n_a)_1}{|m_1|^4} R_e (m_1^2) \left|\frac{T_1}{T_o}\right|^2 |T_o|^2\ e^{-2k\, I_m (M_1) z}, \tag{44f}$$

in which $(n_a)_1$ is given by equation (31) and in which S points along the direction determined by equation (32). From equations (42b) and (43b), one finds that at $z = 0$

$$\frac{|S_{reflected}|}{|S_{incident}|} = \left|\frac{R_o}{T_o}\right|^2 \quad \text{(energy reflectance)}, \tag{45}$$

a result that holds whether or not m_o is complex. One finds from equations (44f) and (42b) that at the interface, $z = 0$, with the approximation $R_e (m_1^2) = n_1^2$

$$\frac{|S_{transmitted}|}{|S_{incident}|} = \frac{(n_a)_1\ n_1^2}{|m_1|^4\ n_o} \left|\frac{T_1}{T_o}\right|^2, \tag{45a}$$

a result that has been specialized to the case $m_o = n_o$. Under the restriction that the transmitted Poynting vector is practically parallel to the wave normals in the second medium, equation (37) holds again. Hence

(1) See P. Drude pg. 368 Theory of Optics, Longmans Green & Co. 1902.
(1a) For a more modern viewpoint relative to cases in which $n^2(1-K^2) = \epsilon < 0$, consult the physical interpretation of negative dielectric constants by Max Born and Emil Wolf, Principles of Optics, Pergamon Press, (1959), pp 618 and 623.

R_o/T_o ^{2} is in fact energy reflectance. The steps leading to equation (38) now yield, instead of equation (38), the result,

$$\text{Energy transmittance} = \frac{(n_a)_1 \; n_1^2 \; n_o}{|m_1|^4} \left| \frac{T_1}{T_o} \right|^2 \frac{\cos i_1}{\cos i_o} , \qquad (45b)$$

wherein T_1/T_o is Fresnel's coefficient of transmission of the H-vector when this vector is perpendicular to the plane of incidence. If $m_1 = n_1$, $(n_a)_1 = n_1$ and

$$\text{Energy transmittance} = \frac{n_o}{n_1} \left| \frac{T_1}{T_o} \right|^2 \frac{\cos i_1}{\cos i_o} , \qquad (45c)$$

equations (45b) and (45c) should be compared with equation (38) for the state of polarization in which the E-vector is perpendicular to the plane of incidence. It will be seen that the ratios of the refractive indices are the inverse of one another. The Fresnel coefficients R_o/T_o and T_1/T_o of equations (24) and (25) are consistent with the requirement that the corresponding flow of energy shall be conserved at the interface $z = 0$.

21.2.7 Summary with respect to the Fresnel coefficients.

21.2.7.1 The Fresnel coefficients r_o/τ_o and τ_1/τ_o given by equations (22) and (23), respectively, determine the reflected and transmitted E-vector when this vector is perpendicular to the plane of incidence. The corresponding incident, reflected and transmitted electromagnetic fields are given by equations (26), (29) and (30), respectively. Whereas $|r_o/\tau_o|^2$ is energy reflectance, the quantity $|\tau_1/\tau_o|^2$ is not, in general, energy transmittance. A study of the incident, reflected and transmitted Poynting vectors shows that energy transmittance across the interface between the two media is given by equation (38).

21.2.7.2 The Fresnel coefficients R_o/T_o and T_1/T_o refer to reflection and transmission of the H-vector when it is perpendicular to the plane of incidence. With this state of polarization the incident, reflected and transmitted electromagnetic fields are determined by equations (42), (43) and (44). Examination of the time-averaged Poynting vectors shows that $|R_o/\tau_o|^2$ is energy reflectance but that $|T_1/T_o|^2$ is not necessarily energy transmittance. Equation (45b) is an approximate one for computing energy transmittance. As the absorption of the second medium vanishes, equation (45b) becomes more exact and approaches the result given by equation (45c) for the case in which neither medium is absorbing.

21.2.7.3 When the first medium is non-absorbing and the second medium is absorbing, equation (31) and (32) show how the wave normals are refracted. This law of refraction is the same for both states of polarization. Whereas the time-averaged Poynting vector points along the wave normals when the E-vector is perpendicular to the plane of incidence, this Poynting vector does not always do so when the H-vector is perpendicular to the plane of incidence.

21.2.8 Normal incidence upon multilayers.

21.2.8.1 Once the Fresnel coefficients for an interface between two media have been established, it is not necessary to resubmit the numerous interfaces of a multilayer to the Maxwell equations and the boundary conditions in order to construct the theory of thin films. Instead, the following instructive method can be utilized. We consider the useful case of normal incidence in order to simplify the presentation. We may suppose, without any essential loss of generality, that the electric vector is perpendicular to the plane X, Z, Figure 21.4. In other words, we choose Z along the normal to the interfaces and take Y as the direction of vibration of the electric vector. Because the incident waves are assumed to be plane, the magnetic vector now vibrates along the X-direction.

21.2.8.2 Waves propagated to the right and left in Figure 21.4 are regarded as transmitted and reflected waves, respectively. In dealing with more than one interface, it is convenient to take τ_ν as a complex number that specifies the amplitude and phase of the "transmitted" electric vector at the right hand boundary of the ν^{th} medium or layer for the range of ν from 0 to N. As indicated in Figure 21.4, τ_{N+1} specifies the amplitude and phase of the wave in the last medium N + 1 at the left hand boundary of this medium. Similarly, r_ν is a complex number that specifies the amplitude and phase of the reflected electric vector at the right hand boundary of the ν^{th} medium or layer. A reflected wave does not exist in the last medium because it extends indefinitely along Z. The ratio r_ν/τ_ν is complex reflectance at the right hand boundary of the ν^{th} medium or layer. We define

$$\rho_\nu \equiv r_\nu/\tau_\nu \quad \text{(complex reflectance).} \qquad (46)$$

21.2.8.3 The Fresnel coefficient of reflectance from medium number 0 to medium number 1 is given by $W_1 = (M_o - M_1)/(M_o + M_1)$ as in equation (22). More generally, the Fresnel coefficient of reflectance at the interface between the $\nu - 1^{th}$ and ν^{th} layer is given by W_ν, equation (20), when the light is incident

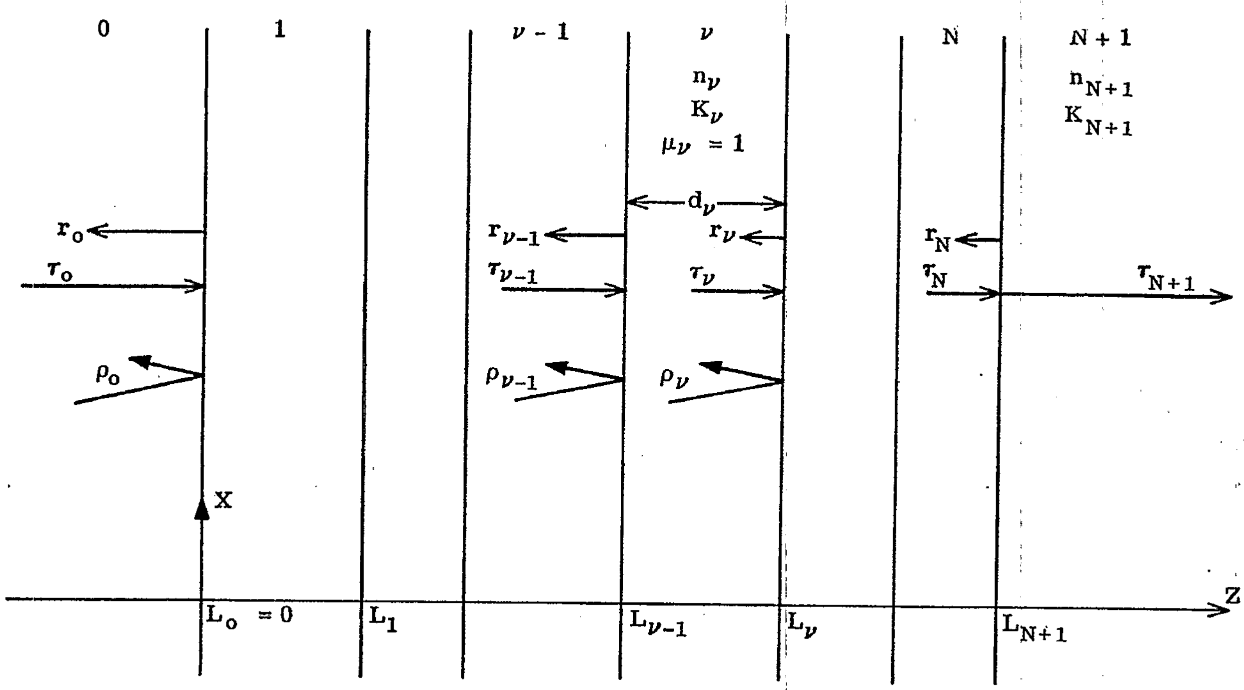

Figure 21. 4- Notation with respect to a system of N layers at normal incidence. The electric vector is taken perpendicular to the XZ plane. Incidence is from the O th medium upon layer #1. Both the initial and final medium #N + 1 are assumed to extend indefinitely along Z.

from the ν-1 th layer upon the ν th layer. When the direction of incidence is reversed, the effect is to interchange the integers ν and ν-1 in equation (20) so that Fresnel's coefficient of reflection becomes $-W_\nu$. In dealing with a system of layers it is therefore convenient to call W_ν the Fresnel coefficient of reflectance, when the electric vector is polarized to vibrate along the Y-direction. The ratios $\rho_\nu = r_\nu/\tau_\nu$ are equal to the Fresnel coefficients of reflectance only in special cases such as, for example, the single interface of Figure 21. 1.

21. 2. 8. 4 We have seen, as in equation (23), that Fresnel's coefficient of transmission across the interface from medium number 0 into medium number 1 is equal to $2M_o/(M_o + M_1)$. More generally, Fresnel's coefficient of transmission across the interface from the $(\nu - 1)$ th into the ν th medium is equal to $2M_{\nu-1}/(M_{\nu-1} + M_\nu)$. Fresnel's coefficient of transmission through this interface in the opposite direction is equal to $2M_\nu/(M_{\nu-1} + M_\nu)$. For normal incidence $M_\nu = m_\nu = n_\nu(1 + i K_\nu)$ since $p_o = 0$ in equation (19). Let

$$\beta_\nu \equiv \frac{4\pi}{\lambda} m_\nu d_\nu = \frac{4\pi}{\lambda} n_\nu d_\nu + i \frac{4\pi}{\lambda} n_\nu K_\nu d_\nu , \qquad (47)$$

where d_ν is the thickness of the ν th layer. When $K_\nu = 0$, β_ν is twice the optical path (in radians) of the ν th layer.

21. 2. 8. 5 The following equilibrium theory for the flow of the electric vector through the multilayer can now be derived in a simple manner. We fix our attention upon the equilibrium flow at the $(\nu-1)$ th and ν th layers as illustrated in Figure 21. 5. Consider the flow described by $r_{\nu-1}$. First, the flow $\tau_{\nu-1}$ is reflected at the interface at the right hand side of the $(\nu-1)$ th layer in the direction of $r_{\nu-1}$ as the flow $\tau_{\nu-1} W_\nu$. Secondly, the flow r_ν arrives at the left side of the ν th layer as the flow $r_\nu e^{i\frac{\beta_\nu}{2}}$ and then passes through the interface subject to the Fresnel coefficient of transmission $2M_\nu/(M_{\nu-1} + M_\nu)$. Hence

$$r_{\nu-1} = \tau_{\nu-1} W_\nu + r_\nu e^{i\frac{\beta_\nu}{2}} \frac{2 M_\nu}{M_{\nu-1} + M_\nu} . \qquad (48)$$

Similarly,

$$\tau_\nu = \tau_{\nu-1} \frac{2 M_{\nu-1}}{M_{\nu-1} + M_\nu} e^{i\frac{\beta_\nu}{2}} - r_\nu W_\nu e^{i\beta_\nu} . \qquad (48a)$$

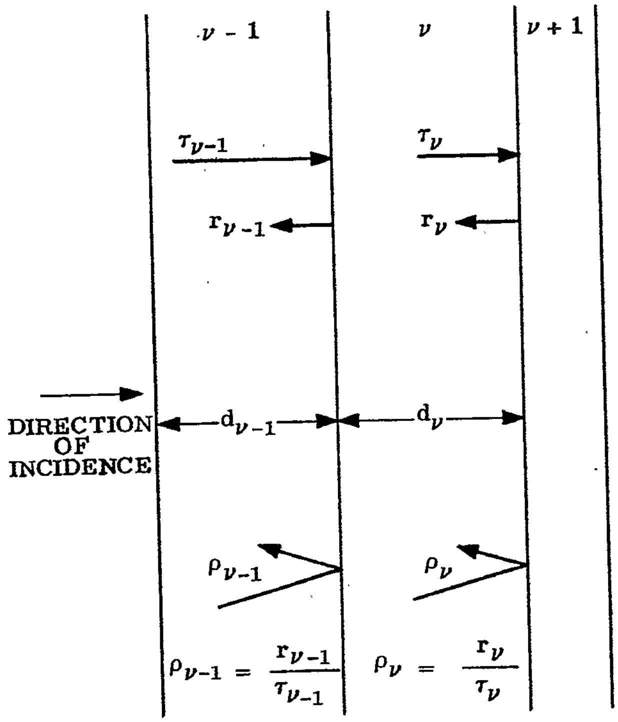

Figure 21.5- Flow conditions at the $(\nu-1)^{th}$ and ν^{th} layers.

The second right hand member has the factor $e^{i\beta\nu}$ because the flow r_ν must cross the ν^{th} layer twice. Furthermore, the flow r_ν is reflected at the left hand interface of the ν^{th} layer so that the corresponding Fresnel coefficient of reflection is $-W_\nu$. From equation (48),

$$\frac{r_{\nu-1}}{\tau_{\nu-1}} = \rho_{\nu-1} = W_\nu + \frac{r_\nu}{\tau_{\nu-1}} \; e^{i\frac{\beta\nu}{2}} \frac{2 M_\nu}{M_{\nu-1} + M_\nu} \; . \tag{48b}$$

From equation (48a),

$$\tau_{\nu-1} = (\tau_\nu + r_\nu W_\nu \; e^{i\beta\nu}) \; \frac{M_{\nu-1} + M_\nu}{2 M_{\nu-1}} \; e^{-i\frac{\beta\nu}{2}} \; . \tag{48c}$$

By eliminating $\tau_{\nu-1}$ from equation (48b) with the aid of equation (48c), one obtains

$$\rho_{\nu-1} = W_\nu + \frac{r_\nu \; e^{i\beta\nu}}{\tau_\nu + r_\nu W_\nu \; e^{i\beta\nu}} \; \frac{4 M_{\nu-1} M_\nu}{(M_{\nu-1} + M_\nu)^2} \; . \tag{48d}$$

Dividing numerator and denominator in the right hand member by τ_ν yields the result

$$\rho_{\nu-1} = W_\nu + \frac{\rho_\nu \; e^{i\beta\nu}}{1 + W_\nu \rho_\nu \; e^{i\beta\nu}} \; \frac{4 M_{\nu-1} M_\nu}{(M_{\nu-1} + M_\nu)^2} \; . \tag{48e}$$

From equation (48e),

$$\rho_{\nu-1} = \frac{\rho_\nu \; e^{i\beta\nu} \left[W_\nu^2 + 4M_{\nu-1} M_\nu / (M_{\nu-1} + M_\nu)^2 \right] + W_\nu}{1 + W_\nu \rho_\nu \; e^{i\beta\nu}} \; . \tag{48f}$$

It follows directly from the definition of W_ν, equation (20), that

$$W_\nu^2 + 4M_{\nu-1} M_\nu / (M_{\nu-1} + M_\nu)^2 = 1 \; . \tag{49}$$

Hence

$$\rho_{\nu-1} = \frac{\rho_\nu \, e^{i\beta\nu} + W_\nu}{1 + W_\nu \, \rho_\nu \, e^{i\beta\nu}} \quad , \tag{50}$$

a recursion formula that enables one to compute $\rho_{\nu-1}$ from ρ_ν and the given properties β_ν and W_ν of the νth layer. Equation (50) is the well known result that follows rigorously from the Maxwell equations and the boundary conditions appropriate to a multilayer.

21.2.8.6 The method for computing the complex reflectance ρ_o of the entire multilayer is now clear. Since $\rho_{N+1} = 0$ because no reflected wave exists in the final medium, number N + 1, it follows from equation (50) that

$$\rho_N = W_{N+1} = \frac{M_N - M_{N+1}}{M_N + M_{N+1}} \quad . \tag{51}$$

This result is to be expected because ρ_N ought to be the Fresnel coefficient of reflection at the last interface of the multilayer. Since ρ_N becomes known, one computes ρ_{N-1} from equation (50). This equation is then applied consecutively to determine $\rho_{N-2}, \rho_{N-3} \cdots\cdots\cdots \rho_o$. If ρ_o is expressed in the form

$$\rho_o = |\rho_o| \; e^{i\theta_o} \; , \tag{52}$$

then $|\rho_o|$ is amplitude reflectance and θ_o is phase change on reflection of the entire multilayer at the first interface z = 0, Figure 21.4. The phase change on reflection is phase retardation (equivalent to an increase in optical path when $\theta_o > 0$).

21.2.8.7 The complex transmittance τ_{N+1}/τ_o of the system of layers is easily derived as follows. By dividing both sides of equation (48c) by τ_ν one obtains

$$\frac{\tau_\nu}{\tau_{\nu-1}} = \frac{2\, M_{\nu-1}}{M_{\nu-1} + M_\nu} \; \frac{e^{i\frac{\beta\nu}{2}}}{1 + W_\nu \, \rho_\nu \, e^{i\beta\nu}} \quad . \tag{53}$$

Now,

$$\frac{\tau_N}{\tau_o} = \frac{\tau_1}{\tau_o} \; \frac{\tau_2}{\tau_1} \cdots\cdots\cdots \frac{\tau_N}{\tau_{N-1}} = \prod_{\nu=1}^{N} \frac{2\, M_{\nu-1}}{M_{\nu-1} + M_\nu} \; \frac{e^{i\frac{\beta\nu}{2}}}{1 + W_\nu \, \rho_\nu \, e^{i\frac{\beta\nu}{2}}} \tag{53a}$$

wherein $\prod$ denotes a product. Furthermore,

$$\frac{\tau_{N+1}}{\tau_N} = \frac{2\, M_N}{M_N + M_{N+1}} \quad , \tag{53b}$$

the Fresnel coefficient of transmission of the last interface. Therefore

$$\frac{\tau_{N+1}}{\tau_o} = \prod_{\nu=1}^{N+1} \frac{2\, M_{\nu-1}}{M_{\nu-1} + M_\nu} \; \prod_{\nu=1}^{N} \frac{e^{\frac{i\beta\nu}{2}}}{1 + W_\nu \, \rho_\nu \, e^{i\beta\nu}} \quad , \tag{54}$$

where τ_{N+1}/τ_o is the complex transmittance from the left hand side of the first interface to the right hand side of the last interface. We observe that the complex transmittance is not merely the product of the Fresnel coefficients of transmission of the N + 1 interfaces and of a factor that includes the optical path through the layers. Consider, for example, the case in which all $K_\nu = 0$ so that $M_\nu = n_\nu$ and $\beta_\nu = \frac{4\pi}{\lambda} n_\nu d_\nu$. From equation (54)

$$\frac{\tau_{N+1}}{\tau_o} = e^{i\frac{2\pi}{\lambda}\sum_{\nu=1}^{N} n_\nu d_\nu} \; \prod_{\nu=1}^{N+1} \frac{2\, n_{\nu-1}}{n_{\nu-1} + n_\nu} \; \prod_{\nu=1}^{N} \frac{1}{1 + W_\nu \, \rho_\nu \, e^{i\beta\nu}} \tag{54a}$$

The quantity $\sum_{\nu=1}^{N} n_\nu d_\nu$ is the optical path through the layer system. The product from $\nu = 1$ to $\nu = N+1$ is the product of the Fresnel coefficients of transmission of the interfaces. The product from $\nu = 1$ to $\nu = N$ is due to interreflections within the multilayer system. Since this product may not be real, the phase change introduced into the wave as it traverses the multilayer is not in general equal to the optical path through the layer. In designing a lens system of high optical quality, it finally becomes necessary to consider the phase changes introduced by transmission through the various layers or multilayers as the number of coated surfaces is increased. Consequently, equation (54) is of interest to both the optical designer and the designer of thin films. We shall see that similar equations hold for oblique incidence.

21.2.8.8 For reasons discussed at the end of Section 21.2.5, $|\rho_o|^2 = |r_o/\tau_o|^2$ is always energy reflectance of the multilayer. Because the incidence is normal, $i_o = i_{N+1} = 0$. Therefore, as in equation (38),

$$\text{Energy transmittance} = \frac{(n_a)_{N+1}}{n_o} \left|\frac{\tau_{N+1}}{\tau_o}\right|^2 \tag{55}$$

in which τ_{N+1}/τ_o is given by equation (54). Because $p_o = 0$ at normal incidence, a result similar to equation (31) gives $(n_a)_{N+1} = R_e(M_{N+1}) = n_{N+1}$. Consequently,

$$\text{Energy transmittance} = \frac{n_{N+1}}{n_o} \left|\frac{\tau_{N+1}}{\tau_o}\right|^2 \tag{55a}$$

at normal incidence when the initial medium is non-absorbing. When the initial and final media are identical and non-absorbing, the energy transmittance of the multilayer is simply $|\tau_{N+1}/\tau_o|^2$.

21.2.9 <u>Oblique incidence upon multilayers; the electric vector perpendicular to the plane of incidence.</u>

21.2.9.1 The theory of Section 21.2.8 applies again with minor changes. Let β_ν be written in the more general form

$$\beta_\nu = \frac{4\pi}{\lambda} M_\nu d_\nu , \tag{56}$$

a result that reduces to equation (47) when $p_o = 0$ (normal incidence). Then,

$$\rho_{\nu-1} = \frac{\rho_\nu e^{i\beta\nu} + W_\nu}{1 + W_\nu \rho_\nu e^{i\beta\nu}} \tag{57}$$

in which W_ν and M_ν are defined by equations (20) and (19). Because ρ_N is given by equation (51), one can compute $\rho_{N-1}, \rho_{N-2} \dots\dots\dots \rho_o$ consecutively from equation (57) to obtain the complex reflectance ρ_o of the multilayer at the left hand boundary of the first interface, $z = 0$, when the electric vector is perpendicular to the plane of incidence. Furthermore, the complex transmittance, τ_{N+1}/τ_o, from the left hand side of the first interface to the right hand side of the last interface, Figure 21.6, is given again by equation (54). The quantity, $|\rho_o|^2$, is energy reflectance of the entire multilayer. The energy transmittance is given by a result similar to that of equation (38), specifically:

$$\text{Energy transmittance} = \frac{(n_a)_{N+1} \cos i_{N+1}}{n_o \cos i_o} \left|\frac{\tau_{N+1}}{\tau_o}\right|^2 , \tag{58}$$

in which

$$(n_a)_{N+1} = \left[n_o^2 \, p_o^2 + R_e^2 (M_{N+1})\right]^{1/2} \tag{59}$$

and

$$(n_a)_{N+1} \sin i_{N+1} = n_o p_o , \tag{60}$$

wherein the wave normals (rays) make the angles i_o and i_{N+1} with Z, in the first and last medium, respectively. See Figure 21.6. It has been assumed that the medium of incidence is non-absorbing in writing equation (58). If the first and last media are identical and non-absorbing, the energy transmittance of the multilayer is given by $|\tau_{N+1}/\tau_o|^2$.

21.2.9.2 For some purposes, it is desirable to find the directions of the wave normals in the various layers. When $U = (0, U_y, 0)$ and the time factor is $e^{-i\omega t}$, the wave equation (4) is satisfied in the ν^{th} medium or layer by a transmitted wave of the form

$$U_y = \tau_\nu \, e^{ikm_o p_o x} \cdot e^{-ikM_\nu (z - L_\nu)} \tag{61}$$

provided that M_ν obeys equation (19) where τ_ν specifies the amplitude and phase of the transmitted E-vector at point $x = 0$ in the interface $z = L_\nu$. Equation (61) may be written in the form

$$U_y = \tau_\nu \, e^{-ik M_\nu L_\nu} \, e^{-k(n_o K_o p_o x + I_m (M_\nu) z)} \, e^{ik(n_o p_o x + R_e (M_\nu) z)} \tag{61a}$$

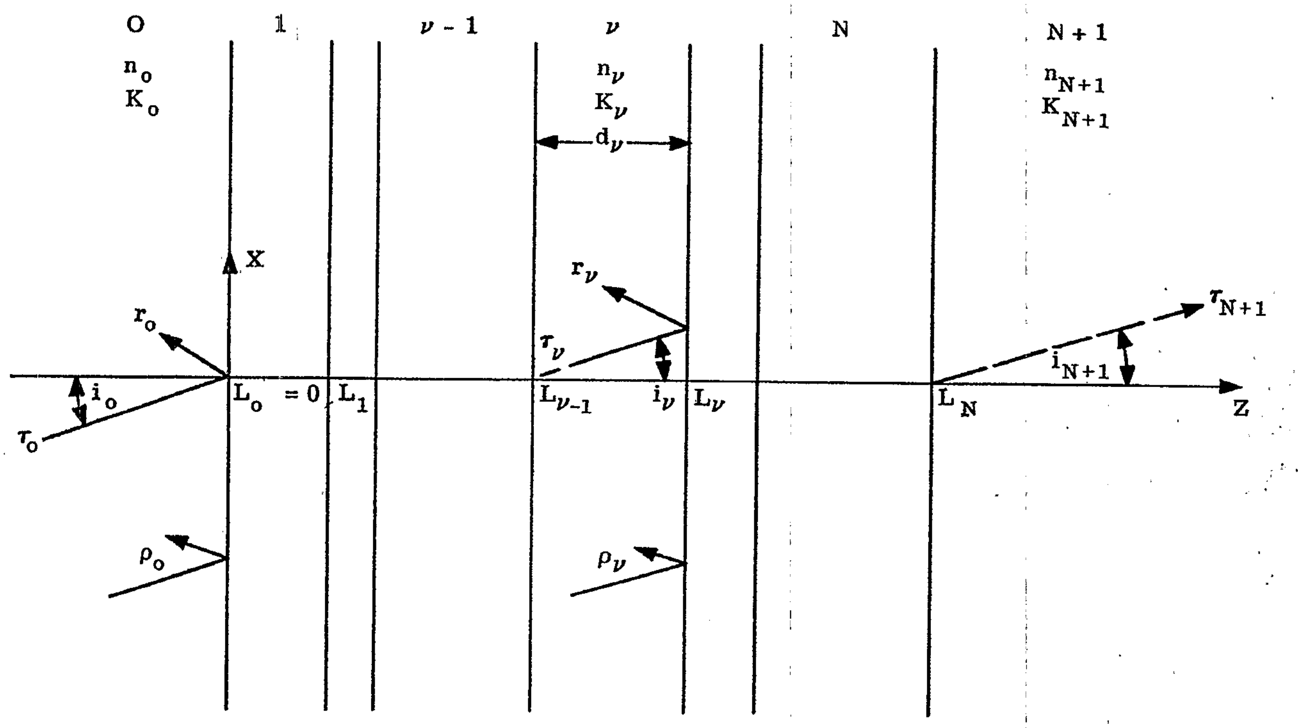

Figure 21. 6- Notation with respect to a multilayer of N films when the incidence is oblique and the E-vector is perpendicular to the plane of incidence. The angles i_ν refer to the directions of the wave normals.

Every equiphase surface in any layer ν is therefore a surface

$$n_o\, p_o\, x + R_e\,(M_\nu)\, z = \text{constant.} \tag{62}$$

If, then, the normals to the equiphase surface make the angle i_ν with Z, it follows directly* from equation (62) that

$$\sin i_\nu = n_o\, p_o / \left[n_o^2\, p_o^2 + R_e^2\,(M_\nu)\right]^{1/2} ,$$

or that

$$(n_a)_\nu \sin i_\nu = n_o \sin i_o , \tag{63}$$

wherein

$$(n_a)_\nu \equiv \left[n_o^2\, p_o^2 + R_e^2\,(M_\nu)\right]^{1/2} \tag{64}$$

is the actual refractive index of the νth layer. Hence the generalized Snell's Law described by equations (63) and (64) holds with respect to all of the media of the system. In particular, equations (59) and (60) are special cases of equations (64) and (63).

21. 2. 9. 3 When required, the electromagnetic field in any layer or medium can be computed as follows. In each medium

$$(E_\nu)_{transmitted} = (0,\ 1,\ 0)\cdot \tau_\nu\ e^{-i\omega t}\ e^{ikm_o p_o x}\ e^{ikM_\nu (z - L_\nu)} , \tag{65}$$

wherein $E_{transmitted}$ is the wave propagated to the right, Figure 21. 6. The corresponding H-vector is now computed from equations (27) and stated with the aid of equation (1). With respect to the wave propagated to the

* See discussions of the normal form of the equation of a straight line in textbooks on analytic geometry.

left in each medium

$$(E_\nu)_{\text{reflected}} = (0,\ 1,\ 0)\ r_\nu\ e^{-i\omega t}\ e^{ikm_o p_o x}\ e^{-ikM_\nu (z - L_\nu)}. \tag{66}$$

The corresponding "reflected" H-vector is computed again from equation (27) and stated with the aid of equation (1). When all ρ_ν's have been determined, each τ_ν can be computed from $\tau_{\nu-1}$ by means of equation (53). Of course, τ_o must be given or assigned. Next, all r_ν's can be calculated from the known ratios $\rho_\nu = r_\nu/\tau_\nu$ and the known values of τ_ν. The theory is therefore a complete theory for the case in which the electric vector is perpendicular to the plane X, Z of incidence, Figure 21. 6.

21. 2. 10 Oblique incidence upon multilayers; the magnetic vector perpendicular to the plane X, Z of incidence.

21. 2. 10. 1 As can be expected, the theory of oblique incidence upon multilayers with the magnetic vector perpendicular to the plane X, Z of incidence differs from the case in which the electric vector is perpendicular to the plane of incidence only in the Fresnel coefficients that become involved. Let

$$\gamma_\nu = R_\nu/T_\nu\ , \tag{67}$$

where T_ν is a complex number that specifies the amplitude and phase of the H-vector propagated to the right, Figure 21. 7, at $x = 0$ in the right hand boundary of the ν^{th} medium, and where R_ν specifies the amplitude and phase of the H-vector propagated to the left at point $x = 0$ in the right hand boundary of the ν^{th} medium, for the range of ν from $\nu = 0$ to $\nu = N$. $R_{N+1} = 0$ because no reflected wave exists in the last medium. T_{N+1} specifies the amplitude and phase of the transmitted H-vector at the left hand boundary of the $(N + 1)^{th}$ medium. Then

$$\gamma_{\nu-1} = \frac{\gamma_\nu\ e^{i\beta\nu} + F_\nu}{1 + F_\nu\ \gamma_\nu\ e^{i\beta\nu}}\ , \tag{68}$$

in which β_ν is given by equation (56) and the Fresnel coefficients of reflection F_ν are given by equation (21).

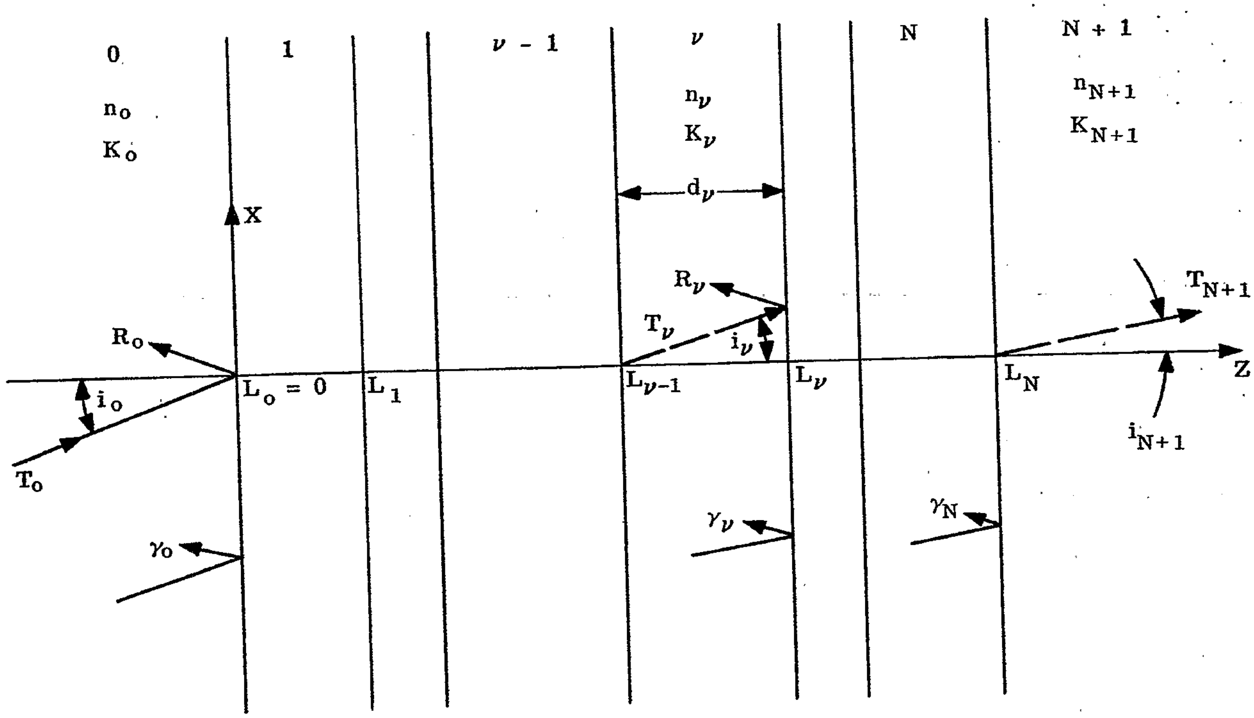

Figure 21. 7- Notation for the case in which the magnetic vector is polarized to vibrate perpendicular to the plane X, Z of incidence. The wave normals have the same angles i_ν with Z as in Figure 21. 5.

Since

$$\gamma_N = F_{N+1}, \tag{69}$$

the recursion formula (68) enables one to compute consecutively all complex reflectances, γ_ν from γ_N down to γ_o. These complex reflectances, γ_ν, are defined at the right hand boundary of the νth medium, as in the case of the complex reflectances ρ_ν.

21.2.10.2 The complex transmittance, T_{N+1}/T_o, from the left hand boundary of the first interface, z = 0, to the right hand boundary of the last interface, $z = L_N$, is given by

$$\frac{T_{N+1}}{T_o} = \prod_{\nu=1}^{N+1} \frac{2 M_{\nu-1} m_\nu^2}{M_{\nu-1} m_\nu^2 + M_\nu m_{\nu-1}^2} \prod_{\nu=1}^{N} \frac{e^{\frac{i\beta\nu}{2}}}{1 + F_\nu \gamma_\nu e^{i\beta\nu}}, \tag{70}$$

in which $2M_{\nu-1} m_\nu^2 / [M_{\nu-1} m_\nu^2 + M_\nu m_{\nu-1}^2]$ is the Fresnel coefficient of transmission in the forward direction through the interface between the $(\nu - 1)$th and νth medium. Equation (70) should be compared with equation (54).

21.2.10.3 The energy reflectance is always given in this state of polarization by $|\gamma_o|^2$. Calculation of the energy transmittance involves difficulties, as discussed at the end of Section 21.2.6, unless the time-averaged Poynting vector is practically parallel to the wave normals. If K_{N+1} is so small that the Poynting vector is sensibly parallel to the wave normals in the last medium, then, as in equation (45b),

$$\text{Energy transmittance} = \frac{n_{N+1}^2 n_o (n_a)_{N+1} \cos i_{N+1}}{|m_{N+1}|^4 \cos i_o} \left|\frac{T_{N+1}}{T_o}\right|^2, \tag{71}$$

in which the medium of incidence is assumed to be non-absorbing. If, also, the last medium is non-absorbing,

$$\text{Energy transmittance} = \frac{n_o \cos i_{N+1}}{n_1 \cos i_o} \left|\frac{T_{N+1}}{T_o}\right|^2. \tag{71a}$$

21.2.10.4 The electromagnetic field in the νth medium is determined as follows. In each medium

$$(H_\nu)_{\text{transmitted}} = (0, 1, 0)\ T_\nu\ e^{-i\omega t}\ e^{ikm_o p_o x}\ e^{ikM_\nu (z - L_\nu)}, \tag{72}$$

for the wave propagated to the right, Figure 21.7. Compare with equation (65). The corresponding E-vector is determined by equation (41) and stated with the aid of equation (1). With respect to the wave propagated to the left in the νth medium,

$$(H_\nu)_{\text{reflected}} = (0, 1, 0)\ R_\nu\ e^{-i\omega t}\ e^{ikm_o p_o x}\ e^{-ikM_\nu (z - L_\nu)}. \tag{73}$$

The corresponding reflected E-vector can be computed from equation (41) and stated with the aid of equation (1). When the method of 21.2.8 is applied to the case in which the H-vector is perpendicular to the X, Z plane, one finds instead of equation (53), that

$$\frac{T_\nu}{T_{\nu-1}} = \frac{2 M_{\nu-1} m_\nu^2}{M_{\nu-1} m_\nu^2 + M_\nu m_{\nu-1}^2} \cdot \frac{e^{i\frac{\beta\nu}{2}}}{1 + F_\nu \gamma_\nu e^{i\beta\nu}}. \tag{74}$$

If all values of γ_ν have been computed from equation (68), every T_ν from T_1 to T_N can be computed from equation (74). Since $\gamma_\nu = R_\nu/T_\nu$, every R_ν can be computed so that the coefficients R_ν and T_ν of the electromagnetic fields become known.

21.2.10.5 Comparison of equations (65) and (66) with equations (72) and (73) shows that the corresponding exponential factors are alike. Consequently, the wave normals are refracted from one layer into the next so as to make the same angle, i_ν, with Z whether the E-vector or the H-vector is perpendicular to the plane of incidence.

21.2.11 <u>An approximate method of computation based upon the complex reflectances.</u>

21.2.11.1 Comparison of equations (50), (57) and (68) shows that the recursion formula (50) for normal incidence can be used as the prototype in dealing with the complex reflectances. It is necessary only to enter the appropriate set of Fresnel coefficients, W_ν or F_ν, and to insert the value of $p_o = \sin i_o$ into β_ν as defined by equations (56) and (19). The following approximation simplifies and expedites the determination of ρ_o. The approximation becomes excellent for films that are intended to exhibit low reflectance and have relatively small Fresnel coefficients, W_ν or F_ν. Inspection of equation (50) reveals at once that when both W_ν and

ρ_ν are small, $\rho_{\nu-1}$ should be given with good approximation by

$$\rho_{\nu-1} = \rho_\nu \; e^{i\beta\nu} + W_\nu . \tag{75}$$

Thus

$$\rho_N = W_{N+1} ,$$
$$\rho_{N-1} = W_{N+1} \; e^{i\beta_N} + W_N ,$$
$$\rho_{N-2} = W_{N+1} \; e^{i(\beta_N + \beta_{N-1})} + W_N \; e^{i\beta_{N-1}} + W_{N-1} ,$$

and therefore,

$$\rho_o = \sum_{\nu=1}^{N+1} W_\nu \exp i \sum_{\mu=1}^{\nu-1} \beta\mu . \tag{76}$$

When the sum (76) is computed as the complex number

$$\rho_o = R_e(\rho_o) + i\, I_m(\rho_o) , \tag{76a}$$
$$|\rho_o|^2 = R_e^2(\rho_o) + I_m^2(\rho_o) . \tag{76b}$$

21.2.11.2 This approximate method for computing ρ_o is used mainly during rapid exploration for likely multilayer systems that do not contain absorbing layers and that are intended to produce low reflectance. When absorption is absent, one may compute $|\rho_o|^2$ directly from the sum

$$|\rho_o|^2 = \sum_{\nu=1}^{N+1} W_\nu + 2 \sum_{\mu=1}^{N} \sum_{\nu=2}^{N+1} W\mu \; W_\nu \cos(\beta_\mu + \beta_{\mu+1} + \cdots\cdots\cdots \beta_{\nu-1}), \tag{76c}$$

in which $\mu < \nu$. Comparison of $|\rho_o|_a^2$ computed from the approximate equation (76c) with the values $|\rho_o|^2$ computed from equation (50) is made in Figure 21.8 for the case of a low reflecting trilayer. The Fresnel coefficients W_2 and W_3 are quite large numerically.

21.2.11.3 The approximate method of equation (76) is the algebraic equivalent of the graphical polygon method used in early calculations on mono and bilayers by C. H. Cartwright [(2)] and others.

21.2.12 Method of admittances; E-vector perpendicular to the plane of incidence.

21.2.12.1 Because the theories of multilayers and transmission lines are similar, many investigators prefer to treat a multilayer as a transmission line. One of the earlier publications dealing with multilayers in terms of the admittances of transmission lines is due to B. Salzberg [(3)]. It will be one aim of the following presentation to unify and to show the relationships that exist between the optical method of the reflectances and the electrical method of the admittances. These two methods are equivalent and complementary. Each method possesses some advantages over the other in dealing with thin films.

21.2.12.2 From equations (65) and (66)

$$(E_{y,\nu})_{\text{transmitted}} = \tau_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{ikM_\nu(z - L_\nu)} , \tag{77}$$

and

$$(E_{y,\nu})_{\text{reflected}} = r_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{-ikM_\nu(z - L_\nu)} , \tag{77a}$$

wherein the subscript ν refers to the wave in the νth medium or layer. Correspondingly from equations (27) and (1),

$$(H_{x,\nu})_{\text{transmitted}} = -\tau_\nu M_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{ikM_\nu(z - L_\nu)} , \tag{77b}$$

and

$$(H_{x,\nu})_{\text{reflected}} = r_\nu M_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{-ikM_\nu(z - L_\nu)} . \tag{77c}$$

(2) C. H. Cartwright et al, U. S. Patent 2, 281, 474.
(3) Bernard Salzberg, J. Opt. Soc. Amer., 40, 465-470 (1950).

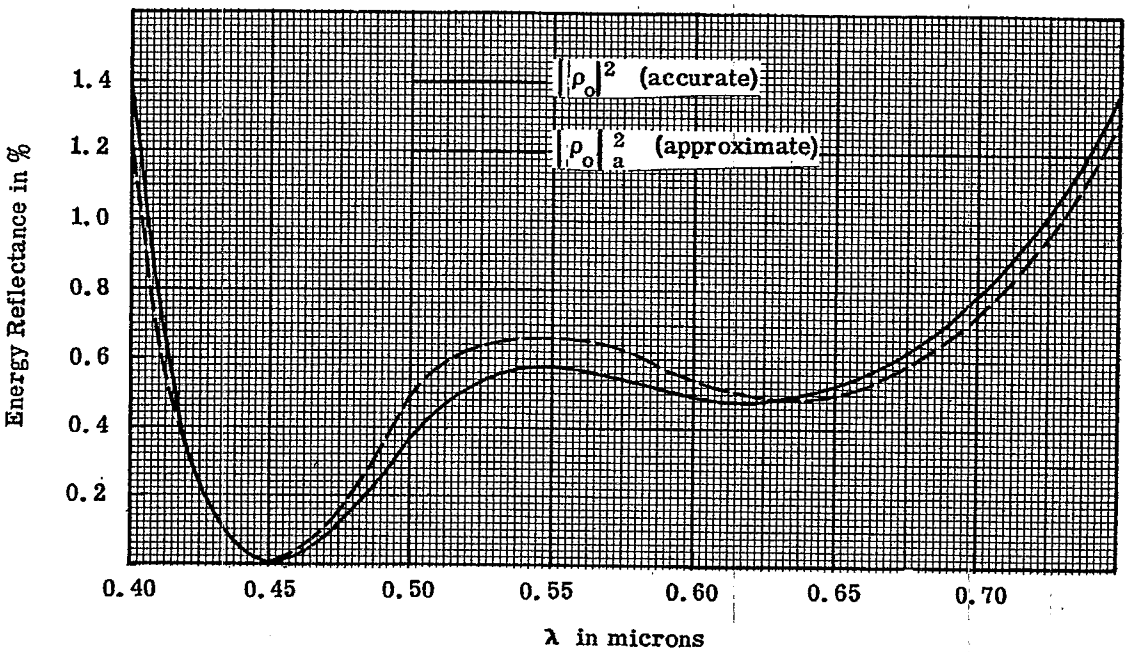

Figure 21. 8- Comparison of the computed energy reflectances $|\rho_o|^2$ and $|\rho_o|_a^2$ for a tri-layer in which $W_1 = -0.15966$; $W_2 = -0.28866$; $W_3 = 0.18765$ and $W_4 = 0.05882$. At the wavelength $\lambda_o = 0.55$ microns, $\beta_1 = 235.8^o$; $\beta_2 = 32^o$ and $\beta_3 = 360^o$.

The admittance Y_ν for the case in which the electric vector is perpendicular to the plane of incidence is defined so that

$$Y_\nu = \frac{(H_{x,\nu})_{\text{transmitted}} + (H_{x,\nu})_{\text{reflected}}}{(E_{y,\nu})_{\text{transmitted}} + (E_{y,\nu})_{\text{reflected}}} , \qquad (78)$$

evaluated at the left hand boundary, $z = L_{\nu-1}$, of the νth medium or layer. Since $L_{\nu-1} - L_\nu = -d_\nu$, equations (77) through (78) give almost directly the result

$$Y_\nu = M_\nu \frac{-\tau_\nu e^{-i\frac{\beta\nu}{2}} + r_\nu e^{i\frac{\beta\nu}{2}}}{\tau_\nu e^{-i\frac{\beta\nu}{2}} + r_\nu e^{i\frac{\beta\nu}{2}}} , \qquad (78a)$$

or

$$Y_\nu = M_\nu \frac{-1 + \rho_\nu e^{i\beta\nu}}{1 + \rho_\nu e^{i\beta\nu}} , \qquad (78b)$$

because $\rho_\nu = r_\nu/\tau_\nu$. Upon solving equation (78b) for $\rho_\nu e^{i\beta\nu}$, one obtains

$$\rho_\nu e^{i\beta\nu} = \frac{M_\nu + Y_\nu}{M_\nu - Y_\nu} . \qquad (78c)$$

Hence the admittances Y_ν can be computed from the reflectances ρ_ν and vice versa.

From equation (57),

$$\rho_o = \frac{\rho_1 e^{i\beta_1} + W_1}{1 + W_1 \rho_1 e^{i\beta_1}} . \qquad (79)$$

Upon eliminating $\rho_1 e^{i\beta_1}$ with the aid of equation (78c) and making use of the identity

$$M_\nu \frac{1 + W_\nu}{1 - W_\nu} = M_{\nu-1} , \qquad (80)$$

one finds that the complex reflectance, ρ_o, of the multilayer, and the admittance, Y_1, are connected by the equation

$$\rho_o = \frac{M_o + Y_1}{M_o - Y_1} . \tag{79a}$$

From equation (78b),

$$Y_{\nu-1} = M_{\nu-1} \frac{\rho_{\nu-1} e^{i\beta_{\nu-1}} - 1}{\rho_{\nu-1} e^{i\beta_{\nu-1}} + 1} . \tag{81}$$

Just as equation (79a) is obtained from equation (79),

$$\rho_{\nu-1} = \frac{M_{\nu-1} + Y_\nu}{M_{\nu-1} - Y_\nu} . \tag{82}$$

By eliminating $\rho_{\nu-1}$ from equation (81) with the aid of equation (82) and utilizing the identity

$$\frac{e^{-ix} - 1}{e^{-ix} + 1} = -\tanh \frac{ix}{2} = -i \tan \frac{x}{2} ,$$

one obtains without difficulty the recursion formula for $Y_{\nu-1}$ in the form

$$Y_{\nu-1} = M_{\nu-1} \frac{Y_\nu - i M_{\nu-1} \tan(\beta_{\nu-1}/2)}{M_{\nu-1} - i Y_\nu \tan(\beta_{\nu-1}/2)} . \tag{81a}$$

Because $\rho_{N+1} = 0$ (no reflected wave in the last medium), it follows from equation (78c) as a boundary condition that

$$Y_{N+1} = - M_{N+1} . \tag{81b}$$

Hence the recursion formula (81a) enables one to compute all Y_ν's from Y_N down to Y_1. With Y_1 thus determined, equation (79a) can be applied to find the complex reflectance, ρ_o, of the multilayer.

21.2.12.3 The complex transmittance, τ_{N+1}/τ_o, of the multilayer is now given awkwardly by equation (54). From equation (78c),

$$1 + W_\nu \rho_\nu e^{i\beta\nu} = 1 + W_\nu \left(\frac{M_\nu + Y_\nu}{M_\nu - Y_\nu}\right) = \frac{M_\nu (1 + W_\nu) - Y_\nu (1 - W_\nu)}{M_\nu - Y_\nu} ;$$

$$= (1 - W_\nu) \frac{M_{\nu-1} - Y_\nu}{M_\nu - Y_\nu} . \tag{82a}$$

Since

$$1 - W_\nu = 2 M_\nu / (M_{\nu-1} + M_\nu) , \tag{82b}$$

$$1 + W_\nu \rho_\nu e^{i\beta\nu} = \frac{2 M_\nu}{M_{\nu-1} + M_\nu} \frac{M_{\nu-1} - Y_\nu}{M_\nu - Y_\nu} . \tag{82c}$$

Let $1 + W_\nu \rho_\nu e^{i\beta\nu}$ be eliminated from equation (54) with the aid of equation (82c). Then

$$\frac{\tau_{N+1}}{\tau_o} = \frac{2 M_N}{M_N + M_{N+1}} \exp i\left(\sum_{\nu=1}^{N} \frac{\beta_\nu}{2}\right) \prod_{\nu=1}^{N} \left[\left(\frac{M_{\nu-1}}{M_\nu}\right) \frac{M_\nu - Y_\nu}{M_{\nu-1} - Y_\nu}\right] , \tag{83}$$

a result that should be compared with equation (54).

21.2.12.4 In summary, when the electric vector is perpendicular to the plane X, Z of incidence, the complex reflectance ρ_o and the complex transmittance τ_{N+1}/τ_o of the multilayer can be computed from equations (79a) and (83) in which all the admittances, Y_ν, from Y_N down to Y_1 are determined by the recursion formula (81a). The admittance, Y_ν, at the point of entry into the νth layer is defined by equations (78) or (78a). β_ν and M_ν are defined by equations (56) and (19), respectively. The results for normal incidence are obtained by setting $p_o = 0$ in equation (19), i.e. by setting $M_\nu = m_\nu = n_\nu (1 + i K_\nu)$.

21.2.13 Method of admittances; H-vector perpendicular to the plane of incidence.

21.2.13.1 When the H-vector is polarized to vibrate at right angles to the plane X, Z of incidence, equations (72) and (73) show that

$$(H_{y,\nu})_{\text{transmitted}} = T_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{ik M_\nu (z - L_\nu)} , \tag{84}$$

and

$$(H_{y,\nu})_{\text{reflected}} = R_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{-ik M_\nu (z - L_\nu)} . \tag{84a}$$

Correspondingly, from equations (41) and (1),

$$(E_{x,\nu})_{\text{transmitted}} = \frac{M_\nu}{m_\nu^2} \; T_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{ik M_\nu (z - L_\nu)} , \tag{84b}$$

and

$$(E_{x,\nu})_{\text{reflected}} = \frac{-M_\nu}{m_\nu^2} \; R_\nu \; e^{-i\omega t} \; e^{ikm_o p_o x} \; e^{-ik M_\nu (z - L_\nu)} . \tag{84c}$$

The admittances are designated by y_ν to distinguish them from the admittances Y_ν of equation (78) and are defined by the equation

$$y_\nu = \frac{(H_{y,\nu})_{\text{transmitted}} + (H_{y,\nu})_{\text{reflected}}}{(E_{x,\nu})_{\text{transmitted}} + (E_{x,\nu})_{\text{reflected}}} , \tag{85}$$

evaluated at the point of entry, $z = L_{\nu-1}$, of the ν^{th} layer. Substitution of equation (84) into equation (85) yields the result

$$y_\nu = \frac{m_\nu^2}{M_\nu} \; \frac{1 + \gamma_\nu \; e^{i\beta\nu}}{1 - \gamma_\nu \; e^{i\beta\nu}} , \tag{85a}$$

in which $\gamma_\nu = R_\nu / T_\nu$ as in equation (67). It follows from equation (85a) that

$$\gamma_\nu \; e^{i\beta\nu} = \frac{y_\nu - m_\nu^2/M_\nu}{y_\nu + m_\nu^2/M_\nu} . \tag{85b}$$

21.2.13.2 Corresponding to the identity (80), one finds from the definition of F_ν, equation (21), that

$$\frac{m_\nu^2}{M_\nu} \; \frac{1 - F_\nu}{1 + F_\nu} = \frac{m_{\nu-1}^2}{M_{\nu-1}} . \tag{86}$$

Let $\gamma_\nu \; e^{i\beta\nu}$ be eliminated from equation (68) with the aid of equation (85b). By utilizing the identity (86) in the result thus obtained, one finds straightforwardly that

$$\gamma_{\nu-1} = \frac{y_\nu - m_{\nu-1}^2/M_{\nu-1}}{y_\nu + m_{\nu-1}^2/M_{\nu-1}} . \tag{87}$$

In particular,

$$\gamma_o = \frac{y_1 - m_o^2/M_o}{y_1 + m_o^2/M_o} , \tag{87a}$$

where γ_o is the complex reflectance of the entire multilayer evaluated at the first interface, $z = 0$, Figure 21.7.

21.2.13.3 From equation (85a),

$$y_{\nu-1} = \frac{m_{\nu-1}^2}{M_{\nu-1}} \; \frac{1 + \gamma_{\nu-1} \; e^{i\beta_{\nu-1}}}{1 - \gamma_{\nu-1} \; e^{i\beta_{\nu-1}}} \tag{88}$$

Let $\gamma_{\nu-1}$ be eliminated from equation (88) with the aid of equation (87). As in the steps leading to equation

(81a), one finds that

$$y_{\nu-1} = \frac{m_{\nu-1}^2}{M_{\nu-1}} \left[\frac{y_\nu - i \dfrac{m_{\nu-1}^2}{M_{\nu-1}} \tan(\beta_{\nu-1}/2)}{\dfrac{m_{\nu-1}^2}{M_{\nu-1}} - i\, y_\nu \tan(\beta_{\nu-1}/2)} \right] . \quad (88a)$$

This recursion formula is similar to equation (81a). It is necessary only to replace $M_{\nu-1}$ by the ratio $m_{\nu-1}^2/M_{\nu-1}$. Since $\gamma_{N+1} = 0$, equation (85b) shows that a boundary condition on y_ν is

$$y_{N+1} = m_{N+1}^2/M_{N+1} . \quad (88b)$$

Equations (88a) and (88b) permit all values of y_ν to be computed from y_N down to y_1 from the optical properties of the multilayer system. At this point, the solution for the complex reflectance, γ_o, of the multilayer can be completed from equation (87a).

21.2.13.4 It remains to develop a more suitable formula to replace equation (70) for the complex transmittance T_{N+1}/T_o of the multilayer. From equation (85b),

$$1 + F_\nu \gamma_\nu e^{i\beta_\nu} = 1 + F_\nu \left(\frac{y_\nu - m_\nu^2/M_\nu}{y_\nu + m_\nu^2/M_\nu} \right) = \frac{y_\nu (1 + F_\nu) + \dfrac{m_\nu^2}{M_\nu}(1 - F_\nu)}{y_\nu + m_\nu^2/M_\nu}$$

$$= (1 + F_\nu) \frac{y_\nu + m_{\nu-1}^2/M_{\nu-1}}{y_\nu + m_\nu^2/M_\nu} ; \quad (89)$$

with

$$1 + F_\nu = \frac{2 m_\nu^2 M_{\nu-1}}{m_\nu^2 M_{\nu-1} + m_{\nu-1}^2 M_\nu} . \quad (89a)$$

Let $1 + F_\nu \gamma_\nu e^{i\beta\nu}$ be eliminated from equation (70) with the aid of equations (89) and (89a). Then

$$\frac{T_{N+1}}{T_o} = \frac{2 M_N m_{N+1}^2}{M_N m_{N+1}^2 + M_{N+1} m_N^2} \exp\left(i \sum_{\nu=1}^{N} \frac{\beta_\nu}{2} \right) \prod_{\nu=1}^{N} \frac{y_\nu + m_\nu^2/M_\nu}{y_\nu + m_{\nu-1}^2/M_\nu} . \quad (89b)$$

21.2.13.5 In summary: When the magnetic vector vibrates at right angles to the plane X, Z of incidence, the complex reflectance γ_o and the complex transmittance T_{N+1}/T_o of the multilayer are determined by equations (87a) and (89b) when the admittances, y_ν, from y_N to y_1 have been computed from the recursion formula (88a). The admittances, y_ν, are defined by equations (85) or (85a), and are different from the complementary admittances, Y_ν, for the state of polarization in which the electric vector is perpendicular to the plane X, Z of incidence. β_ν and M_ν are defined by equations (56) and (19). When desired, the admittances, y_ν, can be computed from the complex reflectances, γ_ν, by means of equation (85a); or the complex reflectances, γ_ν, can be computed from the admittances, y_ν, by means of equation (85b).

21.2.14 Absentee layers.

21.2.14.1 It can be shown easily from the method of admittances that non-absorbing layers behave as if they were absent at wavelengths λ for which

$$n_\nu d_\nu \cos i_\nu = \mu \frac{\lambda}{2} ; \quad \mu = 1, 2, 3, 4, \text{etc.} \quad (90)$$

We shall call such layers absentee layers. For example, at normal incidence the so called half-wave layer, the case $\mu = 1$ and $i_\nu = 0$ in equation (90), is an absentee layer. Condition (90) is satisfied when $\beta_\nu = \mu 2\pi$, i.e. when $\tan(\beta_\nu/2) = 0$ in equations (81a) and (88a). Consequently, $Y_{\nu-1} = Y_\nu$ and $y_{\nu-1} = y_\nu$ when equation (90) is satisfied. This means that the νth layer does not affect the admittance at the point of entry of the $(\nu-1)$th layer whether the E-vector or the H-vector vibrates at right angles to the plane X, Z of incidence. The νth layer does not influence the reflectance or the transmittance of the multilayer at any wavelength for which equation (90) is satisfied. In fact, the layer behaves as if $\mu = 0$, i.e. as if the thickness d_ν of the layer were zero.

21.2.14.2 Consider the behavior, with wavelength λ_μ or with β_ν, of a single, homogeneous, non-absorbing film deposited on any substrate. Because this film is an absentee layer at wavelengths λ_μ for which equation (90) is satisfied, it follows, for example, that the energy reflectances $|\rho_o|^2$ of the coated and uncoated surface should be alike at the wavelengths λ_μ. Actually, these reflectances are not quite alike at λ_μ because the film may absorb, may not be homogeneous, or may scatter appreciably.

21.2.14.3 Absentee layers are often introduced in multilayers for the purpose of altering the behavior of the multilayer at wavelengths λ that do not satisfy equation (90).

21.2.15 The Q-Method.

21.2.15.1 Comparison of the recursion formulae (57), (68), (81a) and (88a) shows that they are all awkward for the purposes of computation. With respect to equation (57), for example, $\rho_{\nu-1}$ is not linear in ρ_ν. This lack of linearity applies to the reflectances and the admittances alike. The reflectances enjoy a small advantage in that approximate expressions such as equation (75) exhibit linearity. It is one of the purposes of the Q-method[4] to circumvent the lack of linearity in the recursion formulae for the reflectances and the admittances.

21.2.15.2 The recurrence formulae connecting successive interfacial reflectances or admittances are of the form

$$A_{\nu-1} = \frac{a_\nu A_\nu + b_\nu}{g_\nu A_\nu + h_\nu} , \qquad (91)$$

in which A_ν can represent ρ_ν, γ_ν, Y_ν or y_ν. Since the denominators of (91) are not zero, let

$$f_\nu \equiv g_\nu A_\nu + h_\nu , \qquad (91a)$$

and set

$$Q_\nu \equiv \prod_{n=\nu}^{n=m+1} f_n , \qquad (91b)$$

in which the upper limit, m, of the product is an integer such that

$$A_{m+1} = 0 ; \quad A_m = 0 . \qquad (91c)$$

Let

$$\gamma_\nu = a_\nu / g_\nu . \qquad (91d)$$

Then it can be shown that Q_ν obeys the linear recursion formula

$$Q_\nu = (\alpha_{\nu+1} g_\nu + h_\nu) Q_{\nu+1} + (b_{\nu+1} - \alpha_{\nu+1} h_{\nu+1}) g_\nu Q_{\nu+2} . \qquad (91e)$$

Furthermore,

$$A_\nu = \left[\alpha_{\nu+1} Q_{\nu+1} + (b_{\nu+1} - \alpha_{\nu+1} h_{\nu+1}) Q_{\nu+2} \right] / Q_{\nu+1} . \qquad (91f)$$

21.2.15.3 Consider, for example, the application of the Q-method to the determination of the complex reflectances $\rho_{\nu-1}$ of equation (57). Thus,

$$A_{\nu-1} = \rho_{\nu-1} = \frac{\rho_\nu e^{i\beta\nu} + W_\nu}{W_\nu \rho_\nu e^{i\beta\nu} + 1} . \qquad (92)$$

Comparison of equations (91) and (92) shows that

$$a_\nu = e^{i\beta\nu} ; \quad g_\nu = W_\nu e^{i\beta\nu} ;$$

$$b_\nu = W_\nu ; \quad h_\nu = 1 . \qquad (92a)$$

Therefore,

$$\alpha_\nu = 1/W_\nu ; \qquad (92b)$$

$$f_\nu = 1 + W_\nu \rho_\nu e^{i\beta\nu} . \qquad (92c)$$

(4) H. Osterberg, J. Opt. Soc. Amer., 43, 728-732 (1953).

Because $\rho_{N+1} = 0$ in a system of multilayers containing N layers, $A_{N+1} = 0$. Hence m = N in equation (91b) so that

$$Q_\nu = \prod_{n=\nu}^{N+1} f_n = \prod_{n=\nu}^{N+1} (1 + W_n \rho_n e^{i\beta n}). \quad (92d)$$

In particular,

$$Q_{N+1} = 1 ; \quad (92e)$$

$$Q_N = 1 + W_N W_{N+1} e^{i\beta_N} ; \rho_N = W_{N+1} . \quad (92f)$$

From equations (91e), (92a) and (92b),

$$Q_\nu = Q_{\nu+1} + \left[Q_{\nu+1} + (W_{\nu+1}^2 - 1) Q_{\nu+2} \right] \frac{W_\nu e^{i\beta\nu}}{W_{\nu+1}} . \quad (92g)$$

Every Q_ν from $\nu = N + 1$ down to $\nu = 1$ is determined from equations (92e) through (92g). Since $A_\nu = \rho_\nu$, equations (91f), (92a) and (92b) now yield ρ_ν in the form

$$\rho_\nu = \frac{Q_{\nu+1} + (W_{\nu+1}^2 - 1) Q_{\nu+2}}{W_{\nu+1} Q_{\nu+1}} . \quad (92h)$$

The complex reflectance, ρ_o, of the entire multilayer is therefore given by

$$\rho_o = \frac{Q_1 + (W_1^2 - 1) Q_2}{W_1 Q_1} . \quad (93)$$

21.2.15.4 With respect to the computation of the complex transmittance, τ_{N+1}/τ_o, of the multilayer, from equation (54), we note that

$$\prod_{\nu=1}^{N} \frac{1}{1 + W_\nu \rho_\nu e^{i\beta\nu}} = f_{N+1} \prod_{\nu=1}^{N+1} \frac{1}{f_\nu} = \frac{1}{Q_1} , \quad (94)$$

because $f_{N+1} = 1$. Hence the complex transmittance, τ_{N+1}/τ_o, of the multilayer is given by the comparatively simple result,

$$\frac{\tau_{N+1}}{\tau_o} = \frac{2^{N+1}}{Q_1} \exp\left(i \sum_{\nu=1}^{N} \frac{\beta_\nu}{2}\right) \prod_{\nu=1}^{N+1} \frac{M_{\nu-1}}{M_{\nu-1} + M_\nu} , \quad (95)$$

when the electric vector is perpendicular to the plane of incidence. Since β_ν is defined by equation (56),

$$\frac{\beta_\nu}{2} = \frac{2\pi d_\nu}{\lambda} \left[R_e (M_\nu) + i \, I_m (M_\nu) \right] . \quad (95a)$$

21.2.15.5 Suppose that the system has no absorption. Then $M_\nu = n_\nu \cos i_\nu$, a real number, and $\frac{\beta_\nu}{2} = \frac{2\pi}{\lambda} n_\nu d_\nu \cos i_\nu$. It will be seen from Figure 21.6 that $\frac{\beta_\nu}{2}$ is the optical path along the rays only when the incidence is normal so that $\cos i_\nu = 1$. If the incidence is normal, and if the system has no absorption, $\sum_{\nu=1}^{N} \frac{\beta_\nu}{2}$ is physically the optical path through the multilayer. But the optical path will not be equal to the phase change suffered by the wave on passing through the multilayer system unless Q_1 is real, a condition that holds only at certain wavelengths for a given multilayer.

21.2.15.6 The phase change (retardation) suffered by the wave in passing through the multilayer is equal to arg (τ_{N+1}/τ_o). The portion determined by arg $(\bar{Q}_1)$ = arg $(1/Q_1)$ can be oscillatory with wavelength and is therefore quite dispersive. We learn that the quantity Q_1 has definite physical significance and that the Q-method is not merely a more convenient method for computing reflectance and transmittance of multilayers.

21.2.15.7 Examination of equation (92g) shows that when every β_ν is an integral multiple of the smallest value of β, each Q_ν will be a terminating exponential series of the Fourier type. Hence the powerful methods of Fourier series may be brought to bear. In the simplest case, each layer can be a quarter-wave layer at a specified wavelength. Furthermore, equation (92g) is a difference equation, consequently the methods of difference equations can often be utilized to simplify the determination of Q_1.

21.2.16 The zero condition.

21.2.16.1 One of the most important applications of single or multilayers is to reduce the energy reflectance of the coated surface. Whereas it is not always practical to attain zero reflectance, the designer of low reflecting films attempts to achieve zero reflectance at one or more wavelengths. Unfortunately, it is not possible to obtain zero reflectance over an extended range of wavelengths. Each method of attack on the design of thin films will include an analytical statement of the zero condition, i.e. the condition for obtaining zero reflectance.

21.2.16.2 With respect to the method involving the interfacial reflectances ρ_ν or γ_ν, the zero condition requires that ρ_o or $\gamma_o = 0$ as indicated in Figure 21.9. From equation (57),

$$\rho_o = \frac{\rho_1 e^{i\beta_1} + W_1}{1 + W_1 \rho_1 e^{i\beta_1}} , \tag{96}$$

and from equation (68),

$$\gamma_o = \frac{\gamma_1 e^{i\beta_1} + F_1}{1 + F_1 \gamma_1 e^{i\beta_1}} . \tag{97}$$

Accordingly, the zero condition assumes the form

$$\rho_1 e^{i\beta_1} = -W_1 , \tag{98}$$

or

$$\gamma_1 e^{i\beta_1} = -F_1 , \tag{98a}$$

depending upon whether the electric or the magnetic vector, respectively, is perpendicular to the plane of incidence. In considering normal incidence, one ordinarily chooses equation (98) and sets $i_o = 0$ in determining M_ν.

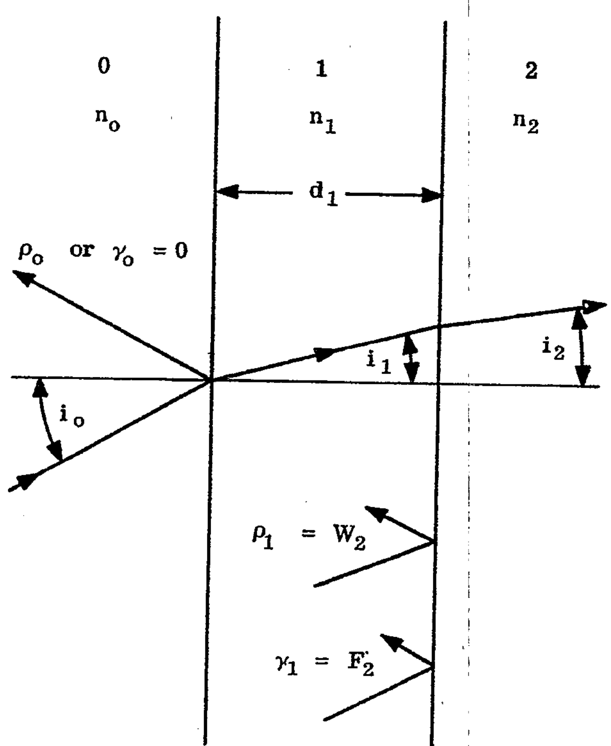

Figure 21.9- A monolayer of refractive index n_1 and thickness d_1 between two media having refractive indices n_o and n_2.

21.2.16.3 In dealing with the admittances Y_ν and y_ν, equations (79a) and (87a) show that the zero condition is

$$Y_1 = -M_o , \tag{99}$$

or

$$y_1 = m_o^2/M_o , \tag{99a}$$

according as the electric vector or the magnetic vector is perpendicular to the plane of incidence.

21.2.16.4 With respect to the Q-method, equation (93) shows that the zero condition is $Q_1 + (W_1^2 - 1) Q_2 = 0$, or

$$Q_1 = (1 - W_1^2) Q_2 , \tag{100}$$

when the electric vector is perpendicular to the plane of incidence.

21.3 ZERO REFLECTANCE FROM NON-ABSORBING MONOLAYERS AND SUBSTRATES

21.3.1 Introduction. The problem is to design a film that produces zero reflectance. The variables of the film are its refractive index, n_1, and its thickness, d_1. The usual restriction of the discussion to normal incidence will not be made because this restriction avoids too many pertinent and practical facts associated with oblique incidence. The following discussion will hinge upon the method of the complex reflectances. The complex reflectances ρ_o and γ_o are given by equations (96) and (97), respectively, with

$$\rho_1 = W_2 , \text{ and } \gamma_1 = F_2 . \tag{101}$$

Equation (101) applies to monolayers, i.e. to cases $N = 1$. From equation (96), the energy reflectance, $|\rho_o|^2$, is given by

$$|\rho_o|^2 = \frac{W_1\overline{W}_1 + W_2\overline{W}_2 \exp(i\beta_1)\overline{\exp(i\beta_1)} + W_1\overline{W}_2\overline{\exp(i\beta_1)} + \overline{W}_1 W_2 \exp(i\beta_1)}{1 + W_1\overline{W}_1 W_2\overline{W}_2 \exp(i\beta_1)\overline{\exp(i\beta_1)} + W_1 W_2 \exp(i\beta_1) + \overline{W}_1\overline{W}_2\overline{\exp(i\beta_1)}} . \tag{102}$$

Similarly, from equations (101) and (97)

$$\gamma_o^{\,2} = \frac{F_1\overline{F}_1 + F_2\overline{F}_2 \exp(i\beta_1)\overline{\exp(i\beta_1)} + F_1\overline{F}_2\overline{\exp(i\beta_1)} + \overline{F}_1 F_2 \exp(i\beta_1)}{1 + F_1\overline{F}_1 F_2\overline{F}_2 \exp(i\beta_1)\overline{\exp(i\beta_1)} + F_1 F_2 \exp(i\beta_1) + \overline{F}_1\overline{F}_2\overline{\exp(i\beta_1)}} . \tag{103}$$

21.3.2 Total internal reflection with M_2 pure imaginary. Let us consider first the class of cases in which $n_o p_o > n_2$ but in which n_1 is chosen so that $n_o p_o < n_1$. Then according to equation (19), M_1 is real but M_2 is pure imaginary. Consequently, from equations (20) and (21), W_1 and F_1 are real and W_2 and F_2 are complex such that

$$W_2\overline{W}_2 = 1 ; \quad F_2\overline{F}_2 = 1 . \tag{104}$$

Furthermore, $\beta_1 = 4\pi M_1 d_1/\lambda$ will be real. Equation (102) assumes now the simplified form

$$\rho_o^{\,2} = \frac{1 + W_1\overline{W}_1 + 2|W_1||W_2| \cos\left[\beta_1 + \arg(W_2) - \arg(W_1)\right]}{1 + W_1\overline{W}_1 + 2|W_1||W_2| \cos\left[\beta_1 + \arg(W_2) + \arg(W_1)\right]} . \tag{105}$$

Because W_1 is real, $\arg(W_1) = 0$ and $|\rho_o|^2 = 1$. Similarly, $|\gamma_o|^2 = 1$ irrespective of the value of β_1. If n_1 is chosen so that $n_1^2 > n_o^2\, p_o^2$, the film cannot alter the energy reflectance of the coated interface and the total reflection remains complete irrespective of the state of polarization of the incident beam. On the other hand, the phase change on reflection can be modified.

21.3.3 Total internal reflection with both M_1 and M_2 pure imaginary. Consider next the class of cases in which $n_o p_o$ exceeds both n_1 and n_2. Both M_1 and M_2 are then pure imaginary. Equations (20) and (21) now show that W_2 and F_2 are real but that W_1 and F_2 are complex such that

$$W_1\overline{W}_1 = 1 ; \quad F_1\overline{F}_1 = 1 . \tag{106}$$

Furthermore, since $\beta_1 = 4\pi M_1 d_1/\lambda$,

$$\exp(i\beta_1) = \overline{\exp(i\beta_1)} = \exp(-4\pi|n_o^2 p_o^2 - n_1^2|^{1/2} d_1/\lambda), \quad (107)$$

an attenuation factor that we shall designate temporarily by A. From equations (102), (106) and (107)

$$\rho_o^{\ 2} = \frac{1 + W_2^2 A^2 + 2A\,|W_1|\,W_2 \cos\left[\arg(W_1) - \arg(W_2)\right]}{1 + W_2^2 A^2 + 2A\,|W_1|\,W_2 \cos\left[\arg(W_1) + \arg(W_2)\right]} . \quad (108)$$

Since W_2 is real, $\arg(W_2) = 0, \pi, 2\pi$, etc. Hence $|\rho_o|^2 = 1$ irrespective of the thickness of the film. A similar conclusion holds when the magnetic vector is perpendicular to the plane of incidence. If i_o is chosen so that $n_o \sin i_o$ exceeds n_1 and n_2, the film cannot modify the energy reflectance of the coated interface.

21.3.4 Total internal reflectance when M_1 is pure imaginary. Total internal reflection may or may not occur when $n_o \sin i_o$ exceeds n_1 but not n_2. In this class of cases, M_1 is pure imaginary and M_2 is real. It can be seen from equations (20) and (21) that the Fresnel coefficients of reflection are complex such that

$$W_1 \overline{W}_1 = 1; \; W_2 \overline{W}_2 = 1; \; F_1 \overline{F}_1 = 1; \; F_2 \overline{F}_2 = 1. \quad (109)$$

In addition, β_1 obeys equation (107). Equation (102) assumes the form

$$|\rho_o|^2 = \frac{1 + A^2 + A(W_1 \overline{W}_2 + \overline{W}_1 W_2)}{1 + A^2 + A(W_1 W_2 + \overline{W}_1 \overline{W}_2)} , \quad (110)$$

in which $A = \exp(i\beta_1)$, a real attenuation factor. Because $A \rightarrow 0$ as $d_1 \rightarrow \infty$, $|\rho_o|^2 \rightarrow 1$ as the thickness d_1 of the film is increased. In fact, A falls very rapidly with increasing d_1. A relatively thin film can therefore produce almost total * internal reflection. One can show that, subject to equation (109),

$$W_1 \overline{W}_2 + \overline{W}_1 W_2 \leqq W_1 W_2 + \overline{W}_1 \overline{W}_2 . \quad (111)$$

Hence $|\rho_o|^2 \leqq 1$.

21.3.5 Zero reflectance; the E-vector perpendicular to the plane of incidence. Zero reflectance is possible with monolayers when $n_o \sin i_o$ is less than n_1 or n_2, i.e. when both M_1 and M_2 are real. Since $\rho_1 = W_2$ for monolayers (case $N = 1$), the zero condition of equation (98) becomes

$$W_2 e^{i\beta_1} = -W_1 . \quad (112)$$

Because W_1 and W_2 are real, equation (112) requires that $\exp(i\beta_1)$ be real. Two choices are possible

$$\beta_1 = \begin{cases} \mu\pi; \; \mu \text{ an odd integer}; & (113) \\ \nu 2\pi; \; \nu \text{ any integer}; & (113a) \end{cases}$$

$$\beta_1 = \frac{4\pi}{\lambda} n_1 d_1 \cos i_1 . \quad (113b)$$

We shall restrict our attention to the more interesting and important** choice (113). For all choices of μ, $\exp(i\beta_1) < 0$. Hence from equation (112) one must have

$$W_1 = W_2 . \quad (114)$$

Therefore from equation (20) and (114),

$$(M_o - M_1)(M_1 + M_2) = (M_o + M_1)(M_1 - M_2) ,$$

so that

$$M_1 = \sqrt{M_o M_2} . \quad (114a)$$

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* Films belonging to the class discussed in this section are often said to be films that frustrate total internal reflection. For the simple case treated here, total reflection is frustrated more thoroughly as the thickness of the film is decreased.

** The choice (113a) leads to the theory of the "soap film." Thus at normal incidence the optical path $n_1 d_1$ is an integral number of half-wavelengths.

At normal incidence equations (113), (113b) and (114a) reduce to the pair of well known and independent conditions

$$n_1 d_1 = \mu \frac{\lambda}{4} , \tag{115}$$

where μ = odd, and $n_1 = \sqrt{n_o n_2}$.

At oblique incidence, it is often advantageous to assemble the independent conditions (113) and (114a) in the more explicit form

$$n_1 d_1 \cos i_1 = \mu \frac{\lambda}{4} \ ; \ \mu \text{ odd} ;$$

and (116)

$$\frac{n_1}{\sqrt{n_o n_2}} = \frac{\sqrt{\cos i_o \ \cos i_2}}{\cos i_1} ,$$

in which, from Snell's law, $n_o \sin i_o = n_1 \sin i_1 = n_2 \sin i_2$. The choice of n_1 and d_2 for zero reflectance depends therefore upon the angle, i_o, of incidence. The first condition of equation (115) is often called the quarter wave condition. The effective "interference path," $n_1 d_1 \cos i_1$, is equal to the optical path, $n_1 d_1$, of the film at normal incidence. The interference path, $n_1 d_1 \cos i_1$, always obeys the quarter wave condition.

21. 3. 6 Zero reflectance; the H-vector perpendicular to the plane of incidence. As in Section 21. 3. 5, zero reflectance is possible when both M_1 and M_2 are real; but, as we shall see, the zero condition corresponding to (114a) is significantly different. Since $\gamma_1 = F_2$ for monolayers, the zero condition (98a) assumes the form

$$F_2 \ e^{i\beta_1} = - F_1 . \tag{117}$$

With non-absorbing media, it will become clear from equation (21) that the Fresnel coefficients, F_1 and F_2, of reflection must be real when M_1 and M_2 are real. Conditions (113) and (113a) apply again. Corresponding to the choice (113), equation (117) requires that

$$F_1 = F_2 . \tag{118}$$

From equations (118) and (21) one obtains straightforwardly,

$$\frac{M_1}{\sqrt{M_o M_2}} = \frac{n_1^2}{n_o n_2} , \tag{118a}$$

a condition that should be compared with equation (114a). Upon introducing $M_1 = n_1 \cos i_1$, $M_2 = n_2 \cos i_2$ and $M_o = n_o \sin i_o$, one finds instead of equation (116) that

$$\frac{n_1}{\sqrt{n_o n_2}} = \frac{\cos i_1}{\sqrt{\cos i_o \ \cos i_2}} . \tag{119}$$

21. 3. 7 Summary.

21. 3. 7. 1 We learn that a film that will produce zero reflectance at oblique incidence when the E-vector is perpendicular to the plane of incidence cannot be expected to produce zero reflectance when the H-vector is perpendicular to the plane of incidence. This conclusion could have been expected; for when the magnetic vector is perpendicular to the plane of incidence, the reflectance is automatically zero at Brewster's angle of incidence without the use of a film whereas,the reflectance is not zero when the electric vector is perpendicular to the plane of incidence. We conclude also that a monolayer cannot in principle produce strictly zero energy reflectance with unpolarized, incident light at other than normal incidence.

21. 3. 7. 2 The reflectance method can be applied in a systematic manner to design bilayers, trilayers, etc. that produce zero reflectance at one or more wavelengths. Except with certain simplified and restricted combinations, the details of the analysis become exceedingly tedious as the number, N, of layers is increased beyond N = 3.

21.4 MATRIX METHODS

21.4.1 Introduction. Matrix methods possess distinct advantages for computation with desk or automatic calculators. For example, the nonlinear recursion formulae that relate successive interfacial reflectances are avoided. We shall construct matrix methods for computing the complex amplitudes, τ_ν and r_ν, for for cases in which the electric vector is perpendicular to the plane of incidence, and for computing the complex amplitudes, T_ν and R_ν, for cases in which the magnetic vector is perpendicular to the plane of incidence. One may treat other states of polarization by splitting the field into two parts, in one of which the E-vector is perpendicular to the plane of incidence, and in the other of which the magnetic vector is perpendicular to the plane of incidence. The complex amplitudes, τ_ν and r_ν, retain the same physical significance as described in Sections 21.2.8 and 21.2.9. The complex amplitudes, T_ν and R_ν, retain the same significance as in Section 21.2.10.

21.4.2 Matrix methods; the E-vector perpendicular to the plane of incidence.

21.4.2.1 Equations (65) and (66) describe the electric vector propagated to the right and left, respectively, in the ν^{th} layer or medium, Figure 21.6. The electric vector has only the y-component. Thus,

$$(E_{y,\nu})_{\text{transmitted}} = \tau_\nu\, e^{-i\omega t}\, e^{ikm_o p_o x}\, e^{ikM_\nu (z - L_\nu)},$$
$$(E_{y,\nu})_{\text{reflected}} = r_\nu\, e^{-i\omega t}\, e^{ikm_o p_o x}\, e^{-ikM_\nu (z - L_\nu)}. \tag{120}$$

From equation (27) the corresponding tangential component $H_{x,\nu}$ of the magnetic vector is determined by

$$H_{x,\nu} = \frac{i}{k}\frac{\partial E_{y,\nu}}{\partial z}. \tag{121}$$

Hence,

$$(H_{x,\nu})_{\text{transmitted}} = -M_\nu\,(E_{y,\nu})_{\text{transmitted}},$$
$$(H_{x,\nu})_{\text{reflected}} = M_\nu\,(E_{y,\nu})_{\text{reflected}}. \tag{122}$$

Let the total tangential components in the νth layer or medium be denoted by $H_{T,\nu}$ and $E_{T,\nu}$. Then, by definition,

$$E_{T,\nu} = (E_{y,\nu})_{\text{transmitted}} + (E_{y,\nu})_{\text{reflected}},$$
$$H_{T,\nu} = (H_{x,\nu})_{\text{transmitted}} + (H_{x,\nu})_{\text{reflected}}. \tag{123}$$

The total tangential components $E_{T,\nu}$ and $H_{T,\nu}$ are continuous across every interface of the multilayer. From equations (120) and (122), $E_{T,\nu}$ and $H_{T,\nu}$ are continuous across the interface $z = L_{\nu-1}$ provided that

$$r_{\nu-1} + \tau_{\nu-1} = r_\nu\, e^{i\frac{\beta\nu}{2}} + \tau_\nu\, e^{-i\frac{\beta\nu}{2}}, \tag{124}$$
$$r_{\nu-1} - \tau_{\nu-1} = \frac{M_\nu}{M_{\nu-1}}\left[r_\nu\, e^{i\frac{\beta\nu}{2}} - \tau_\nu\, e^{-i\frac{\beta\nu}{2}}\right], \tag{124a}$$

since $k\,M_\nu\,(L_\nu - L_{\nu-1}) = k\,M_\nu\,d_\nu = \beta_\nu/2$. Therefore,

$$2r_{\nu-1} = \frac{r_\nu}{M_{\nu-1}}\, e^{i\frac{\beta\nu}{2}}(M_{\nu-1} + M_\nu) + \frac{\tau_\nu}{M_{\nu-1}}\, e^{-i\frac{\beta\nu}{2}}(M_{\nu-1} - M_\nu), \tag{125}$$

$$2\tau_{\nu-1} = \frac{r_\nu}{M_{\nu-1}}\, e^{i\frac{\beta\nu}{2}}(M_{\nu-1} - M_\nu) + \frac{\tau_\nu}{M_{\nu-1}}\, e^{-i\frac{\beta\nu}{2}}(M_{\nu-1} + M_\nu). \tag{125a}$$

It may be noted that division of equations (125) leads to the result of equation (57).

21.4.2.2 The following matrix algebra is all that is needed in executing the matrix method. A matrix describes a linear transformation from one pair of variables x_1, y_1 to a second pair x_2, y_2. Thus the linear transformation

$$x_2 = a_{11}x_1 + a_{12}y_1,$$
$$y_2 = a_{21}x_1 + a_{22}y_1, \tag{126}$$

is written in matrix notation as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} . \tag{127}$$

Suppose that a further transformation to the variables x_3, y_3 is given by

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} . \tag{128}$$

One can verify by eliminating x_2, y_2 that the matrix product of equation (128) is a matrix such that

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} a_{11} + b_{12} a_{21} & b_{11} a_{12} + b_{12} a_{22} \\ b_{21} a_{11} + b_{22} a_{21} & b_{21} a_{12} + b_{22} a_{22} \end{bmatrix} . \tag{129}$$

Equation (129) gives the rule for multiplication. In performing the multiplication $[b_{kl}] \times [a_{mn}]$, to obtain the element in row i and column j of the product matrix take the scalar product of the i^{th} row of matrix b, and the j^{th} column of matrix a. The continued product of any number of matrices can be performed by repeating the rule. Multiplication is not commutative, i.e. $[b_{kl}] \times [a_{mn}] \neq [a_{mn}] \times [b_{kl}]$.

21.4.2.3 Returning to equations (125), we observe that in matrix notation

$$\begin{bmatrix} r_{\nu-1} \\ \tau_{\nu-1} \end{bmatrix} = \frac{\mathcal{M}_\nu}{2 M_{\nu-1}} \begin{bmatrix} r_\nu \\ \tau_\nu \end{bmatrix} \tag{130}$$

where $\mathcal{M}_\nu$ denotes the square matrix

$$\mathcal{M}_\nu = \begin{bmatrix} (M_{\nu-1} + M_\nu)\, e^{i\frac{\beta\nu}{2}} & (M_{\nu-1} - M_\nu)\, e^{-i\frac{\beta\nu}{2}} \\ (M_{\nu-1} - M_\nu)\, e^{i\frac{\beta\nu}{2}} & (M_{\nu-1} + M_\nu)\, e^{-i\frac{\beta\nu}{2}} \end{bmatrix} . \tag{130a}$$

Therefore

$$\begin{bmatrix} r_{\nu-1} \\ \tau_{\nu-1} \end{bmatrix} = \frac{\mathcal{M}_\nu}{2 M_{\nu-1}} \frac{\mathcal{M}_{\nu+1}}{2 M_\nu} \begin{bmatrix} r_{\nu+1} \\ \tau_{\nu+1} \end{bmatrix} , \tag{131}$$

whence,

$$\begin{bmatrix} r_{\nu-1} \\ \tau_{\nu-1} \end{bmatrix} = \frac{1}{2^{N-\nu+1}} \prod_{j=\nu-1}^{N-1} \frac{1}{M_j} \prod_{j=\nu}^{N-1} \mathcal{M}_j \begin{bmatrix} r_N \\ \tau_N \end{bmatrix} , \tag{131a}$$

and

$$\begin{bmatrix} r_o \\ \tau_o \end{bmatrix} = \frac{1}{2^N} \prod_{j=0}^{N-1} \frac{1}{M_j} \prod_{j=1}^{N} \mathcal{M}_j \begin{bmatrix} r_N \\ \tau_N \end{bmatrix} . \tag{131b}$$

But r_N / τ_N is the Fresnel coefficient of reflection at the last interface, so that

$$r_N = \tau_N W_{N+1} = \tau_N \frac{M_N - M_{N+1}}{M_N + M_{N+1}} . \tag{132}$$

Furthermore, τ_{N+1}/τ_N is the Fresnel coefficient of transmission of the last interface. Hence,

$$\tau_N = \frac{M_N + M_{N+1}}{2 M_N} \tau_{N+1} ; \tag{132a}$$

$$r_N = \frac{M_N - M_{N+1}}{2 M_N} \tau_{N+1} ; \tag{132b}$$

$$\begin{bmatrix} r_N \\ \tau_N \end{bmatrix} = \frac{1}{2 M_N} \begin{bmatrix} (M_N - M_{N+1})\, \tau_{N+1} \\ (M_N + M_{N+1})\, \tau_{N+1} \end{bmatrix} = \frac{\tau_{N+1}}{2 M_N} \begin{bmatrix} (M_N - M_{N+1}) \\ (M_N + M_{N+1}) \end{bmatrix} . \tag{132c}$$

Finally, from equations (131b) and (132c), one obtains

$$\begin{bmatrix} r_o \\ \tau_o \end{bmatrix} = \frac{\tau_{N+1}}{2^{N+1}} \prod_{j=0}^{N} \frac{1}{M_j} \prod_{j=1}^{N} \mathfrak{M}_j \begin{bmatrix} (M_N - M_{N+1}) \\ (M_N + M_{N+1}) \end{bmatrix} . \tag{133}$$

In this equation the unknowns are usually r_o and τ_{N+1}. τ_o is a complex number that specifies the amplitude and phase of the incident electric vector at $x = 0$ at the left hand side of the first interface $z = 0$, Figure 21.6. One may set $\tau_o = 1$. r_o specifies the amplitude and phase of the reflected vector at $x = 0$ at the left hand side of the first interface of the multilayer. τ_{N+1} specifies the amplitude and phase of the transmitted electric vector at $x = 0$ at the right hand side of the last interface $z = L_N$, Figure 21.6. N is the number of layers.

21.4.2.4 One important advantage of this matrix method is that it enables the exploration of the effects of changing the thickness and refractive index of the ν^{th} layer without recomputing the entire matrix product beyond the $(\nu-1)^{th}$ layer. Changing thickness of the νth layer alters only the matrix $\mathfrak{M}_\nu$. Because

$$\prod_{j=1}^{N} \mathfrak{M}_j = \prod_{j=1}^{\nu-1} \mathfrak{M}_j \times \mathfrak{M}_\nu \times \prod_{j=\nu+1}^{N} \mathfrak{M}_j , \tag{134}$$

the first and third matrix products in the right hand member can be computed as matrices that remain fixed during the exploration of the effect of changing thickness. Changing refractive index of the ν^{th} layer alters both $\mathfrak{M}_\nu$ and $\mathfrak{M}_{\nu+1}$. In this case one utilizes instead of equation (134),

$$\prod_{j=1}^{N} \mathfrak{M}_j = \prod_{j=1}^{\nu-1} \mathfrak{M}_j \times \mathfrak{M}_\nu \mathfrak{M}_{\nu+1} \times \prod_{j=\nu+2}^{N} \mathfrak{M}_j . \tag{135}$$

This case illustrates also the manipulation of sub-products of matrices where these sub-products may remain fixed or may be varied. A particular fixed sub-product may be shifted to different positions in the complete matrix product -- corresponding to shifting a group of layers in the multilayer. The use of the matrix method for designing multilayers with periodic structure has been discussed by W. Weinstein.(5)

21.4.2.5 A further interpretation of the matrices $\mathfrak{M}_\nu$ is appropriate. It is not a trivial fact that the matrix $\mathfrak{M}_\nu$ of equation (130a) can be expressed as the matrix product

$$\mathfrak{M}_\nu = \begin{bmatrix} M_{\nu-1} + M_\nu & M_{\nu-1} - M_\nu \\ M_{\nu-1} - M_\nu & M_{\nu-1} + M_\nu \end{bmatrix} \begin{bmatrix} e^{i\frac{\beta\nu}{2}} & 0 \\ 0 & e^{-i\frac{\beta\nu}{2}} \end{bmatrix} , \tag{136}$$

$$= \mathcal{S}_\nu \mathcal{T}_\nu \tag{136a}$$

wherein $\mathcal{S}_\nu$ is the first right hand matrix in equation (136) and $\mathcal{T}_\nu$ is the second right hand matrix. $\mathcal{S}_\nu$ and $\mathcal{T}_\nu$ are matrices that depend, respectively, upon the optical constants, M_ν, and the interference path, β_ν, of the νth layer. With reference to equation (133), one may write when desired

$$\prod_{j=1}^{N} \mathfrak{M}_j = \prod_{j=1}^{N} \mathcal{S}_j \mathcal{T}_j \tag{136b}$$

21.4.2.6 Let us now consider solution in terms of the amplitude-sums. With respect to equations (124), let

$$A_\nu = r_\nu + \tau_\nu ; \qquad B_\nu = M_\nu (r_\nu - \tau_\nu) . \tag{137}$$

Then

$$r_\nu = \frac{1}{2}\left(A_\nu + \frac{B_\nu}{M_\nu} \right) ; \qquad \tau_\nu = \frac{1}{2}\left(A_\nu - \frac{B_\nu}{M_\nu} \right) . \tag{137a}$$

By eliminating r_ν and τ_ν from equations (124) and (124a) with the aid of equation (137a), one finds that

$$A_{\nu-1} = A_\nu \cos(\beta_\nu/2) + B_\nu \frac{i \sin(\beta_\nu/2)}{M_\nu} ;$$
$$B_{\nu-1} = A_\nu \, i M_\nu \sin(\beta_\nu/2) + B_\nu \cos(\beta_\nu/2) . \tag{137b}$$

(5) Walter Weinstein, Vacuum, 4, 3-18 (1954).

Hence the amplitude-sums, A_ν and B_ν, obey the relation

$$\begin{bmatrix} A_{\nu-1} \\ B_{\nu-1} \end{bmatrix} = \begin{bmatrix} \cos(\beta_\nu/2) & \dfrac{i\sin(\beta_\nu/2)}{M_\nu} \\ i M_\nu \sin(\beta_\nu/2) & \cos(\beta_\nu/2) \end{bmatrix} \begin{bmatrix} A_\nu \\ B_\nu \end{bmatrix}. \tag{137c}$$

From equation (132)

$$A_N = r_N + \tau_N = \tau_{N+1}; \qquad B_N = M_N (r_N - \tau_N) = -M_{N+1}\,\tau_{N+1}. \tag{137d}$$

Therefore

$$\begin{bmatrix} A_o \\ B_o \end{bmatrix} = \tau_{N+1} \prod_{\nu=1}^{N} \begin{bmatrix} \cos(\beta_\nu/2) & \dfrac{i\sin(\beta_\nu/2)}{M_\nu} \\ i M_\nu \sin(\beta_\nu/2) & \cos(\beta_\nu/2) \end{bmatrix} \begin{bmatrix} 1 \\ -M_{N+1} \end{bmatrix}. \tag{137e}$$

The method of computation based upon the use of equations (137e) and (137a) is especially advantageous in dealing with non-absorbing multilayers and substrates; for then all M_ν and β_ν are real. Computation of the matrix product of equation (137e) becomes relatively simple. It should be noted that the determinant of each matrix is unity. As pointed out by W. Weinstein,(6) the determinant of the product of matrices is the product of their determinants. Therefore the determinant of products of these matrices is unity -- a valuable fact for the purpose of checking calculations. The complex amplitudes, r_o and τ_{N+1}, are computed at the end, with τ_o assigned a convenient value such as unity. The reflectance, $\rho_o = r_o/\tau_o$, and the transmittance, τ_{N+1}/τ_o, become known.

21.4.3 Matrix methods; the H-vector perpendicular to the plane of incidence.

21.4.3.1 When the magnetic vector is perpendicular to the plane of incidence, the tangential components of E and H are given by equations (84). As in equation (123), we form the total tangential components consisting of the transmitted and reflected waves. Application of the continuity condition for the total tangential components of E and H at the interface $z = L_{\nu-1}$ leads, as in equation (124), to the result

$$R_{\nu-1} + T_{\nu-1} = R_\nu\, e^{i\frac{\beta\nu}{2}} + T_\nu\, e^{-i\frac{\beta\nu}{2}}; \tag{138}$$

$$R_{\nu-1} - T_{\nu-1} = \frac{M_\nu\, m_{\nu-1}^2}{M_{\nu-1}\, m_\nu^2}\left[R_\nu\, e^{i\frac{\beta\nu}{2}} - T_\nu\, e^{-i\frac{\beta\nu}{2}}\right]. \tag{138a}$$

Therefore

$$2R_{\nu-1} = \frac{R_\nu\, e^{i\frac{\beta\nu}{2}}}{M_{\nu-1}\, m_\nu^2}\left(M_{\nu-1}\, m_\nu^2 + M_\nu\, m_{\nu-1}^2\right) + \frac{T_\nu\, e^{-i\frac{\beta\nu}{2}}}{M_{\nu-1}\, m_\nu^2}\left(M_{\nu-1}\, m_\nu^2 - M_\nu\, m_{\nu-1}^2\right) \tag{139}$$

$$2T_{\nu-1} = \frac{R_\nu\, e^{i\frac{\beta\nu}{2}}}{M_{\nu-1}\, m_\nu^2}\left(M_{\nu-1}\, m_\nu^2 - M_\nu\, m_{\nu-1}^2\right) + \frac{T_\nu\, e^{-i\frac{\beta\nu}{2}}}{M_{\nu-1}\, m_\nu^2}\left(M_{\nu-1}\, m_\nu^2 + M_\nu\, m_{\nu-1}^2\right). \tag{139a}$$

Hence,

$$\begin{bmatrix} R_{\nu-1} \\ T_{\nu-1} \end{bmatrix} = \frac{\mathcal{M}_\nu}{2M_{\nu-1}\, m_\nu^2}\begin{bmatrix} R_\nu \\ T_\nu \end{bmatrix}, \tag{140}$$

in which $\mathcal{M}_\nu$ is the matrix

$$\mathcal{M}_\nu = \begin{bmatrix} \left(M_{\nu-1}\, m_\nu^2 + M_\nu\, m_{\nu-1}^2\right) e^{i\frac{\beta\nu}{2}} & \left(M_{\nu-1}\, m_\nu^2 - M_\nu\, m_{\nu-1}^2\right) e^{-i\frac{\beta\nu}{2}} \\ \left(M_{\nu-1}\, m_\nu^2 - M_\nu\, m_{\nu-1}^2\right) e^{i\frac{\beta\nu}{2}} & \left(M_{\nu-1}\, m_\nu^2 + M_\nu\, m_{\nu-1}^2\right) e^{-i\frac{\beta\nu}{2}} \end{bmatrix}. \tag{140a}$$

(6) ibid p 8

Therefore

$$\begin{bmatrix} R_o \\ T_o \end{bmatrix} = \frac{1}{2^N} \prod_{j=0}^{N-1} \frac{1}{M_j\, m_{j+1}^2} \prod_{j=1}^{N} \mathfrak{M}_j \begin{bmatrix} R_N \\ T_N \end{bmatrix}, \tag{140b}$$

however,

$$T_N = \frac{m_{N+1}^2 M_N + m_N^2 M_{N+1}}{2\, M_N\, m_{N+1}^2}\, T_{N+1}, \tag{140c}$$

and

$$R_N = F_N\, T_N = \frac{m_{N+1}^2 M_N - m_N^2 M_{N+1}}{2\, M_N\, m_{N+1}^2}\, T_{N+1}, \tag{140d}$$

with F_N given by equation (21). Hence,

$$\begin{bmatrix} R_o \\ T_o \end{bmatrix} = \frac{T_{N+1}}{2^{N+1}} \prod_{j=0}^{N} \frac{1}{M_j\, m_{j+1}^2} \prod_{j=1}^{N} \mathfrak{M}_j \begin{bmatrix} m_{N+1}^2 M_N - m_N^2 M_{N+1} \\ m_{N+1}^2 M_N + m_N^2 M_{N+1} \end{bmatrix}, \tag{141}$$

in which the matrices $\mathfrak{M}_j$ are given by equation (140a). The complex amplitudes R_ν and T_ν retain the same physical significance as in 21.2.10.

21.4.3.2 Equation (141) serves to determine T_{N+1} from T_o and R_o from T_o and T_{N+1}. In most problems, one may set $T_o = 1$. Finally, the complex reflectance γ_o is computed from its definition $\gamma_o = R_o/T_o$. It will be observed that equations (133) and (141) determine the complex transmittances, τ_{N+1} and T_{N+1}, of the multilayer quite directly without necessitating additional multiplications such as those of equation (54) or (70).

21.4.3.3 The solution in terms of the amplitude sums can be obtained as follows. Let

$$C_\nu = R_\nu + T_\nu; \qquad D_\nu = \frac{M_\nu}{m_\nu^2}(R_\nu - T_\nu). \tag{142}$$

Then

$$R_\nu = \frac{1}{2}\left(C_\nu + \frac{m_\nu^2}{M_\nu} D_\nu\right); \qquad T_\nu = \frac{1}{2}\left(C_\nu - \frac{m_\nu^2}{M_\nu} D_\nu\right). \tag{142a}$$

By eliminating R_ν and T_ν from equation (138) with the aid of equation (142a), one obtains

$$C_{\nu-1} = C_\nu \cos\frac{\beta_\nu}{2} + i\, D_\nu \frac{m_\nu^2}{M_\nu} \sin\frac{\beta_\nu}{2}; \qquad D_{\nu-1} = i\, C_\nu \frac{M_\nu}{m_\nu^2} \sin\frac{\beta_\nu}{2} + D_\nu \cos\frac{\beta_\nu}{2}. \tag{142b}$$

Therefore

$$\begin{bmatrix} C_{\nu-1} \\ D_{\nu-1} \end{bmatrix} = \begin{bmatrix} \cos\frac{\beta_\nu}{2} & i\,\frac{m_\nu^2}{M_\nu}\sin\frac{\beta_\nu}{2} \\ i\,\frac{M_\nu}{m_\nu^2}\sin\frac{\beta_\nu}{2} & \cos\frac{\beta_\nu}{2} \end{bmatrix} \begin{bmatrix} C_\nu \\ D_\nu \end{bmatrix}. \tag{142c}$$

From equations (140c) and (140d),

$$C_N = R_N + T_N = T_{N+1};$$
$$D_N = \frac{M_N}{m_N^2}(R_N - T_N) = -\frac{M_{N+1}}{m_{N+1}^2} T_{N+1}. \tag{142d}$$

Therefore, upon forming $\begin{bmatrix} C_o \\ D_o \end{bmatrix}$ from equation (142c), one obtains

$$\begin{bmatrix} C_o \\ D_o \end{bmatrix} = T_{N+1} \prod_{\nu=1}^{N} \begin{bmatrix} \cos\frac{\beta_\nu}{2} & i\frac{m_\nu^2}{M_\nu}\sin\frac{\beta_\nu}{2} \\ i\frac{M_\nu}{m_\nu^2}\sin\frac{\beta_\nu}{2} & \cos\frac{\beta_\nu}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -M_{N+1}/m_{N+1}^2 \end{bmatrix}, \tag{142e}$$

a result that should be compared with equation (137e). The remarks in the paragraph following equation (137e) apply again to equation (142e).

21.5 QUATERNION METHODS

21.5.1 Introduction. A computing program based upon quarternions instead of matrices has been introduced by Dr. Gordon L. Walker * for analyzing thin films with the aid of automatic computers. The relative advantages of matrices and quaternions depend mainly upon circumstances at the location of the automatic calculator. For example, wherever a programmed matrix formulation is available, it may be simpler to adapt a matrix method. As has been seen in Section 21.4, it is both natural and direct to state the solution to the problems of thin films in matrix notation. Because many solutions in matrix form have been given, we shall restrict our considerations to showing how any matrix solution can be transformed into the corresponding quaternion form.

10.5.2 Quaternions. A quaternion Q is the sum of a scalar and a vector. Thus,

$$Q = q_o\sigma_o + q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3, \tag{143}$$

in which σ_1, σ_2 and σ_3 are unit vectors and $\sigma_o = 1$. Coefficients q_o, q_1, q_2 and q_3 are scalars that can be complex imaginary. With respect to summation,

$$P + Q = Q + P = (p_o + q_o)\sigma_o + (p_1 + q_1)\sigma_1 + (p_2 + q_2)\sigma_2 + (p_3 + q_3)\sigma_3. \tag{143a}$$

If b is a scalar,

$$bQ = Qb = bq_o\sigma_o + bq_1\sigma_1 + bq_2\sigma_2 + bq_3\sigma_3. \tag{143b}$$

In forming the product of two quaternions, P and Q, one observes the following rules of operation with respect to the unit vectors.

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -1; \qquad \sigma_o^2 = 1;$$
$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3;$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \sigma_1;$$
$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = \sigma_2. \tag{143c}$$

It follows that $PQ \neq QP$. Let p_ν and q_ν be the coefficients of quaternions P and Q, respectively. Set

$$PQ = R = R_o\sigma_o + R_1\sigma_1 + R_2\sigma_2 + R_3\sigma_3. \tag{143d}$$

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*This unpublished scheme has been used by Drs. G. L. Walker, H. Jupnik and A. Traub at the American Optical Company. A method that resembles the method of quaternions in several respects has been discussed by M. Andre Herpin, Comptes Rendus Acad. Sci., 225, 182-183 (1947).

Then,

$$R_o = p_o q_o - p_1 q_1 - p_2 q_2 - p_3 q_3 ;$$
$$R_1 = p_o q_1 + q_o p_1 + p_2 q_3 - p_3 q_2 ;$$
$$R_2 = p_o q_2 + q_o p_2 + p_3 q_1 - p_1 q_3 ;$$
$$R_3 = p_o q_3 + q_o p_3 + p_1 q_2 - p_2 q_1 . \quad (143e)$$

21.5.3 Corresponding matrices and quaternions. Let

$$\mathcal{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (144)$$

be a given matrix. Let $\tilde{\mathcal{M}}$ denote the corresponding quaternion. Then

$$\tilde{\mathcal{M}} = m_o \sigma_o + m_1 \sigma_1 + m_2 \sigma_2 + m_3 \sigma_3 , \quad (144a)$$

wherein

$$m_o = \frac{m_{11} + m_{22}}{2} ; \qquad m_2 = \frac{m_{12} - m_{21}}{2} ;$$
$$m_1 = i \frac{m_{11} - m_{22}}{2} ; \qquad m_3 = i \frac{m_{12} + m_{21}}{2} . \quad (144b)$$

On the other hand, let $\tilde{\mathcal{M}}$ be the given quaternion. In order to obtain the corresponding matrix, one replaces $\sigma_o = 1$ and the unit vectors by the matrices

$$\sigma_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \qquad \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ;$$
$$\sigma_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} ; \qquad \sigma_3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} ; \quad (144c)$$

in which σ_o is a unit matrix. Thus with $\tilde{\mathcal{M}}$ regarded as the given quaternion, the corresponding matrix is

$$\mathcal{M} = m_o \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + m_1 \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + m_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + m_3 \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} ;$$
$$= \begin{bmatrix} m_o - i m_1 & m_2 - i m_3 \\ -(m_2 + i m_3) & m_o + i m_1 \end{bmatrix} . \quad (144d)$$

The matrices of equations (144c) satisfy the requirements of equations (143c). The quaternion corresponding to the product of two matrices taken in a specified order is the product of the quaternions corresponding to each of the two matrices taken in the same order. Thus

$$\text{Quaternion } (\mathcal{M}_1 \; \mathcal{M}_2) = \tilde{\mathcal{M}}_1 \tilde{\mathcal{M}}_2 . \quad (144e)$$

Similarly,

$$\text{Matrix } (\tilde{\mathcal{M}}_1 \; \tilde{\mathcal{M}}_2) = \mathcal{M}_1 \mathcal{M}_2 . \quad (144f)$$

For the purposes of this text, equation (144e) is far more important than equation (144f). Repeated application of equation (144e) shows that

$$\text{Quaternion } \left(\prod_{\nu=1}^{N} \mathcal{M}_\nu\right) = \prod_{\nu=1}^{N} \tilde{\mathcal{M}}_\nu , \quad (144g)$$

in which $\tilde{\mathcal{M}}_\nu$ is the quaternion that corresponds to matrix $\mathcal{M}_\nu$.

21.5.4 Replacements for matrix equations. Let $\mathcal{M}$ be the square matrix

$$\mathcal{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \tag{145}$$

that connects the quantities r, τ, A and B according to the law

$$\begin{bmatrix} r \\ \tau \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}, \tag{145a}$$

as in equations (126) and (127). One can verify almost directly from equations (126) to (129) that it is permissible to set

$$\begin{bmatrix} r \\ \tau \end{bmatrix} = \begin{bmatrix} r & 0 \\ \tau & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}.$$

Hence equation (145a) can be written in the form

$$\begin{bmatrix} r & 0 \\ \tau & 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}. \tag{145b}$$

The quaternion form of matrix equation (145b) is now obtained by replacing each of the three square matrices by its corresponding quaternion with the aid of equation (144b). Let U denote the quaternion corresponding to the left hand member of equation (145b), i.e. let

$$U = \text{Quaternion} \begin{bmatrix} r & 0 \\ \tau & 0 \end{bmatrix}. \tag{145c}$$

By applying the correspondence rules of equation (144b) to equation (145c), one finds that

$$U = \frac{r}{2}\sigma_o + i\frac{r}{2}\sigma_1 - \frac{\tau}{2}\sigma_2 + i\frac{\tau}{2}\sigma_3. \tag{145d}$$

Similarly, with respect to

$$S_o = \text{Quaternion} \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix},$$

$$S_o = \frac{A}{2}\sigma_o + i\frac{A}{2}\sigma_1 - \frac{B}{2}\sigma_2 + i\frac{B}{2}\sigma_3. \tag{145e}$$

Matrix equation (145b) is therefore replaced by the quaternion equation

$$U = \tilde{\mathcal{M}}\, S_o, \tag{145f}$$

in which

$$\tilde{\mathcal{M}} = \text{Quaternion} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$= m_o\sigma_o + m_1\sigma_1 + m_2\sigma_2 + m_3\sigma_3, \tag{145g}$$

as in equations (144a) and (144b). It is instructive to examine the quaternion product $\tilde{\mathcal{M}}\, S_o$. One finds from equations (143d), (143e), (145e) and (145g) that with

$$U = \tilde{\mathcal{M}}\, S_o = \sigma_o U_o + U_1\sigma_1 + U_2\sigma_2 + U_3\sigma_3, \tag{145h}$$

$$U_o = \frac{A}{2}(m_o - i\, m_1) + \frac{B}{2}(m_2 - i\, m_3);$$
$$U_2 = \frac{A}{2}(m_2 + i\, m_3) - \frac{B}{2}(m_o + i\, m_1); \tag{145i}$$

together with

$$U_1 = i\, U_o; \quad U_3 = -i\, U_2. \tag{145j}$$

The relations between U_o , U_2 and m_{kl} are obtained from equations (144b) and (145i). One finds that

$$U_o = \frac{A}{2} m_{11} + \frac{B}{2} m_{12} ;$$
$$U_2 = \frac{A}{2} m_{21} - \frac{B}{2} m_{22} . \qquad (145k)$$

21.5.5 Quaternion solutions for r and τ. The solutions for r and τ are found by comparing equations (145d) and (145h). These equations require that

$$r = 2 U_o ;$$
$$\tau = -2 U_2 ; \qquad (146)$$

in which U_o and U_2 are the coefficients of σ_o and σ_2 in the quaternion product $\tilde{\mathcal{M}} S_o$ of equation (145f). Equation (146) forms a simple way of computing r and τ once U_o and U_2 have been found.

21.5.6 A check on the quaternion method. Equations (146) and (145k) can be combined to form a simple check on the correctness of the quaternion method. One finds directly that

$$r = m_{11} A + m_{12} B ;$$
$$\tau = m_{21} A + m_{22} B . \qquad (147)$$

These solutions for r and τ are those of the matrix equation (145a). If therefore equation (145a) is correct, equation (146) is correct.

21.5.7 A more useful statement of the formulation. In stating the matrix that corresponds to a multilayer, it is rarely convenient to specify the matrix in the form of equation (145a). Instead, it is convenient to express the matrix in the form

$$\begin{bmatrix} r \\ \tau \end{bmatrix} = F \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \qquad (148)$$

in which factor F is a scalar. For example, in the matrix of equation (137e) the factor $F = \tau_{N+1}$. Re-examination of the argument leading from equation (145a) to (146) shows that one obtains instead of equation (146) the slightly modified result

$$r = 2 F U_o ;$$
$$\tau = -2 F U_2 ; \qquad (149)$$

in which U_o and U_2 are the coefficients of σ_o and σ_2 , respectively, of the quaternion corresponding to the matrix product of equation (148).

21.5.8 Special conditions. Depending upon the nature of the multilayer and upon the manner in which the corresponding matrix problem has been solved, special conditions may exist among the matrix elements m of the matrix $\mathcal{M}$ associated with the multilayer. For example, with respect to the matrix

$$\mathcal{M} = \prod_{\nu=1}^{N} \mathcal{M}_\nu = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} , \qquad (150)$$

of equation (133), it follows that at normal incidence upon a system of non-absorbing layers

$$m_{22} = \bar{m}_{11} ; \quad m_{21} = \bar{m}_{12} ; \qquad (151)$$

because the diagonal elements of $\mathcal{M}_\nu$ of equation (130a) are then conjugate complex numbers. Correspondingly, from equation (144b)

$$m_o = R_e (m_{11}) ; \quad m_2 = i \mathcal{I}_m (m_{12}) ;$$
$$m_1 = -\mathcal{I}_m (m_{11}) ; \quad m_3 = i R_e (m_{12}) . \qquad (152)$$

It will not, however, be a purpose of this discussion to enumerate the various special conditions together with their consequences.

21.5.9 Application to the matrix solution of equation (133). The matrix solution of equation (133) applies to cases in which the electric vector is polarized so as to vibrate at right angles to the plane of incidence. This example will illustrate procedures that may be followed in converting any matrix solution for thin films into a solution in terms of quaternions. With respect to efficiency, it is undoubtedly preferable to choose as factor F the product

$$\frac{\tau_{N+1}}{2^{N+1}} \prod_{\nu=0}^{N} \frac{1}{M_\nu}$$

of equation (133). Let us demand, however, that the Fresnel coefficients W_ν of equation (20) shall appear as parameters. This can be accomplished by writing $\mathcal{M}_\nu$ of equation (130a) in the form

$$\mathcal{M}_\nu = (M_{\nu-1} + M_\nu) \begin{bmatrix} e^{i\frac{\beta\nu}{2}} & W_\nu\, e^{-i\frac{\beta\nu}{2}} \\ W_\nu\, e^{i\frac{\beta\nu}{2}} & e^{-i\frac{\beta\nu}{2}} \end{bmatrix} . \qquad (153)$$

Let also the factor $M_N + M_{N+1}$ be removed from the second right hand matrix of equation (133). Then,

$$\begin{bmatrix} r_o \\ \tau_o \end{bmatrix} = F \prod_{\nu=1}^{N} \begin{bmatrix} e^{i\frac{\beta\nu}{2}} & W_\nu\, e^{-i\frac{\beta\nu}{2}} \\ W_\nu\, e^{i\frac{\beta\nu}{2}} & e^{-i\frac{\beta\nu}{2}} \end{bmatrix} \begin{bmatrix} W_{N+1} \\ 1 \end{bmatrix} ; \qquad (154)$$

in which

$$F = \frac{\tau_{N+1}}{2^{N+1}} \; \frac{M_N + M_{N+1}}{M_o} \prod_{\nu=1}^{N} \frac{M_\nu + M_{\nu-1}}{M_\nu} . \qquad (155)$$

We take $\tilde{\mathcal{M}}_\nu$ as the quaternion corresponding to the ν^{th} matrix of the product from $\nu = 1$ to $\nu = N$ of equation (154). Explicitly, we take

$$\tilde{\mathcal{M}}_\nu = \cos\frac{\beta\nu}{2}\,\sigma_o - \sin\frac{\beta\nu}{2}\,\sigma_1 - i\,W_\nu \sin\frac{\beta\nu}{2}\,\sigma_2 + i\,W_\nu \cos\frac{\beta\nu}{2}\,\sigma_3 . \qquad (156)$$

The quaternion S_o of the last matrix of equation (154) is obtained from equation (145e) by setting

$$A = W_{N+1}; \qquad B = 1 . \qquad (157)$$

We form and compute the quaternion

$$\tilde{\mathcal{M}} = \prod_{\nu=1}^{N} \tilde{\mathcal{M}}_\nu \qquad (158)$$

with $\tilde{\mathcal{M}}_\nu$ given in terms of the physical properties β_ν and W_ν of the ν^{th} layer by equation (156). Next, as in equation (145f), we compute the quaternion U as the product

$$U = \frac{1}{2}\,\tilde{\mathcal{M}} \left[W_{N+1}\,\sigma_o + i\,W_{N+1}\,\sigma_1 - \sigma_2 + i\,\sigma_3 \right] ; \qquad (159)$$

$$= \prod_{\nu=1}^{N} \tilde{\mathcal{M}}_\nu \; S_o$$

In making the last computation of equation (159), it is necessary to compute only U_o and U_2, the coefficients of σ_o and σ_2, respectively. With U_o and U_2 thus determined, it follows from equation (149) that

$$r_o = 2\,F\,U_o ; \qquad \tau_o = -2\,F\,U_2 ; \qquad (160)$$

in which F is given by equation (155).

21.5.10 Determination of the complex reflectance and transmittance of the multilayer. The complex reflectance of the multilayer is given by the ratio $\rho_o = r_o / \tau_o$. From equation (160)

$$\rho_o = - U_o / U_2 , \qquad (161)$$

a result that is independent of the factor F and that is evaluated at the right hand boundary of the medium of incidence with the electric vector perpendicular to the plane of incidence. The complex transmittance of the multilayer is given by the ratio τ_{N+1}/τ_o. From equations (160) and (155)

$$\tau_{N+1}/\tau_o = -2^{N+1} \frac{M_o}{M_N + M_{N+1}} \prod_{\nu=1}^{N} \frac{M_\nu}{M_{\nu-1} + M_\nu} \frac{1}{2U_2}, \tag{162}$$

a result evaluated at the point of entry into the last medium with the electric vector perpendicular to the plane of incidence. Computation of the complex reflectance and transmittance is relatively simple when the coefficients U_o and U_2 of the composite quaternion U of the multilayer have been calculated.

21.5.11 Application to the monolayer. The uninitiated reader will find it a useful exercise to verify from equations (158) and (159) that for the monolayer (the case N = 1)

$$U_o = \frac{W_2}{2} e^{i\frac{\beta_1}{2}} + \frac{W_1}{2} e^{-i\frac{\beta_1}{2}};$$

$$-U_2 = \frac{1}{2} e^{-i\frac{\beta_1}{2}} + \frac{W_1 W_2}{2} e^{i\frac{\beta_1}{2}}. \tag{163}$$

It is then shown easily from equations (161), (162) and (163) that

$$\rho_o = \frac{W_2 e^{i\beta_1} + W_1}{1 + W_1 W_2 e^{i\beta_1}}; \tag{164}$$

$$\frac{\tau_{N+1}}{\tau_o} = \frac{\tau_2}{\tau_o} = \frac{4 M_o M_1}{(M_o + M_1)(M_1 + M_2)} \frac{e^{i\frac{\beta_1}{2}}}{1 + W_1 W_2 e^{i\beta_1}}. \tag{165}$$

Equation (164) agrees, for example, with the result of equations (50) and (51) for cases N = 1. Likewise, equation (165) agrees with the result of equation (54). Matrix methods and quaternion methods do not possess advantages over recursion methods such as those of Sections 21.2.8 and 21.2.9 until the number N of layers in the multilayer exceeds 2. In fact, the methods of equations (164) and (165) are to be preferred as regards simplicity and convenience for the monolayer.

21.5.12 Comments. In the interesting method constructed by Gordon L. Walker, the quaternion corresponding to S_o of equations (145) is rendered incomplete in that the coefficients of the unit vectors σ_1 and σ_2 are zero. The method presented here does not involve more quaternion multiplication than the method due to Walker and requires slightly less algebra at the last steps for computing the complex reflectance ρ_o and the complex transmittance τ_{N+1}/τ_o of the multilayer. Furthermore, Walker has preferred to apply the quaternion method to factored matrices $\mathcal{M}_\nu$ of the type described by equations (136). For example with respect to the matrices of equations (133) and (136), one can take

$$F = \frac{\tau_{N+1}}{2^{N+1}} \prod_{\nu=0}^{N} \frac{1}{M_\nu}; \tag{166}$$

$$A = M_N - M_{N+1}; \quad B = M_N + M_{N+1}; \tag{166a}$$

and compute the quaternion

$$U = \prod_{\nu=1}^{N} \tilde{\mathcal{S}}_\nu \tilde{\mathcal{J}}_\nu S_o, \tag{166b}$$

wherein $\tilde{\mathcal{S}}_\nu$ and $\tilde{\mathcal{J}}_\nu$ are the quaternions corresponding to matrices $\mathcal{S}_\nu$ and $\mathcal{J}_\nu$, respectively, of equation (136a) and S_o is determined from equation (145a). The complex reflectance ρ_o can be computed from equation (161) and the complex transmittance τ_{N+1}/τ_o can be computed from equation (160). It will be found from equations (136) and (144b) that the quaternions $\tilde{\mathcal{S}}_\nu$ and $\tilde{\mathcal{J}}_\nu$ are incomplete. The product $\tilde{\mathcal{S}}_\nu \tilde{\mathcal{J}}_\nu$ is, however, a complete quaternion.

21.6 MONOLAYER COATINGS

21.6.1 Introduction. Monolayer coatings serve many specialized purposes. One broad class of monolayers consists of dielectric substances deposited or formed upon absorbing or non-absorbing substrates. A second broad class consists of metallic or semiconducting films on absorbing or non-absorbing substrates. We shall not be concerned with that class of monolayers whose sole function is to alter the mechanical, chemical or electrical properties of a surface. Monolayers may occur naturally as, for example, in the tarnishing of silver by a layer of silver sulphide. The practical range in thickness of a layer may vary from molecular dimensions to centimeters or even meters depending mainly upon the wavelength of the radiation involved. This radiation may extend from the ultraviolet into radar. Not all monolayers should be regarded as homogeneous but a large group of monolayers can be considered homogeneous in constructing an approximate theory for interpreting their "optical" behavior. The approximations thus afforded are often in excellent agreement with experiment.

21.6.2 Methods of computation.

21.6.2.1 Wherever automatic calculators have been programmed on the basis of matrix or quaternion methods, this program can serve for computing monolayers although such programs become unduly elaborate. The following method is one of the useful and accurate methods for desk calculators. This method will be presented for the case in which the incident electric vector is perpendicular to the plane of incidence. When the electric vector vibrates in the plane of incidence, it is necessary to replace the Fresnel coefficients, W_ν, of equation (20) by the Fresnel coefficients, F_ν, of equation (21) as shown in Section 21.2.10. By setting $\nu = 1$ in equation (57) and noting that $\rho_1 = W_2$ from equation (51), one finds that the complex reflectance, ρ_o, of the monolayer is given by

$$\rho_o = \frac{W_2 \, e^{i\beta_1} + W_1}{1 + W_1 W_2 \, e^{i\beta_1}} , \qquad (167)$$

as in equation (164). Likewise, by setting $N = 1$ in equation (54) one obtains the complex transmittance in the form

$$\frac{\tau_2}{\tau_o} = \frac{4 M_o M_1}{(M_o + M_1)(M_1 + M_2)} \; \frac{e^{i\frac{\beta_1}{2}}}{1 + W_1 W_2 \, e^{i\beta_1}} , \qquad (168)$$

as in equation (165). Explicitly,

$$W_\nu = \frac{M_{\nu-1} - M_\nu}{M_{\nu-1} + M_\nu} \; ; \; \nu = 1, 2; \qquad (169)$$

$$\beta_1 = \frac{4\pi}{\lambda} M_1 d_1 , \qquad (169a)$$

in which M_ν is defined by equation (19). ρ_o and τ_2 are evaluated by equations (167) and (168) at the left hand side of the first interface and at the right hand side of the second interface, respectively, as indicated in Figure 21.10. Marked simplification occurs at normal incidence for which $i_o = 0$ so that $p_o = \sin i_o = 0$.

$$M_\nu = m_\nu = n_\nu (1 + i K_\nu) \; ; \; i_o = 0 . \qquad (169b)$$

Furthermore, at normal incidence no distinction need be made among the directions of polarization of the electric vector when the film and substrate are isotropic. Equations (167) and (168) then serve for all states of polarization.

21.6.2.2 Phase changes that occur on reflection and transmission are given as phase retardations by arg (ρ_o) and arg (τ_2/τ_o), respectively. When these quantities are wanted, ρ_o and τ_2/τ_o must be evaluated as complex numbers by the operations indicated by equations (167) and (168).

21.6.2.3 The parameter β_1 is awkward to handle whenever m_1 is complex $(K_1 \neq 0)$. This awkwardness is increased when the angle of incidence $i_o \neq 0$. At normal incidence

$$\beta_1 = \frac{4\pi}{\lambda} n_1 d_1 (1 + i K_1) , \qquad (170)$$

and

$$e^{i\beta_1} = e^{-\frac{4\pi}{\lambda} n_1 K_1 d_1} \; e^{i\frac{4\pi}{\lambda} n_1 d_1} , \qquad (170a)$$

in which $4\pi n_1 d_1/\lambda$ is twice the optical path of the film and the exponent, $-4\pi n_1 K_1 d_1/\lambda$, is an attenuation

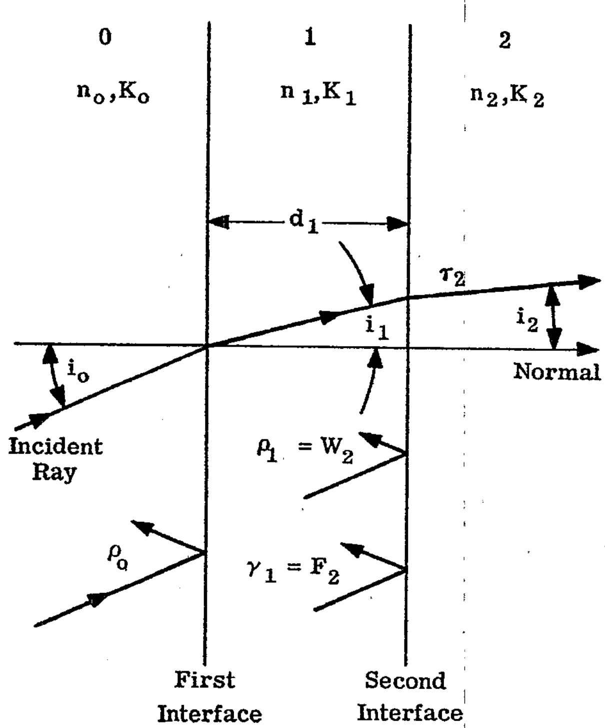


Figure 21. 10- Convention and notation with respect to the monolayer (case N = 1).

factor. When the medium of incidence and the monolayer do not absorb, and when $n_1^2 > n_o^2 p_o^2$,

$$M_1 = \left[n_1^2 - n_o^2 p_o^2\right]^{1/2} = n_1 \cos i_1 ,$$

where i_1 is the angle of refraction in the film. Correspondingly,

$$\beta_1 = \frac{4\pi}{\lambda} n_1 d_1 \cos i_1 ; \quad K_o = K_1 = 0. \tag{170b}$$

21.6.2.4 The energy reflectance R of the monolayer is given by $R = |\rho_o|^2$ and is most easily computed from equation (167) in the form

$$R = |\rho_o|^2 = \frac{|W_2 e^{i\beta_1} + W_1|^2}{|1 + W_1 W_2 e^{i\beta_1}|^2} . \tag{171}$$

Suppose that the system is non-absorbing. In such cases (a class of wide interest) the Fresnel coefficients W_1 and W_2 are real and β_1 is given by equation (170b). Correspondingly,

$$R = \frac{W_1^2 + 2 W_1 W_2 \cos \beta_1 + W_2^2}{1 + 2 W_1 W_2 \cos \beta_1 + W_1^2 W_2^2} . \tag{172}$$

Differentiation of R with respect to β_1 in equation (172) shows that the extreme values of $R(\beta_1)$ occur when

$$\beta_1 = \nu\pi ; \quad \nu \text{ an integer.} \tag{173}$$

Hence when the electric vector is perpendicular to the plane of incidence, when $n_1 > n_o p_o$ and when $K_o = K_1 = K_2 = 0$, the maxima and minima, R_m, are given by

$$R_m = \frac{(W_1 \pm W_2)^2}{(1 \pm W_1 W_2)^2} . \tag{174}$$

The values of W_1 and W_2 (and hence R_m) depend upon the angle of incidence. But at fixed angles of incidence, $R_{max.}$ and $R_{min.}$ remain fixed and R is a periodic function of the thickness d_1. Departures from periodicity can occur when β_1 is altered by changing wavelength because the refractive indices (and hence W_1 and W_2) are dispersive with λ. It is not difficult to see that $R(\beta_1)$ cannot be periodic when the film absorbs. Equation (170a) shows that the $\exp(i\beta_1)$ approaches zero as d_1 approaches infinity. Consequently, from equation (171) R oscillates with increasing β_1 such that

$$R \rightarrow |W_1|^2 , \tag{175}$$

the Fresnel coefficient of reflectance of the first interface, Figure 21.10.

21.6.2.5 When the system contains no absorbing media, the energy transmittance, T_e, of the monolayer can be found in terms of the energy reflectance, R, from the law of conservation of energy, namely,

$$T_e + R = 1 , \tag{176}$$

irrespective of the refractive indices of the first and last media. When absorption occurs, one can compute the complex transmittance τ_2 / τ_o from equation (168) and the energy transmittance from equation (58). If only the film is absorbing, $(n_a)_2 = n_2$ so that equation (58) yields the result,

$$T_e = \frac{n_2 \cos i_2}{n_o \cos i_o} \left| \frac{\tau_2}{\tau_o} \right|^2 , \tag{177}$$

for cases in which the electric vector is perpendicular* to the plane of incidence.

21.6.3 <u>Non-absorbing systems; normal incidence.</u>

21.6.3.1 The behavior of the energy reflectances, $|\rho_o|^2$, for non-absorbing monolayers on non-absorbing substrates is illustrated in Figure 21.11 for cases in which the refractive index $n_o = 1$. When no absorption occurs, reversal of the direction of incidence leaves $|\rho_o|^2$ unchanged. For reasons explained in Section 21.2.14, the monolayer is an absentee layer at points $\beta_1 = \nu 2\pi$ where $\nu = 0$ or an integer. At these points the energy reflectance is that of the uncoated glass.

21.6.3.2 With respect to cases $n_o < n_1 < n_2$, it follows from equation (169) that $W_1 < 0$ and $W_2 < 0$. Correspondingly from equation (174),

$$R_m = R_{min.} ,$$

when

$$\beta_1 = \mu \pi ; \quad \mu \text{ an odd integer.} \tag{178}$$

By introducing

$$W_1 = \frac{n_o - n_1}{n_o + n_1} \quad ; \quad W_2 = \frac{n_1 - n_2}{n_1 + n_2}$$

into equation (174), one finds that

$$R_{min.} = |\rho_o|^2_{min.} = \left(\frac{n_o n_2 - n_1^2}{n_o n_2 + n_1^2} \right)^2 . \tag{179}$$

In particular, $R_{min.} = 0$ when β_1 obeys equation (178) and n_1 is chosen so that

$$n_1 = \sqrt{n_o n_2} . \tag{180}$$

Equations (178) and (180) agree with the more general equation (115) and (116) at normal incidence. Equation (179) is important for two reasons. First, it enables one to estimate the minimum value of the energy reflectance. Secondly, it enables one to compute the refractive index n_1 of the film from the measured**value of $R_{min.}$ and the known values of n_o and n_2. The refractive indices of evaporated monolayers are usually less than those of the bulk materials and vary with the conditions of evaporation.

* See equations (70) and (71) for cases in which the electric vector vibrates in the plane of incidence.

**The spectral energy reflectance, R_p, of a coated plate is ordinarily obtained with a spectrophotometer. Suppose, for example, that the absorption of the plate is negligible and that the energy reflectance of the back surface is B. On account

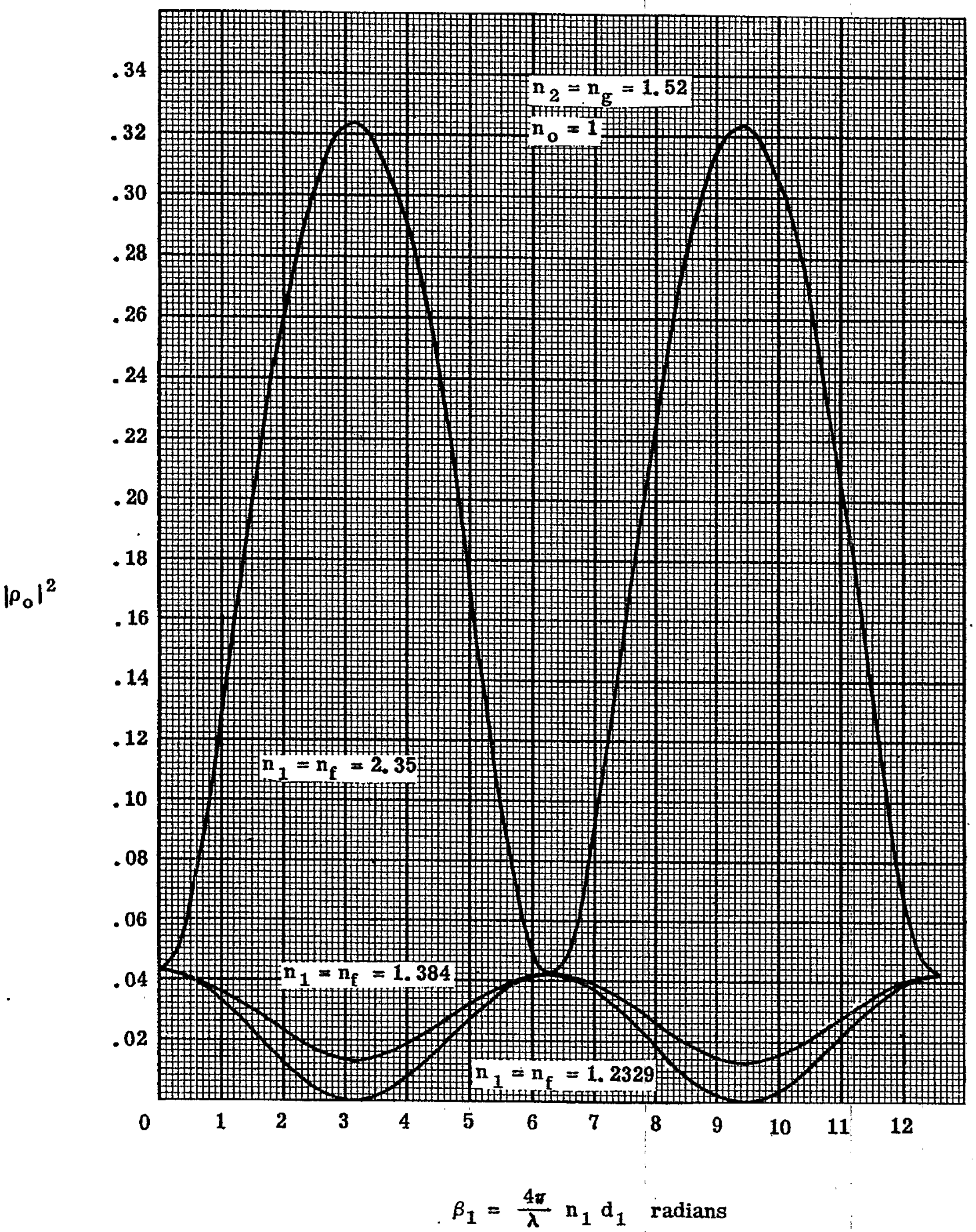


Figure 21. 11- Energy reflectances $|\rho_o|^2$ vs β_1 in radians. These curves illustrate the periodic behavior of nonabsorbing monolayers on nonabsorbing substrates. β_1 is now twice the optical path of the monolayer. The curves drawn for $n_f = 2.35$ and $n_f = 1.384$ with $n_g = 1.52$ correspond to zinc sulphide and magnesium fluoride, respectively, on spectacle crown glass. The curves illustrate the important characteristic that $|\rho_o|^2 \gtrless$ reflectance of the uncoated substrate according as $n_f \gtrless n_g$ at points $\beta_1 \neq \nu\pi$ where $\nu = 0$ or an integer. The curve for $n_f = 1.2329 = \sqrt{n_g} = \sqrt{1.52}$ illustrates how zero reflectance is achieved at points $\beta_1 = \mu\pi$ where μ is an odd integer.

of the interreflections that occur within the plate, it follows that

$$R_p = \frac{R + B - 2RB}{1 - RB}, \qquad (1)$$

where R is the surface reflectance $R = |\rho_o|^2$. Hence

$$R = |\rho_o|^2 = \frac{R_p - B}{1 + R_p B - 2B}. \qquad (2)$$

If the plate is a glass plate in air with its back surface uncoated, $B = [(n_g - 1)/(n_g + 1)]^2$, where n_g is the refractive index of the glass plate.

21.6.3.3 When $n_o < n_1 > n_2$, equation (169) shows that Fresnel's coefficients W_1 and W_2 have opposite sign. At points $\beta_1 = \mu\pi$ (μ odd), the extreme reflectances R_m of equation (174) are now maxima with the choice of the negative sign. Also $|\rho_o|^2_{max.}$ is therefore given by the right hand member of equation (179). $|\rho_o|^2_{max.}$ approaches unity with increasing n_1.

21.6.3.4 In Figure 21.12, $|\rho_o|^2$ is plotted against λ in the visible region for the indicated values of $n_1 = n_f$ and $n_2 = n_g$ with $n_o = 1$. The family of curves illustrates the effectiveness of various monolayers in reducing surface reflectances of spectacle crown glass. The thicknesses of the monolayers are chosen so that the optical path of each monolayer is one-fourth wavelength at 0.550 microns.

21.6.3.5 Energy reflectances $|\rho_o|^2$ are plotted against wavelength in the infrared region in Figure 21.13 for $n_o = 1$, $n_2 = n_s = 3.450$ and for the indicated values of $n_1 = n_f$. This family of curves illustrates the effectiveness of various monolayers in reducing the reflectance of a surface of silicon. As applied to silicon, the effects of absorption by the substrate are not included in Figure 21.13. Absorption of silicon is low in the infrared region. The curve for $n_f = 1.52$ has been added to illustrate what occurs when n_f is smaller than the value required for producing zero reflectance, i.e. when $n_f < \sqrt{n_o n_2}$. Let n_f and n_f' denote the refractive indices of two different non-absorbing monolayers such that $n_f > \sqrt{n_o n_2}$ and $n_f' < \sqrt{n_o n_2}$. It is not difficult to show that when the refractive indices are independent of wavelength, the two monolayers produce the same spectral reflectance curves provided that n_f and n_f' are chosen in accordance with the relation

$$n_f \, n_f' = n_1 \, n_1' = n_o \, n_2. \qquad (181)$$

For example, a monolayer having the refractive index $n_f' = 1.57$ is equivalent to the monolayer having the refractive index $n_f = 2.200$ when n_o and n_2 have been chosen as in Figure 21.13.

21.6.4 Dielectric monolayers on opaque, metallic substrates; normal incidence. Dielectric layers of hard materials such as magnesium fluoride and silicon monoxide are often deposited upon surfaces of aluminum, silver and other metals for the purpose of protecting these surfaces from abrasion. The effects of monolayers of MgF_2 and SiO upon the surface reflectance and phase change on reflection are illustrated in Figure 21.14 for opaque substrates of aluminum. Appreciable losses in reflectance can occur when the monolayers are so thin that their optical paths $n_1 d_1 < \lambda/4$. For maximum reflectance, the optical path of the monolayer should be slightly less than $\lambda/2$. This maximum reflectance can exceed that of the uncoated surface provided that the refractive index n_1 of the monolayer is high enough. With MgF_2[7] and SiO,[8] one obtains a maximum reflectance that departs imperceptibly from that of the uncoated substrate. The phase change on reflection varies markedly with the optical path of the monolayer. This property is of importance to interferometry.

21.6.5 Absorbing monolayers on opaque, metallic substrates; normal incidence. The effects of absorbing monolayers upon the surface reflectance of opaque, metallic substrates are illustrated in Figure 21.15 by a series of monolayers that have a fixed and relatively high refractive index $n_1 = 4.0$. The curve for the non-absorbing film $n_1 K_1 = 0$ shows that the maximum reflectance can exceed * that of the uncoated surface appreciably when n_1 becomes high. Increasing the absorption of the monolayer within the range of $n_1 K_1$ of Figure 21.15 reduces both the maximum and the minimum reflectances. The minimum reflectances occur for optical paths p_1 near 45° or $\lambda/8$. The family of curves suggest that tarnishing of silver is due to the formation of a monolayer.

(7) Monolayers of MgF_2 have become valuable for increasing the reflectance of aluminum in the ultraviolet. For a discussion of the region around 1216A, see P.H. Berning, G. Hass and R. P. Madden; Paper T51 given during 44th Annual Meeting of OSA, Ottawa, October 1959.

(8) The refractive indices and absorption of so called SiO films depend markedly upon conditions of evaporation. Slow deposition in the presence of air or oxygen produces "SiO" films that can have refractive indices near 1.7 with low absorption. See G. Hass, Vacuum, 2, p 338 (1952).

* Compare with Figure 21.14

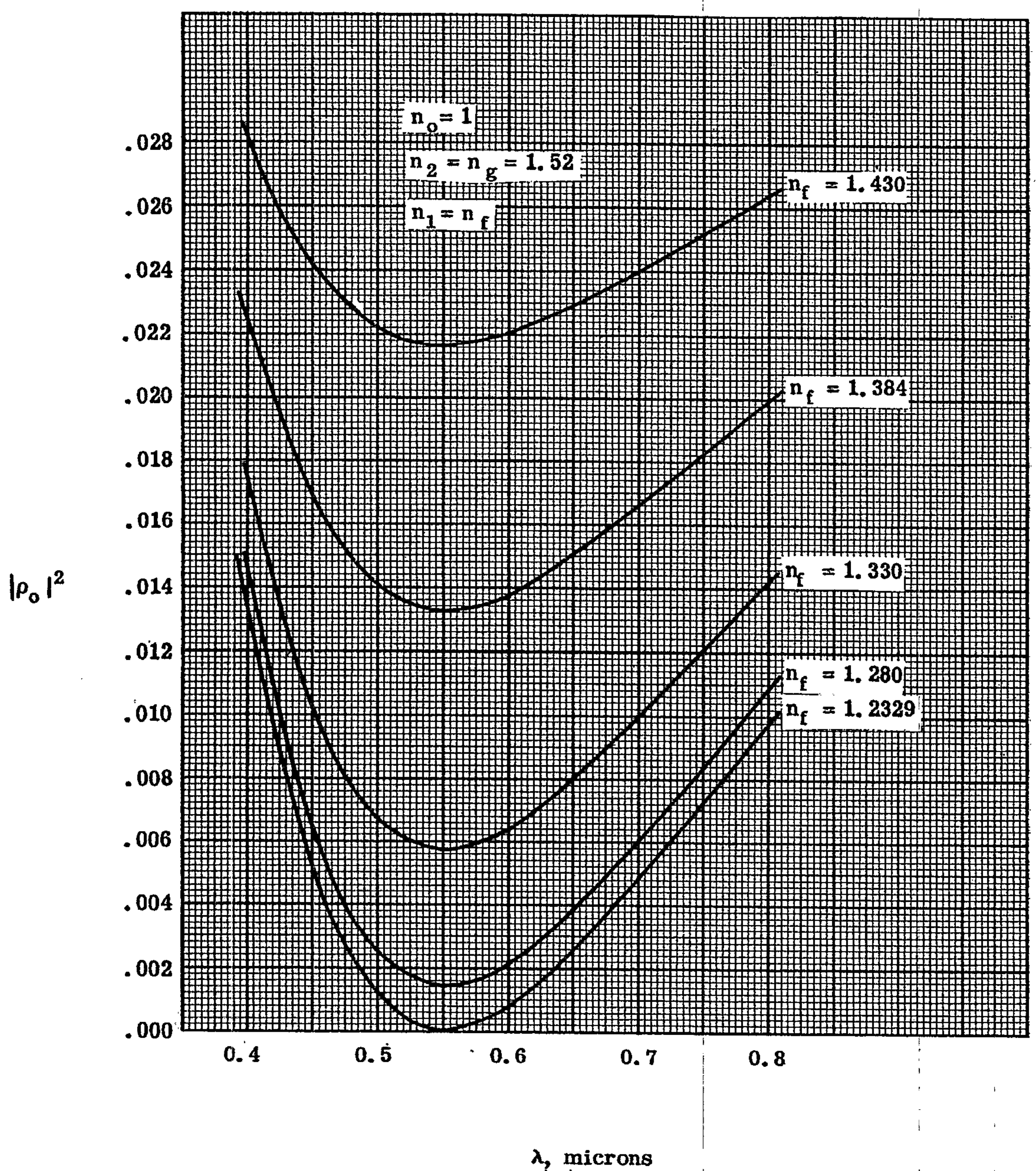


Figure 21. 12- Plot of energy reflectances vs wavelength in microns for various low reflecting monolayers on spectacle crown glass. Each monolayer has the optical path $\lambda/4$ at $\lambda = 0.550$ microns.

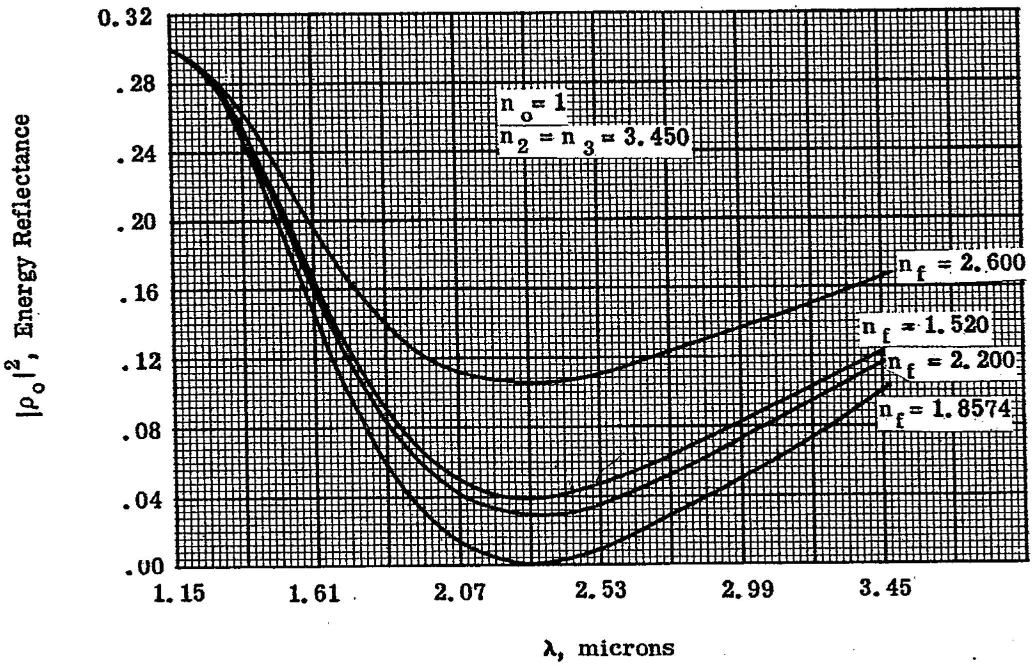


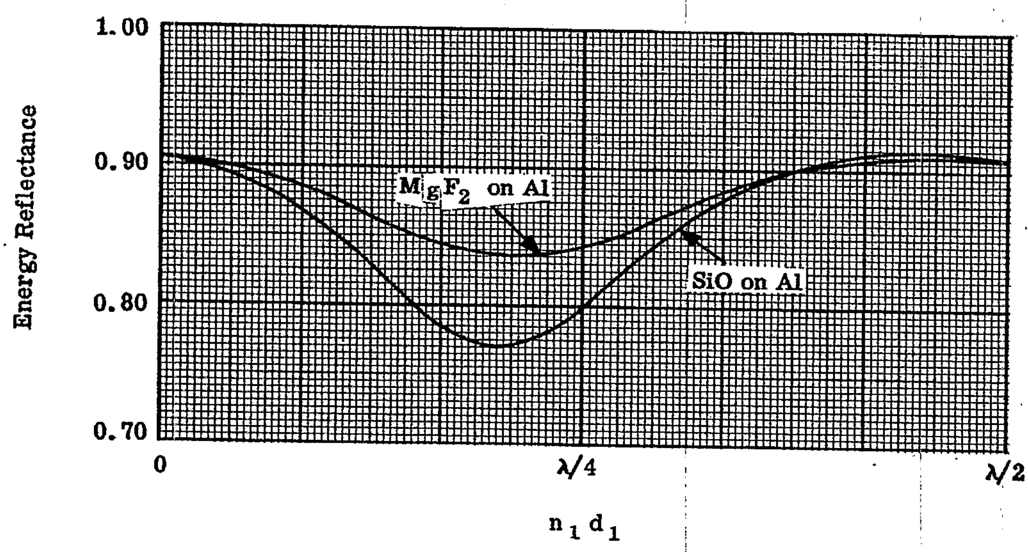
Figure 21. 13- Low reflecting monolayers on substrates having high refractive index.

21.6.6 Metallic monolayers on glass; normal incidence. The curves of Figure 21.16 a and b illustrate the optical behavior of monolayers of silver on non-absorbing substrates such as glass. Layers of silver differ from layers of other metals mainly in that layers of silver have remarkably low * absorption although the nK -value is moderately high. Silver is sensibly opaque of thicknesses near 0.15λ. It should be observed that the reflectance at the glass to film interface is less than that at the air to film interface -- a characteristic of most metals. As the thickness of the silver layer is increased, the phase changes on reflection pass into the third quadrant for reasons discussed following equation (16) in Section 21.2.3. The reflectance curve for reflection from glass to silver exhibits theoretically a minimum at a thickness between zero and 0.01λ.

21.6.7 Non-absorbing systems; oblique incidence. The manner in which low reflecting, non-absorbing monolayers modify the reflectances $|\rho_o|^2$ and $|\gamma_o|^2$ of surfaces of non-absorbing substrates is illustrated ** by Figures 21.17, 21.18, and 21.19 for the 45° angle of incidence. The spectral reflectance curves of Figure 21.17 for $|\rho_o|^2$ and $|\gamma_o|^2$ correspond to electric vectors that vibrate respectively perpendicular and parallel to the plane of incidence for a monolayer of $M_g F_2$ on a surface of spectacle crown glass. Whereas a monolayer of $M_g F_2$ is effective in reducing both $|\rho_o|^2$ and $|\gamma_o|^2$ for spectacle crown glass, the reflectance $|\rho_o|^2$ is still quite high. Figure 21.18 shows that the refractive index of the monolayer must approach the value $n_1 = 1.2$ in order to reduce $|\rho_o|^2$ below 1% for spectacle crown glass. Rugged films having refractive indices below 1.30 are not available. On the other hand, Figure 21.19 shows that one can choose a relatively high and available value of n_1 for reducing both $|\rho_o|^2$ and $|\gamma_o|^2$ to values below 1.2% when the refractive index n_2 of the substrate is markedly higher than that of spectacle crown glass. Figure 21.19 illustrates also the fact that $|\rho_o|^2_{min.} = |\gamma_o|^2_{min.}$, when $n_1 = \sqrt{n_o n_2}$. The effect of increasing or decreasing the angle of incidence from 45° is to raise or to lower, respectively, the reflectance at the crossing point C of Figure 21.19. It is to be expected that the reduction of $|\rho_o|^2$ presents the more formidable problem; for $|\gamma_o|^2$ is automatically zero at Brewster's angle of incidence.

*The amount of absorption is altered considerably by the conditions of evaporation. L. G. Schultz gives the values $n = 0.055$ and $nK = 3.32$ for silver at $\mu = 0.55$.

**As with many other subjects of this text, the possible number of instructive illustrations is restricted in the interests of conserving space.



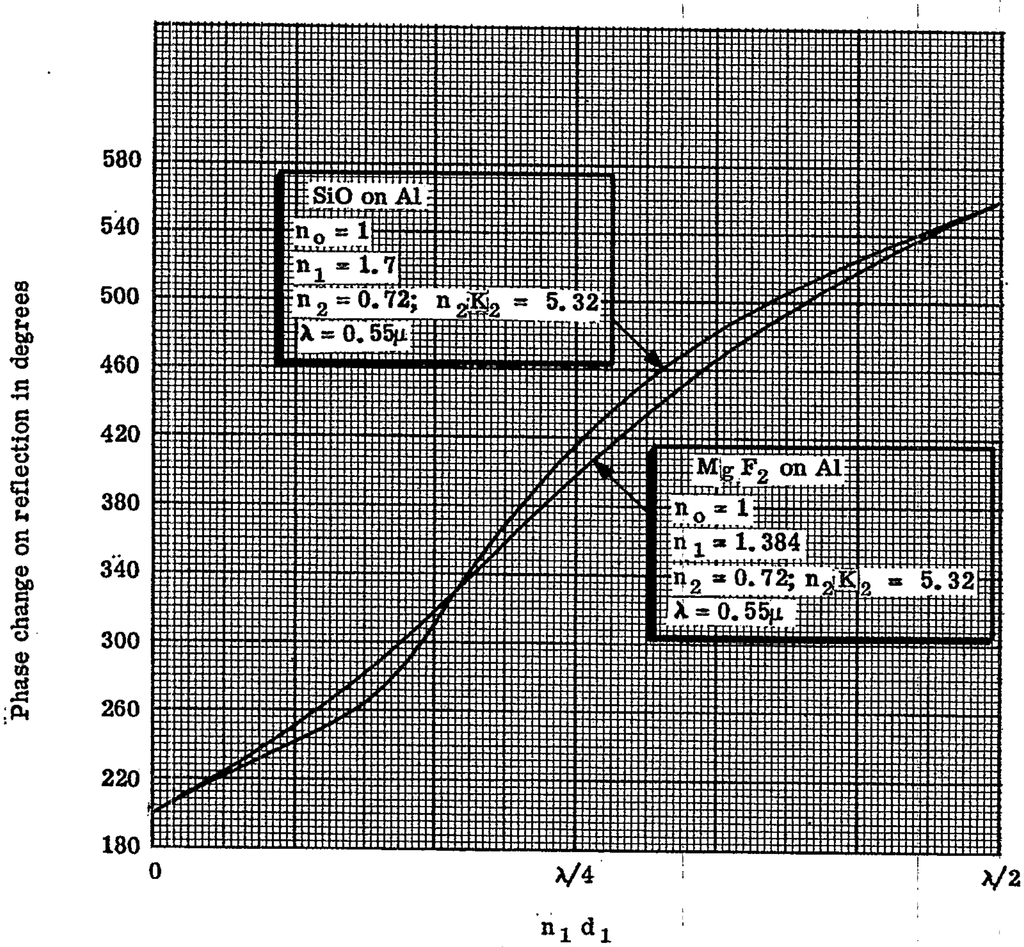


Figure 21. 14- Energy reflectances and phase changes on reflection vs optical path of the film in wavelengths for MgF_2 and SiO on opaque substrates of aluminum. Phase changes on reflection appear as phase retardations. Absorption by SiO monolayers has been neglected. The optical constants of aluminum are those given by L. G. Schultz, J. Opt. Soc. Amer., 44, 357-368 (1954).

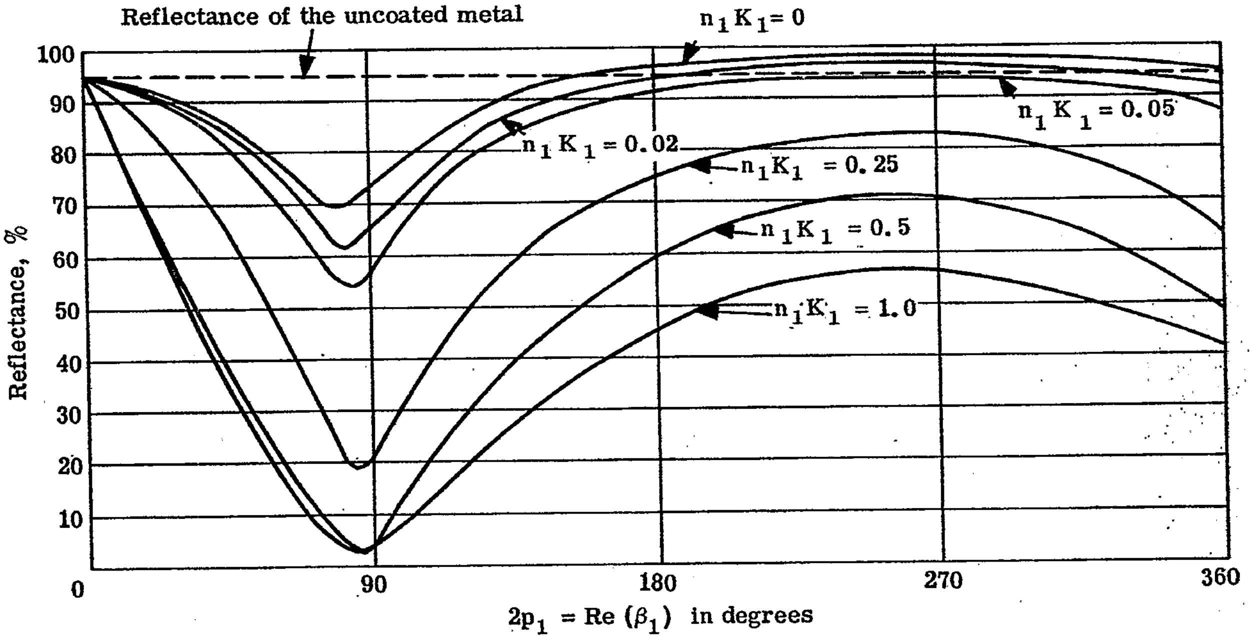


Figure 21. 15- Reflectances from various absorbing monolayers on a metallic substrate as functions of $R_e(\beta_1) = 2p_1$ where p_1 is the optical path through the monolayer. The metallic substrate corresponds to silver with the optical constants $n_2 = 0.15$ and $n_2K_2 = 3.28$. The monolayers have the high refractive index $n_1 = 4.0$ and the indicated values of n_1K_1. The case $n_1K_1 = 0$ is the nonabsorbing monolayer.

21.7 BILAYER COATINGS

21.7.1 Introduction. Bilayers possess advantages over monolayers for decreasing or increasing the reflectance of surfaces of dielectrics or metals. As examples, zero reflectance can be obtained at an assigned wavelength by suitable choice of the thickness ratio of the two layers and a marked degree of control over the distribution of spectral reflectance or transmittance becomes possible. Bilayers are superior to monolayers for beam splitters and for control of phase changes that occur on reflection or transmission. Whereas absorbing bilayers are used for such purposes as controlling the transmittance of sunglasses, the most important bilayers are predominantly dielectric. Low reflecting bilayers fall into well defined groups whose characteristics will be discussed.

21.7.2 Methods of computation. Matrix and quaternion methods have been treated in Sections 21.4 and 21.5. The reader who is inclined toward watching the interfacial reflectances and transmittances and toward using recursion formulae may choose the method of Sections 21.2.8 to 21.2.11. The convention and notation with respect to bilayers is shown in Figure 21.20. Suppose that the electric vector is perpendicular to the plane of incidence. We obtain the complex reflectance, ρ_o, in the form

$$\rho_o = \frac{\rho_1 e^{i\beta_1} + W_1}{1 + W_1 W_2 e^{i\beta_1}}, \qquad (182)$$

wherein

$$\rho_1 = \frac{W_3 e^{i\beta_2} + W_2}{1 + W_2 W_3 e^{i\beta_2}}. \qquad (182a)$$

The Fresnel coefficients of reflection W_ν are defined, as usual, by equations (19) and (20). β_ν ($\nu = 1, 2$) is defined by equation (56). The complex transmittance, τ_3, corresponding with ρ_o is obtained by setting $N = 2$, $\tau_o = 1$ and $\rho_2 = W_3$ in equation (54). The complex reflectance, γ_o, for cases in which the electric vector vibrates in the plane of incidence can be computed from equation (182), after replacing ρ_ν by γ_ν and W_ν by F_ν as defined by equation (21). The corresponding complex transmittance is denoted by T_3 and is obtained by setting $N = 2$, $T_o = 1$ and $\gamma_2 = F_3$ in equation (70). At normal incidence, $\gamma_o = \rho_o$ and $T_3 = \tau_3$. Closed formulae are available for the energy reflectances. These formulae simplify markedly

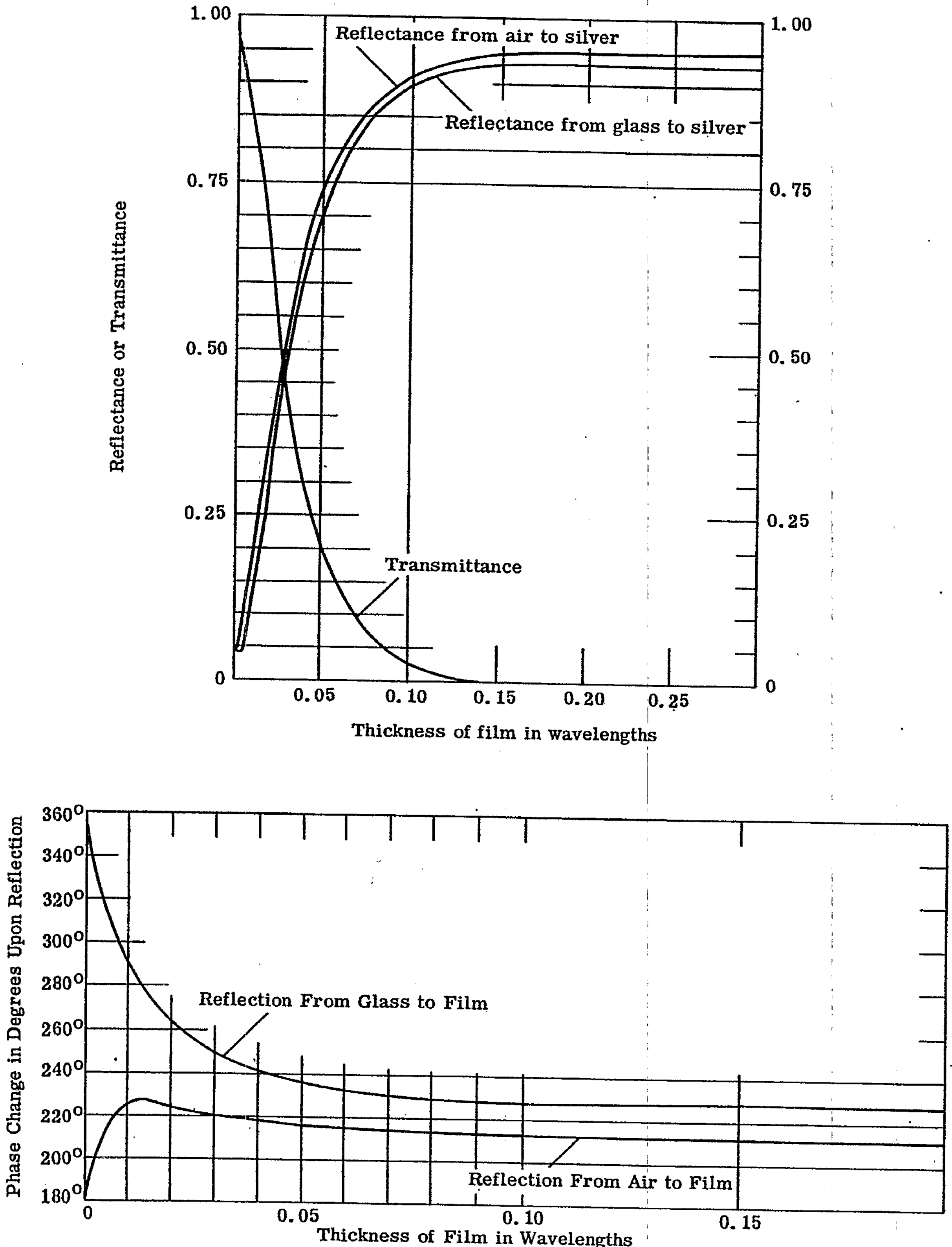


Figure 21. 16- Curves of reflectance, transmittance and phase changes on reflection vs thickness of the monolayer in wavelengths for silver films on spectacle crown glass. These curves have been computed for glass having the refractive index 1.52 and for an absorbing film having the optical constants $n_f = 0.15$ and $n_f K_f = 3.28$. These optical constants predate those measured for silver by L. G. Schultz and belong to a wavelength near 0.55 microns. The phase changes of Figure 21.16b are retardations. Phase retardations of 230° can be regarded, if desired, as phase advances of 130°.

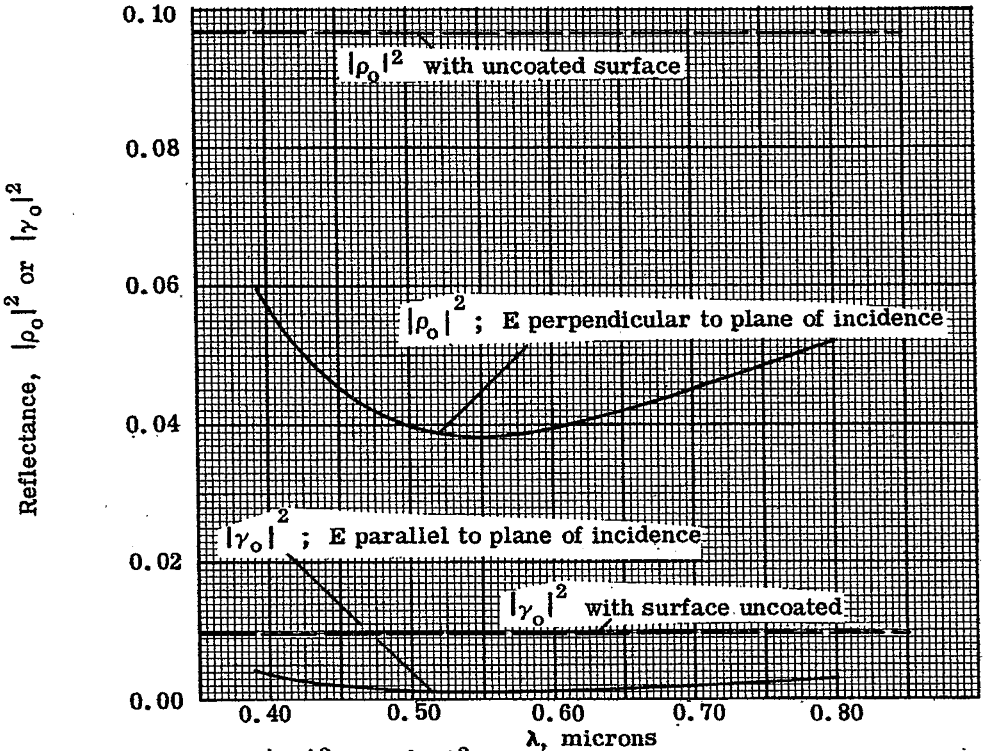


Figure 21. 17- The reflectances $|\rho_o|^2$ and $|\gamma_o|^2$ vs wavelength for a layer of magnesium fluoride on spectacle crown glass. The refractive indices are $n_o = 1$; $n_1 = 1.384$ and $n_2 = 1.52$ and the angle of incidence is 45^o. The thickness of the film has been chosen so that $\beta_1 = 4\pi n_1 d_1 \cos i_1/\lambda = \pi$ at $\lambda = 0.55\mu$.

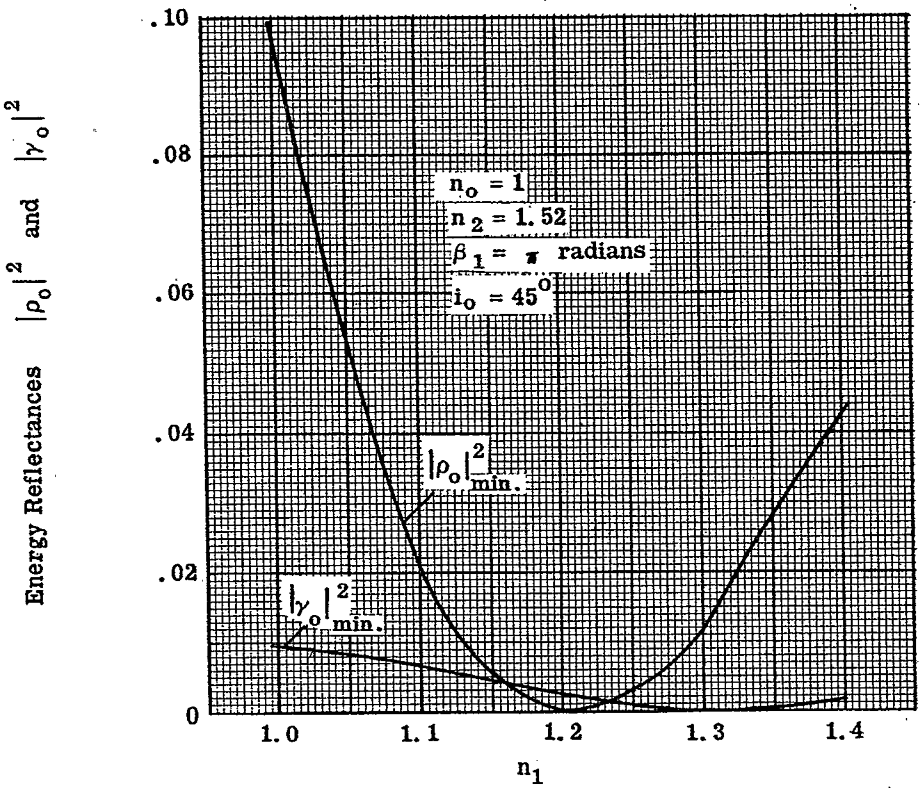


Figure 21. 18- Plot of the minimum reflectances $|\rho_o|^2_{min.}$ and $|\gamma_o|^2_{min.}$ against the refractive indices n_1 of the monolayers on spectacle crown glass at 45^o angle of incidence.

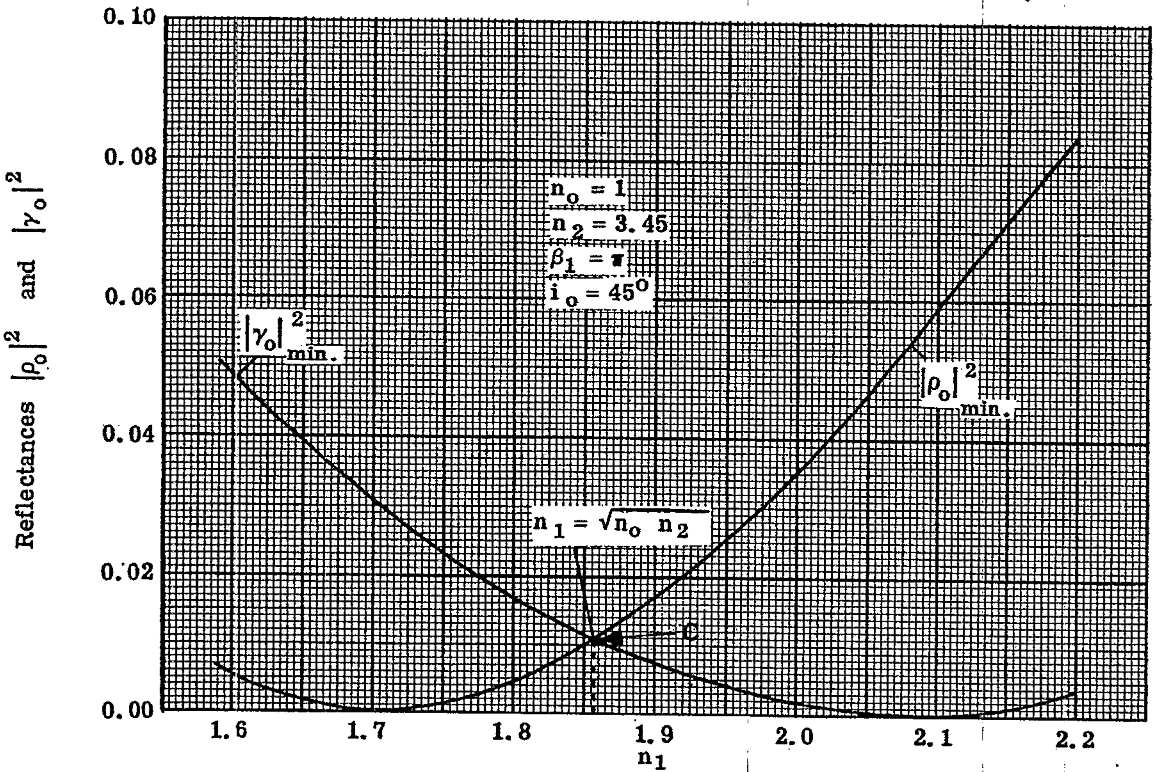


Figure 21. 19- Plot of the minimum reflectances $|\rho_o|^2_{min.}$ and $|\gamma_o|^2_{min.}$ against n_1 for monolayers on a nonabsorbing substrate of high refractive index $n = 3.45$ at the fixed angle 45^o of incidence.

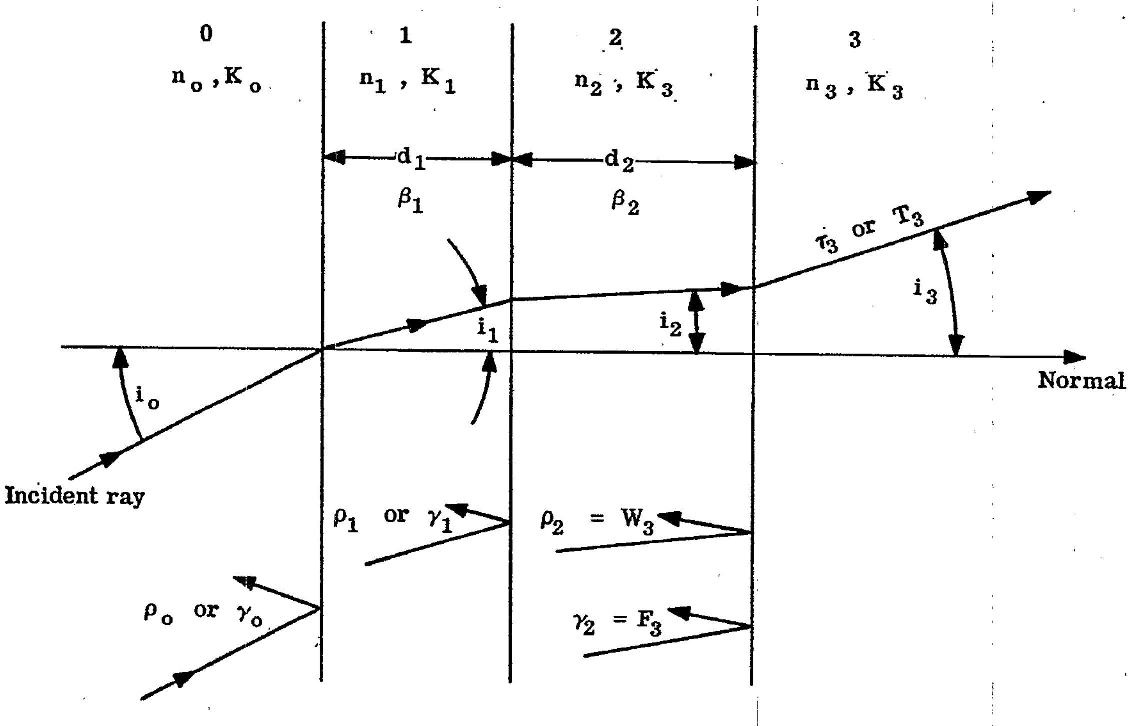


Figure 21. 20- Notation and convention with respect to bilayers (case N = 2).

only when all of the media and layers are non-absorbing and when total internal reflectance need not be considered. Thus,

$$|\rho_o|^2 = 1 - \frac{(1-W_1^2)(1-W_2^2)(1-W_3^2)}{D}, \quad (183)$$

where

$$D = 1 + W_1^2 W_2^2 + W_1^2 W_3^2 + W_2^2 W_3^2 + 2W_1 W_2(1 + W_3^2)\cos\beta_1 + 2W_2 W_3(1 + W_1^2)\cos\beta_2 + 2W_1 W_3 \cos(\beta_1 + \beta_2) + 2W_1 W_3 W_2^2 \cos(\beta_1 - \beta_2). \quad (183a)$$

W_ν and β_ν are now real with

$$\beta_\nu = \frac{4\pi}{\lambda} n_\nu d_\nu \cos i_\nu ; \quad \nu = 1, 2. \quad (183b)$$

Equation (183) can be used to compute $|\gamma_o|^2$ by replacing ρ_o by γ_o and W_ν by F_ν. Because absorption has been excluded, the sum of the energy reflectance and the energy transmittance must be unity when

$$|\tau_o| = |T_o| = 1.$$

21.7.3 The simplest low-reflecting bilayer. Monolayers such as $M_g F_2$ on glass fail to produce zero reflectance because $n_1 > \sqrt{n_o n_2}$. This class of monolayers is characterized by the fact that $n_o < n_1 < n_2$. Consequently, $W_1 < 0$ and $W_2 < 0$ at normal incidence. The following conclusions are not difficult to ascertain from equation (169) for normal incidence.

$$|W_2| < |W_1| \quad \text{when} \quad n_1 > \sqrt{n_o n_2} ; \quad (184)$$

$$|W_2| > |W_1| \quad \text{when} \quad n_1 < \sqrt{n_o n_2} . \quad (184a)$$

Hence the absolute value of the Fresnel coefficient W_2 at the film-to-glass interface is too small to satisfy the condition for zero reflectance when $n_1 > \sqrt{n_o n_2}$; for $W_1 = W_2$ at the point of zero reflectance. These considerations suggest that it should be possible to decrease the energy reflectance from monolayers for which $n_1 > \sqrt{n_o n_2}$ by depositing either a thin dielectric film that has high refractive index or a thin film of highly reflecting metal upon the interface between media no. 1 and 2 as illustrated in Figure 21.21. The author(9) has shown that metals can be used to augment the interfacial reflectance for obtaining zero reflectance. An advantage of this method is that a wide range of dielectric materials can be used as the monolayer by making suitable choices of the thickness of the thin metallic film. A disadvantage of employing metallic or absorbing films for augmenting the interfacial reflectance is that the energy reflectance for incidence from glass to film can be quite high even when the energy reflectance in the opposite direction from air to film has been made zero. This disadvantage as regards the irreversibility of the reflectance can be avoided by using dielectric materials of high refractive index as the reflectance augmenting layer in the manner described by Osterberg, Kashdan and Pride.(10) The use of dielectrics having high refractive index instead of metal yielded some unexpected advantages that will now be discussed.

21.7.3.1 Summary of the theory. In summarizing the results of an unpublished theory of the author, let us utilize the convention of Figure 21.22 in which layer no. 1 is to be the thin dielectric layer of high refractive index. All of the media are non-absorbing and the incidence is to be normal. One can show from the zero condition ($\rho_o = 0$) of equation (98) that the value of β_1 required for zero reflectance is given by

$$\cos\beta_1 = \frac{W_1^2 + W_2^2 - W_3^2(1 + W_1^2 W_2^2)}{2W_1 W_2 (W_3^2 - 1)}, \quad (185)$$

in which

$$\beta_1 = \frac{4\pi}{\lambda} n_1 d_1 \text{ radians.} \quad (185a)$$

(9) U. S. Patent 2,366,687 (Jan. 2, 1945).

(10) See abstracts of papers published in J. Opt. Soc. Amer., 42, p. 291 (1952).

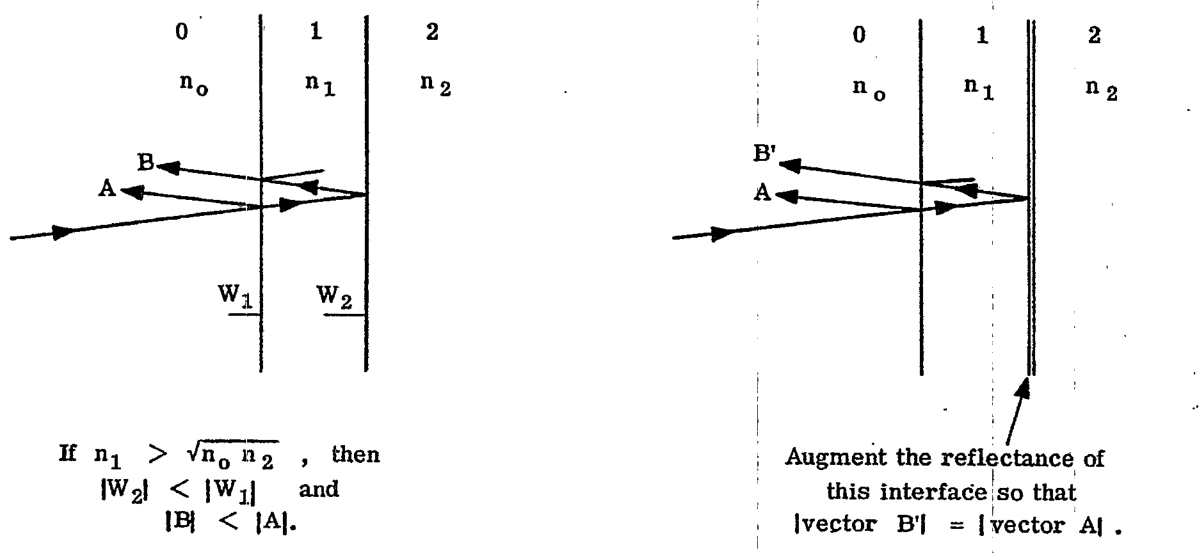


Figure 21.21- Illustration of the principle of augmenting one of the interfacial reflectances of a monolayer so as to obtain zero or decreased reflectance by depositing a thin, highly reflecting layer upon the interface.

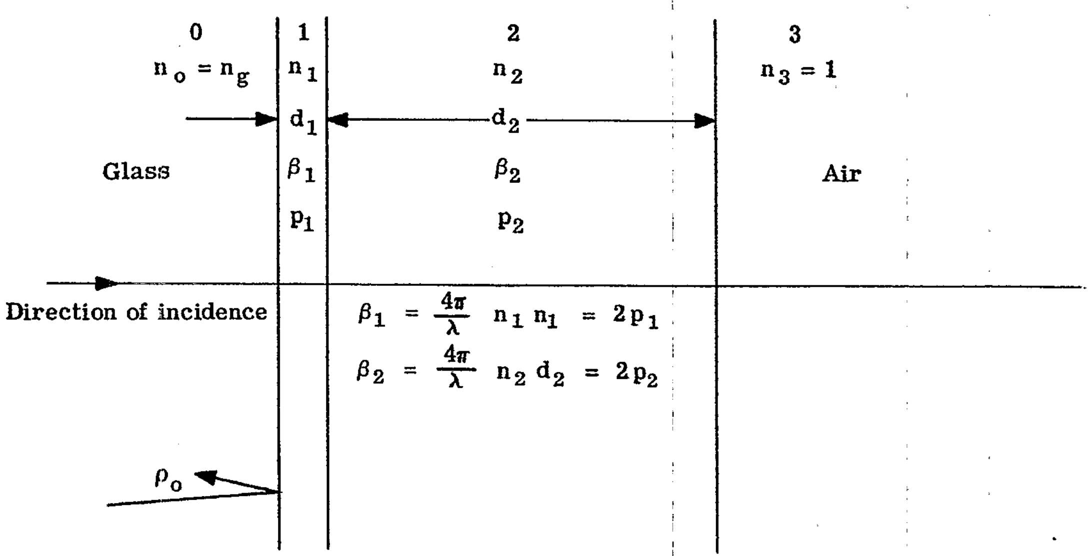


Figure 21.22- Notation with respect to the simplest low reflecting bilayer. Layer no. 1 is a thin film of high refractive index n_1 . p_1 and p_2 are the optical paths of the two layers.

Correspondingly, in terms of the refractive indices

$$\cos \beta_1 = 1 + \frac{2n_1^2 (n_g - 1)(n_g - n_2^2)}{(n_1^2 - n_g^2)(n_1^2 - n_2^2)} , \quad (185b)$$

for cases in which $n_3 = n_{air} = 1$. For solutions to exist, it is necessary that

$$\cos \beta_1 \leqq 1, \quad (185c)$$

a condition that places restrictions upon the refractive indices of equation (185b). Of these restrictions, the one of greatest interest is

$$n_1^2 > n_2^2 \; n_g > \left\{ \begin{matrix} n_2^2 \\ n_g^2 \end{matrix} \right. > n_g . \quad (185d)$$

If equation (185b) has a solution $\beta_1 = \beta_{10}$, it has other solutions that differ from β_{10} by integral multiples of 2π. We are to choose the smallest possible solution for which

$$0 < \beta_1 < \pi . \quad (185e)$$

Correspondingly, the optical path p_1 of the thin layer will be less than $\pi/2$. It can be much less than $\pi/2$, as we shall see. Having computed β_1, one can compute the associated value of β_2 from β_1 in the following manner. Let

$$\theta_1 = \tan^{-1} \frac{-W_1 \sin \beta_1}{-(W_1 \cos \beta_1 + W_2)} ; \quad (186)$$

$$\theta_2 = \tan^{-1} \frac{W_1 W_2 \sin \beta_1}{1 + W_1 W_2 \cos \beta_1} . \quad (186a)$$

Then

$$-\beta_2 = \theta_1 - \theta_2 ; \quad (\theta_2 > \theta_1). \quad (186b)$$

There exists no ambiguity as to the quadrant in which θ_1 and θ_2 fall because

$$\sin \theta_1 : -W_1 \sin \beta_1 ; \qquad \sin \theta_2 : W_1 W_2 \sin \beta_1 ;$$
$$\cos \theta_1 : -(W_1 \cos \beta_1 + W_2); \qquad \cos \theta_2 : 1 + W_1 W_2 \cos \beta_1 ; \quad (186c)$$

in which the factor of proportionality is greater than zero. Equations (185) and (186) thus enable one to compute * the doubled optical paths β_1 and β_2, Figure 21.12, of the two layers when n_1 has been suitably chosen with respect to n_g and n_2.

21.7.3.2 Let us now consider the case: $n_1 = 2.50$; $n_2 = 1.38$; $n_3 = 1$. This case applies to bilayers comprised of a thin inner layer of TiO_2 and an outer layer of MgF_2 under conditions at which the refractive indices of TiO_2 and MgF_2 are 2.50 and 1.38, respectively. The optical paths p_1 and p_2 required for vanishing reflectance have been computed from equations (185) and (186) and plotted as functions of n_g in Figure 21.23 (a) and (b). Curiously, the required optical path p_1 of the inner layer with refractive index $n_1 = 2.50$ is substantially 15.5° or 0.043λ for the range n_g of ordinary glasses. This fact has been confirmed experimentally and means that a TiO_2 layer of fixed thickness will serve for practically all glasses. This thickness is only $0.043/2.5 = 0.0172\lambda$. The required thickness approaches zero as n_g approaches $n_2^2 = 1.38^2$. The corresponding optical path p_2 of the outer layer must be chosen with some care since the slope of the curve of Figure 21.23(b) is quite marked. The optical path, p_2, exceeds the quarter wave condition considerably in the range, n_g, for ordinary glasses. However, the main effect of altering the optical path of the outer layer is to alter the wavelength at which minimum reflectance occurs. This class of bilayer is therefore relatively easy to produce. It is extremely durable. Cerium oxide is to be preferred to titanium dioxide as the inner layer because cerium oxide can be evaporated as a dielectric material without need for subsequent heating and oxidation. Equations (185) and (186) should be applied to redetermine p_1 and p_2 when cerium oxide is used. Comparisons of the bilayer (theoretical and experimental) with a monolayer of

*Frequently, one can be quite certain about the quadrant in which β_2 must fall. Under such circumstances it is simpler to compute β_2 from the formula

$$\cos \beta_2 = \frac{W_2^2 + W_3^2 - W_1^2 (1 + W_2^2 W_3^2)}{2 W_2 W_3 (W_1^2 - 1)} .$$

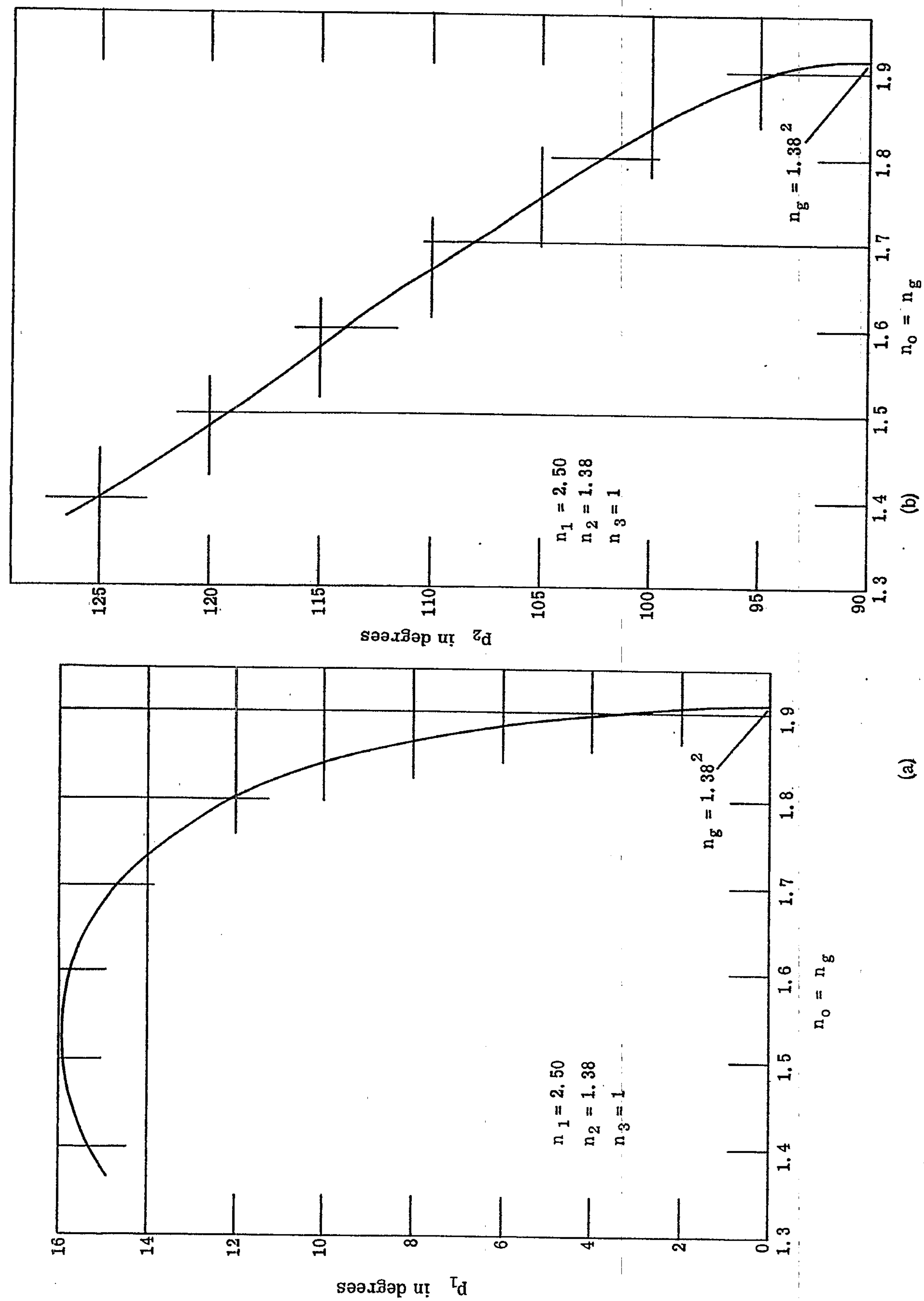


Figure 21.23- Plot of the optical paths p_1 and p_2 against the refractive index n_g of the substrate. p_1 and p_2 are the optical paths required to obtain zero reflectance for the layer of high refractive index $n_1 = 2.50$ and the layer of low refractive index n_2, respectively.

$M_g F_2$ on spectacle crown glass of refractive index $n_g = 1.52$ are made in Figure 21.24. The theoretical and experimental curves for the bilayer are in good agreement and exhibit much lower reflectances than monolayers of $M_g F_2$ near the wavelength for minimum reflectance. The dispersion of the reflectance is, however, markedly greater for the bilayer.

21.7.3.3 With reference to white light reflectance of the bilayer, T. Sawaki and H. Kubota investigated bilayers having low white light reflectance and found that those bilayers which simulate monolayers having augmented interfacial reflectance rank among the bilayers exhibiting the lowest white light reflectances.

21.7.4 Achromatized bilayers.

21.7.4.1 Curves of the computed * spectral reflectances at normal incidence for various achromatized bilayers on spectacle crown glass are illustrated in Figures 21.25 and 21.26. The outer layer is exemplified by $M_g F_2$. It has a refractive index n_1 which fails to meet the zero condition, $n_1 = \sqrt{n_o n_2}$, for monolayers. The outer and inner layers have the optical path $\lambda/4$ and $\lambda/2$, respectively, at a chosen ** wavelength, λ_m. The inner layer of refractive index n_2 is therefore an absentee layer at $\lambda = \lambda_m$. Consequently, the reflectance R_m at $\lambda = \lambda_m$ is due to the outer layer and the substrate of refractive index $n_3 = n_g$. When $n_2 > n_1$, the reflectance R_m is a maximum for wavelengths near λ_m. Achromatization occurs because the energy reflectance must drop before it can rise as λ is varied on either side of λ_m. With respect to the curves of Figure 21.25, the minimum reflectances on each side of point R_m are equal. Bilayers exhibiting this property will be classified as isoachromatic. The two minima of the spectral reflectance curve for the case $n_1 = 1.38$, $n_2 = 1.86$ and $n_2 = 1.52$ are zeros. Isoachromatic bilayers displaying zero minima will be called null-isoachromatic bilayers. The publications by A. F. Turner (11) and A. Vasicek (12) deal with null-isoachromatic bilayers.

21.7.4.2 The spectral reflectance curve for $n_2 = 1.52$, i.e. for the comparison monolayer of $M_g F_2$, of Figures 21.25 and 21.27 lies above the other spectral reflectance curves except at extreme values of λ and β_1. To substitute any one of the isoachromatic bilayers of Figures 21.25 and 21.27 in place of the monolayer of refractive index n_1 will therefore reduce the white light reflectance. The class of bilayer discussed in Section 10.7.3 and its subsections is, however, much superior for reducing white light reflectance. Except for specialized applications, achromatic bilayers are to be preferred when increased neutrality of spectral reflectance becomes important. For this purpose, the null-isoachromatic bilayers are not as suitable as the isoachromatic bilayers exemplified by the curve for $n_2 = 1.55$ of Figure 21.27. The flatness of this curve is quite remarkable. Comparison of Figures 21.25 and 21.26 shows that a wide range of distributions of spectral reflectances can be attained by means of achromatized bilayers. Figure 21.28 has been included to show how the separation of the minima at points A and B can be reduced by choosing n_1 nearer to the value $n_1 = \sqrt{n_o n_g}$.

21.7.4.3 The theoretical design of null-achromatic bilayers will now be developed algebraically in a manner that illustrates one use of the method of admittances. From equations (81a) and (81b), one obtains for normal incidence the results,

$$Y_3 = -n_3; \tag{187}$$

$$Y_1 = n_1 \frac{Y_2 - in_1 \tan p_1}{n_1 - in_2 \tan p_1}; \quad p_1 = \frac{\beta_1}{2}; \tag{187a}$$

$$Y_2 = n_2 \frac{Y_3 - in_2 \tan p_2}{n_2 - iY_3 \tan p_2}; \quad p_2 = \frac{\beta_2}{2}. \tag{187b}$$

The condition for zero reflectance is $\rho_o = 0$ or, from equation (79a),

$$Y_1 = -M_o = -n_o = -1 \tag{188}$$

since we shall suppose that the medium of incidence is vacuum. By eliminating the admittances Y_1, Y_2 and Y_3, one obtains quite directly the condition for zero reflectance in its general form for non-absorbing bilayers, namely,

$$n_1 \frac{1 - in_1 \tan p_1}{n_1 - i \tan p_1} = n_2 \frac{n_3 + in_2 \tan p_2}{n_2 + in_3 \tan p_2}; \tag{188a}$$

* The materials of this discussion have been taken from unpublished research notes of the author.

** λ_m is often chosen as 0.55μ when the bilayer is intended for the visible region.

(11) A. F. Turner, J. de Phys., 11, 444 (1950).

(12) A. Vasicek, Optica Acta, May 1951, special issue, pp. 20-25.

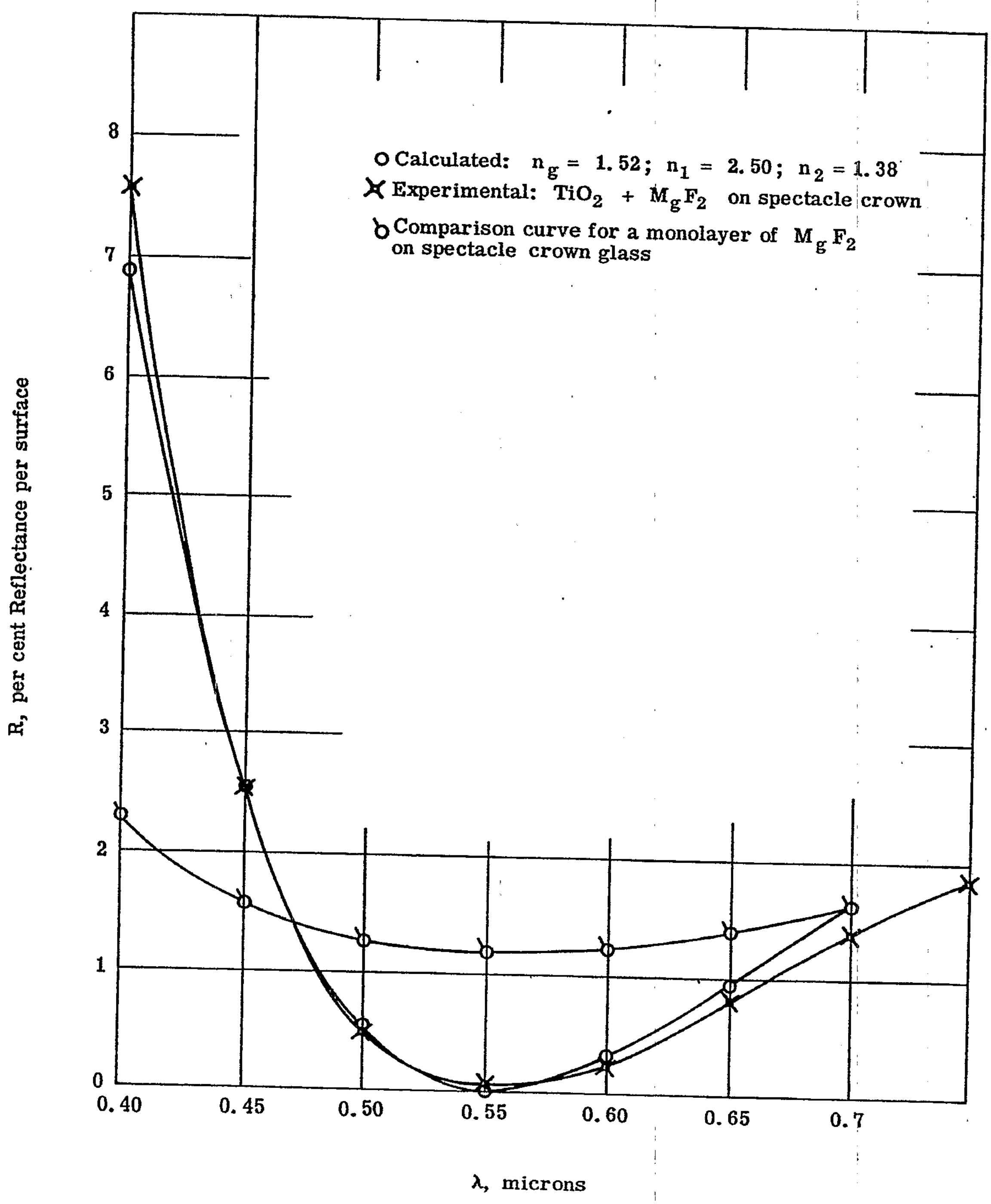


Figure 21.24- Comparison of the computed and experimental reflectances from bilayers of $TiO_2 + M_gF_2$ on spectacle crown glass of refractive index 1.52. In making the computations it has been assumed that the refractive indices remain fixed at the values $n_g = 1.52$; $n_1 = 2.50$ and $n_2 = 1.38$. A spectral reflectance curve for monolayers of M_gF_2 on spectacle crown glass is included for further comparison.

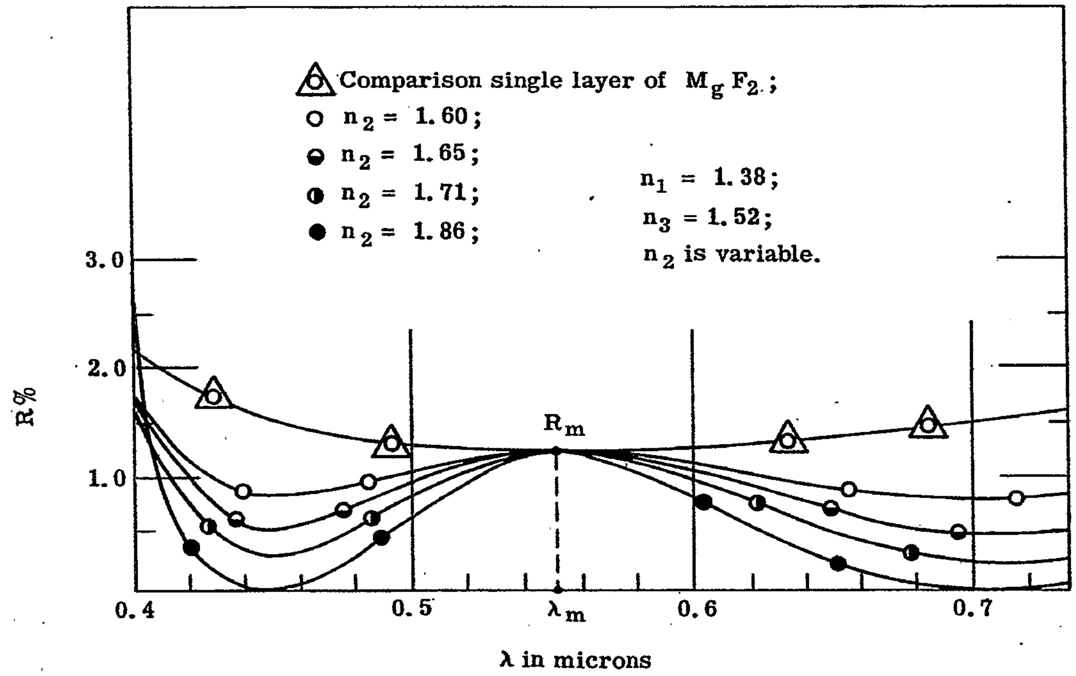


Figure 21.25- Spectral reflectance curves for a family of isoachromatic bilayers on spectacle crown glass of refractive index 1.52.

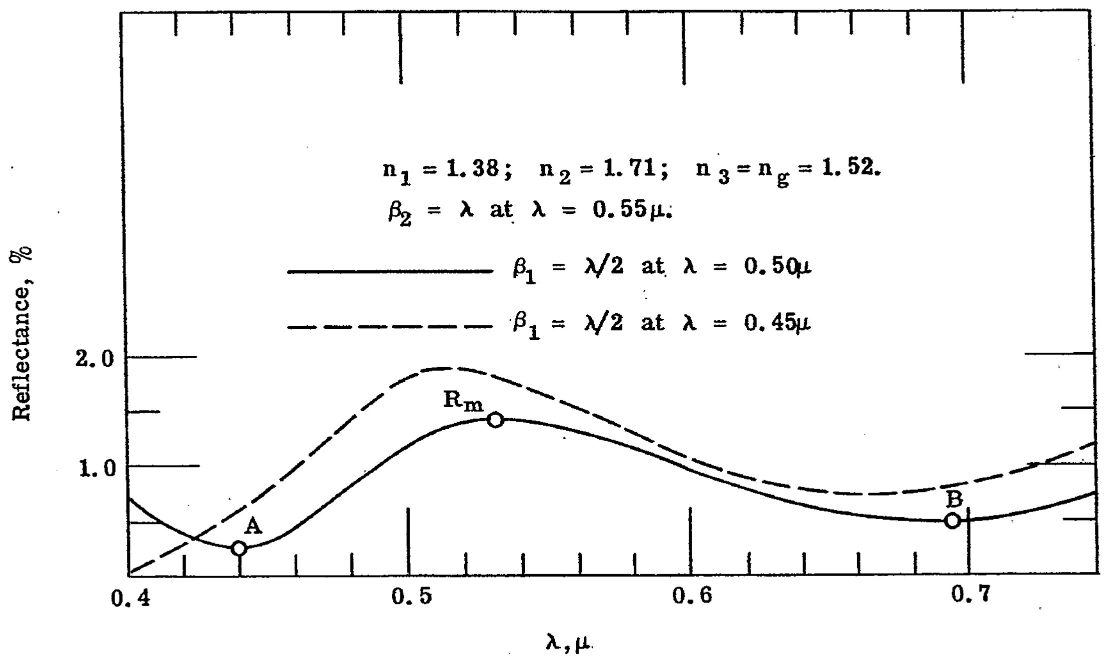


Figure 21.26- Spectral reflectance curves for two achromatic but not isoachromatic bilayers on spectacle crown glass. These curves illustrate the trend wherein the minimum reflectances at A and B become unlike when the outer layer of low refractive index and the inner layer of higher refractive index are not quarter wave and half wave, respectively, in optical path at the same wavelength.

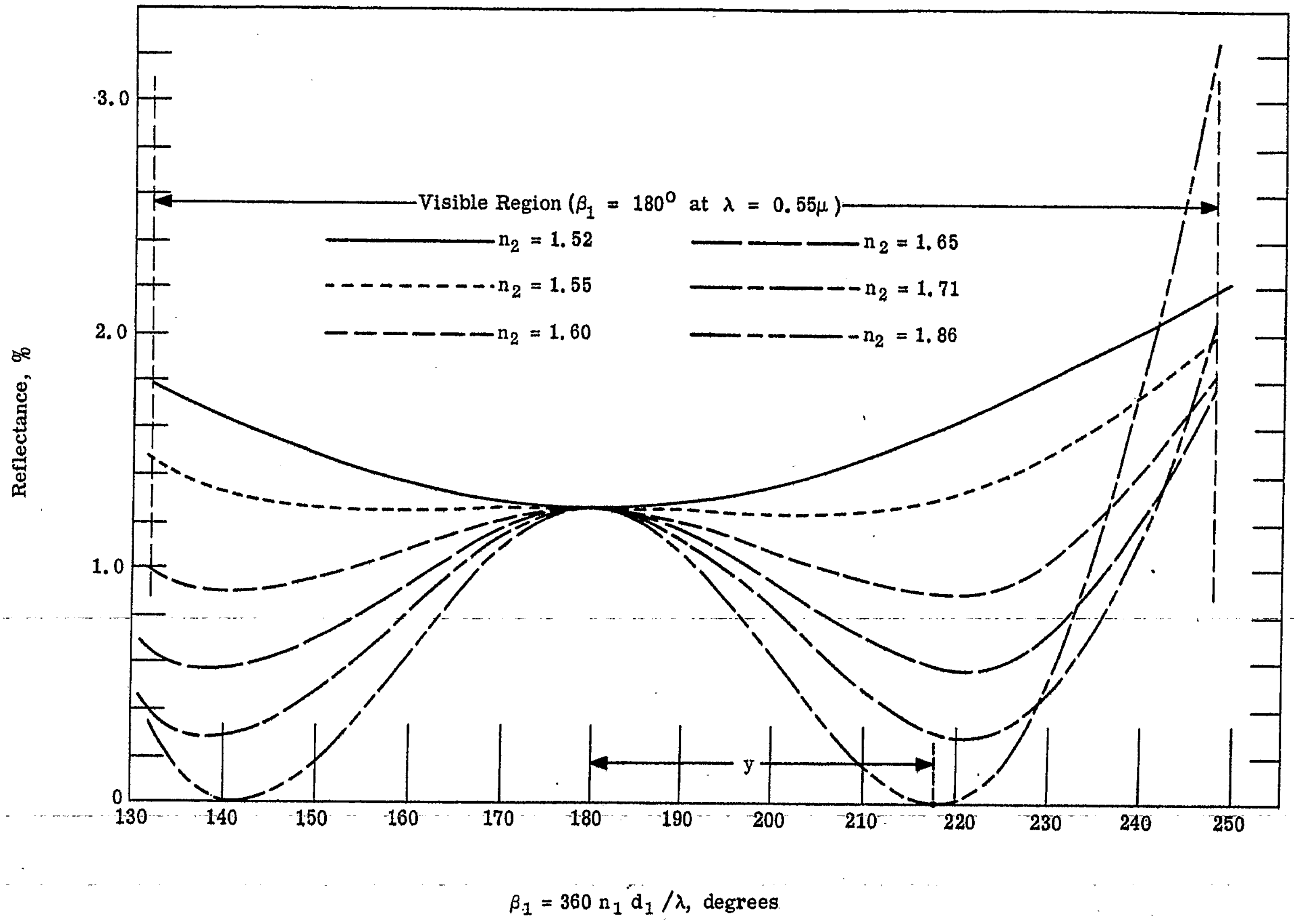


Figure 21.27- Curves illustrating the symmetry of the energy reflectances of isoachromatic bilayers about the point $\beta_1 = 180^\circ$ where β_1 is twice the optical path of the outer layer of low refractive index $n_1 = 1.38$ (corresponding to MgF_2). The refractive index of the substrate is 1.52 as in Table 21.14.

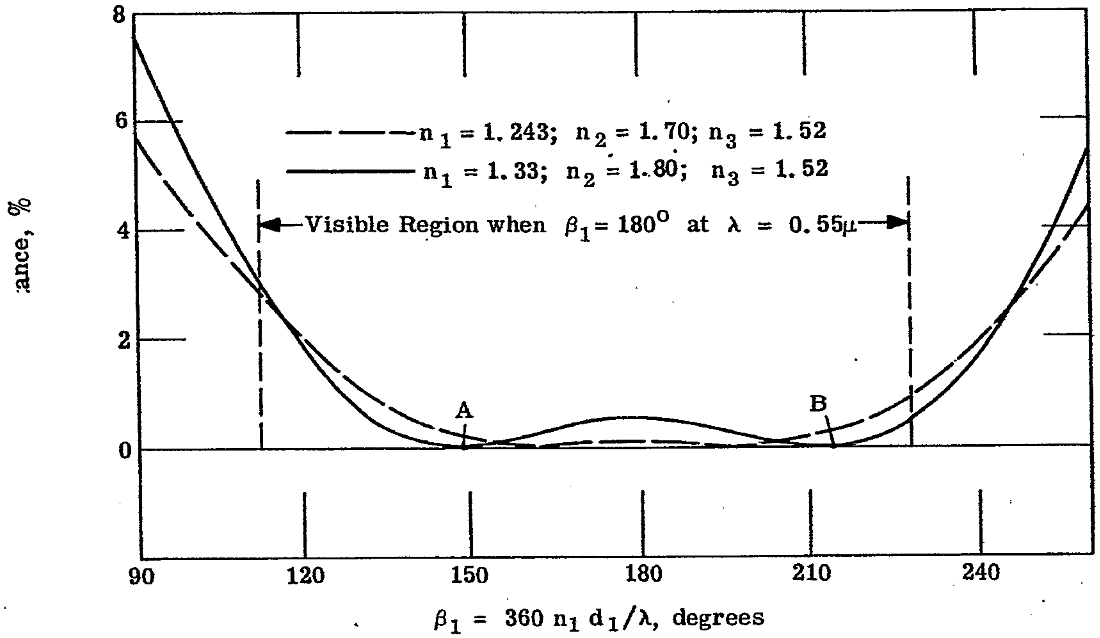


Figure 21.28- Curves showing the effects of reducing the refractive index n_1 of the outer layer at fixed values of n_3.

in which

$$p_1 = 2\pi n_1 d_1/\lambda; \quad p_2 = 2\pi n_2 d_2/\lambda. \tag{188b}$$

We suppose at this point that the ratio n_2/n_1 does not depend upon wavelength and introduce

$$p_2 = f\, p_1, \tag{188c}$$

in which f can be assigned any desired value. Investigation shows that null-isoachromatic bilayers are possible mathematically for the choice $f = 1$, but require values of n_1 that are usually too small to obtain physically. The choice $f = 2$ leads to the null-isoachromatic bilayers that are of interest to this section. Equating real and imaginary parts on each side of equation (188a), after clearing fractions and introducing

$$p_2 = 2p_1 = 2p, \tag{189}$$

one obtains the zero condition in the form

$$n_1(1 - n_3) + \left[\frac{2n_1^2 n}{n_2} - 2n_2 - n_1(1 - n_3)\right] \tan^2 p = 0; \tag{189a}$$

$$\frac{2n_1 n_3}{n_2} - 2n_1 n_2 + n_3 - n_1^2 - (n_3^2 - n_1^2)\tan^2 p = 0; \tag{189b}$$

By eliminating $\tan^2 p$, one obtains the result

$$n_2 n_3 (n_1^2 + 1)(n_1 + n_2) - 2n_1(n_2^3 + n_1 n_3^2) = 0, \tag{190}$$

an equation for determining n_2 from n_1 and n_3 or n_2 from n_1 and n_3. One obtains also

$$\cos 2p = \cos \beta_1 = n_1 \frac{n_2^2 - n_3}{n_1(n_3 - n_2^2) + n_2(n_3 - n_1^2)}. \tag{190a}$$

Introduce

$$2p = \beta_1 = 180^\circ \pm y \, , \tag{190b}$$

where y is indicated in Figure 21.27. Then the solution for y becomes

$$\cos y = n_1 \frac{n_2^2 - n_3}{n_1 (n_2^2 - n_3) + n_2 (n_1^2 - n_3)} \, . \tag{190c}$$

The required refractive indices and the optical paths, p_1 and p_2, of the two members of the bilayer thus become known algebraically.

21.7.5 Quarter wave bilayers.

21.7.5.1 Non-absorbing, quarter wave bilayers on non-absorbing substrates comprise a third, important class of bilayers. In the interests of brevity, the following discussion will be restricted to normal incidence.

21.7.5.2 Let us suppose that layer number two, Figure 21.20, is a quarter wave layer. Then $\beta_2 = \pi$ and ρ_1, equation (182a), will be real for non-absorbing systems. Consequently from equation (182),

$$|\rho_o|^2 = \frac{\rho_1^2 + W_1^2 + 2\rho_1 W_1 \cos \beta_1}{1 + \rho_1^2 W_1^2 + 2\rho_1 W_1^2 \cos \beta_1} \, . \tag{191}$$

By differentiating $|\rho_o|^2$ with respect to β_1, one finds that the condition for maxima and minima is $\sin \beta_1 = 0$ or $\beta_1 = \nu\pi$ where ν is an integer. The choice $\nu = 1$ makes layer number one a quarter wave layer. If, therefore, both members of the bilayer are quarter wave layers at a given wavelength, λ_o, then the energy reflectance $|\rho_o|^2$ is either a minimum or a maximum for wavelengths in the immediate neighborhood of λ_o. Maxima and minima can occur at wavelengths removed from λ_o but at these wavelengths the bilayer will not be a quarter wave bilayer.

21.7.5.3 Consider now the recursion formula (81a) at wavelength λ_o for which $\beta_1 = \beta_2 = \pi$. One obtains

$$Y_{\nu-1} = \frac{M_{\nu-1}^2}{Y_\nu} = \frac{n_{\nu-1}^2}{Y_\nu} \, . \tag{192}$$

Whence

$$Y_1 = \frac{n_1^2}{Y_2} \; ; \quad Y_2 = \frac{n_2^2}{Y_3} \, . \tag{192a}$$

From equation (81b) we note that $Y_3 = -n_3$. Therefore the admittance Y_1 of the quarter wave bilayer is given by

$$Y_1 = -\frac{n_1^2}{n_2^2} n_3 \tag{192b}$$

at $\lambda = \lambda_o$.

21.7.5.4 Equations (79a) and (192b) give the complex reflectance ρ_o of the non-absorbing bilayer on a non-absorbing substrate at $\lambda = \lambda_o$ and at normal incidence in the form

$$\rho_o = \frac{n_o - \left(\frac{n_1}{n_2}\right)^2 n_3}{n_o + \left(\frac{n_1}{n_2}\right)^2 n_3} \, , \tag{193}$$

from which the energy reflectance $|\rho_o|^2$ is either a maximum or a minimum. The condition that must exist among the refractive indices to obtain zero reflectance is

$$n_o n_2^2 = n_3 n_1^2 \, . \tag{194}$$

Available materials[13] do not ordinarily meet the zero condition of equation (194).

21.7.5.5 Let us now consider the case $n_o = 1$. In most applications, the medium of incidence is air for which n_o may be set at the approximate value unity. Then

$$R = |\rho_o|^2 = \left(\frac{1 - \left(\frac{n_1}{n_2}\right)^2 n_3}{1 + \left(\frac{n_1}{n_2}\right)^2 n_3} \right)^2 ; \quad \lambda = \lambda_o ; \qquad (195)$$

where R denotes energy reflectance. The manner in which R depends on the choice of n_1 and n_2 is illustrated in Figure 21.29 for the case $n_3 = n_g = 1.52$. Whereas only restricted generalizations can be made about combinations n_1 and n_2 that produce reflectances less than the reflectance of the uncoated substrate, we may conclude that the reflectance of the bilayer exceeds that of the uncoated substrate whenever $n_1 > n_2$ for the case $n_o = 1$. To obtain a high reflecting bilayer, one should deposit first the layer having the lower refractive index n_2.

21.7.6 Non-quarter wave bilayers.

21.7.6.1 Quarter wave bilayers will rarely satisfy the zero condition (194). However, when equation (194) is not satisfied by the refractive indices n_1 and n_2, it may be possible to choose β_1 and β_2 different from 180° in such a manner as to meet the more general zero condition. The corresponding bilayers belong to a fourth important class. The bilayers of Section 21.7.3 modified by adding $\lambda/2$ to the optical path of the inner layer of high refractive index are examples of this fourth class of bilayers. Of greatest interest are those bilayers for which β_1 and β_2 depart only slightly from 180°. The exact method of equations (185) to (185b) and equations (186) can be applied to find members of this fourth class.

(13) For a discussion of methods of chemical deposition that utilize a mixture of materials having high and low refractive indices in order to obtain films having refractive indices in the approximate range 1.44 to 2.1, see U. S. Pat. 2466119, April 15, 1949 by H. R. Moulton and E. D. Tillyer.

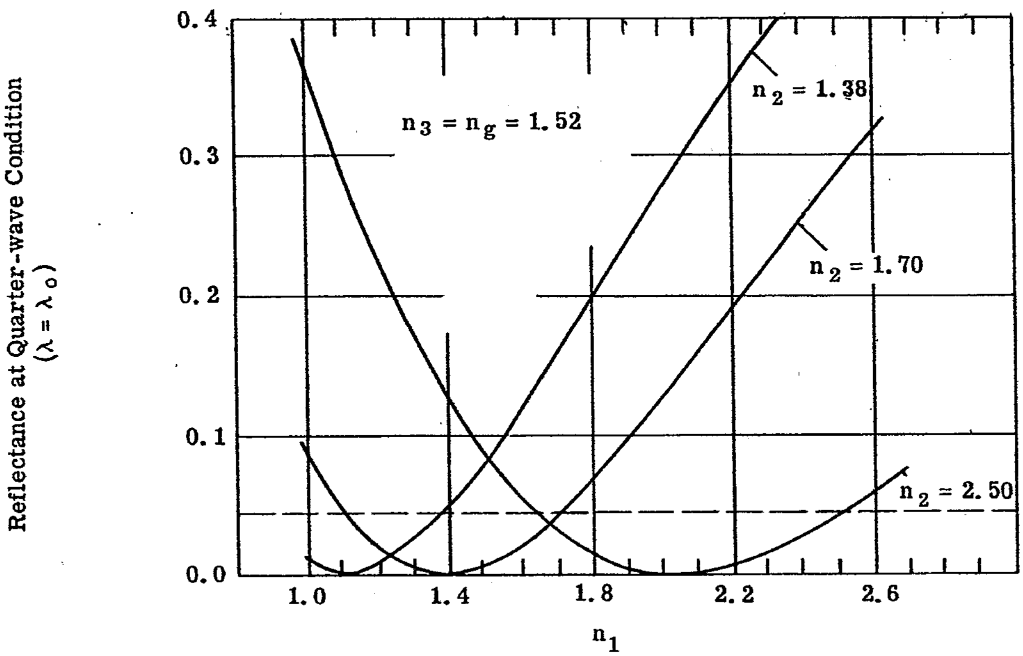


Figure 21.29- Plots of energy reflectances at $\lambda = \lambda_o$ vs the refractive index n_1 of the outer layer for the indicated values of the refractive indices n_2 of the inner layer with $n_3 = n_g = 1.52$. The broken line indicates the reflectance of the uncoated substrate, i.e. reflectance from air to the uncoated glass of refractive index $n_g = 1.52$.

21.7.6.2 The early investigators of thin films utilized a graphical method, involving closed polygons, for finding multilayers having zero reflectance. The approximate method of Section 21.2.11 is the algebraic equivalent of the method of closed polygons when applied to the zero condition of equation (98). By inserting ρ_1 from equation (75) into equation (98), one finds that

$$W_3 \; e^{i(\beta_1 + \beta_2)} + W_2 \; e^{i\beta_1} + W_1 = 0. \qquad (196)$$

By equating the real and imaginary parts of the left hand member of equation (196) to zero, one obtains the zero condition in the form

$$\sin(\beta_1 + \beta_2) = -\frac{W_2}{W_3} \sin\beta_1 \qquad (197)$$

$$W_3 \cos(\beta_1 + \beta_2) + W_2 \cos\beta_1 + W_1 = 0. \qquad (197a)$$

Elimination of $(\beta_1 + \beta_2)$ from equation (197a) with the aid of equation (197) yields straightforwardly the result

$$\cos\beta_1 = \frac{W_3^2 - W_1^2 - W_2^2}{2 W_1 W_2} . \qquad (197b)$$

Equations (197) and (197b) determine in a simple manner first β_1 and then β_2. As the first example, consider the case in which $n_o = 1$, $n_1 = 1.38$, $n_2 = 1.72$ and $n_3 = 1.52$. With reference to the curve for $n_2 = 1.70$, Figure 21.29, n_2 is then a little too high to obtain zero reflectance when $\beta_1 = \beta_2 = 180^o$. Equations (197b) and (197) yield four pairs of solutions for β_1 and β_2. These are

$$\beta_1 = 164^o \; 16' \; ; \; \beta_2 = \begin{cases} 224^o \; 36' \\ 346^o \; 52' \end{cases} ; \qquad (198)$$

and

$$\beta_1 = 199^o \; 44' \; ; \; \beta_2 = \begin{cases} 135^o \; 24' \\ 13^o \; 08' \end{cases} . \qquad (198a)$$

The case $\beta_1 = 199^o \; 44'$ and $\beta_2 = 13^o \; 08'$ belongs to the classification of Section 21.7.3. In case $\beta_1 = 164^o \; 16'$ and $\beta_2 = 224^o \; 36'$, the optical paths $\beta_1/2$ and $\beta_2/2$ are most nearly equal to 90^o. On the other hand, when one examines the examples $n_o = 1$, $n_2 = 1.68$, $n_3 = 1.52$ with $n_1 = 1.38$ and 1.384, he finds that $|\cos\beta_1| > 1$. Although the physical parameters have been changed only slightly, the method of closed polygons does not admit solution.

21.7.7 High reflecting bilayers on metals. Bilayers of non-absorbing films can be used for gaining significant increases in reflectance from metals. For example, a very durable bilayer of silicon monoxide and titanium oxide for increasing the reflectance of an evaporated aluminum mirror has been described by G. Hass.[14]

21.8 TRILAYERS

21.8.1 Introduction. The main interest in trilayers has centered on the possibilities which they provide for obtaining lower and flatter curves of spectral reflectances than is feasible with bilayers. Trilayers can exhibit three minima in spectral reflectance over the visible region. In extreme cases all three of these minima can be zero minima. The so called quarter-half-three quarter wave trilayer [15] is advantageous for achromatization. *

21.8.2 Low reflectance trilayers. The behavior of the quarter-half-quarter wave type of low reflecting trilayer is indicated in Figure 21.30. It should be noted that the central layer is a half-wave (and hence absentee) layer at a wavelength λ_o at which the inner and outer layers are quarter wave layers. Consequently, the trilayer behaves in effect as a quarter wave bilayer at $\lambda = \lambda_o$. The condition for zero reflectance can be found by considering equation (194) for quarter wave bilayers. Thus, $n_o^2 \; n_3^2 = n_4 \; n_1^2$ when the refractive indices

(14) Georg Hass, Vacuum, 2, p 339 (1952).

(15) For an example and discussion of this class of trilayer see O. S. Heavens, Optical Properties of Thin Solid Films, Butterworths Scientific Publications, London (1955), pp 213-215.

* The term apochromatization would be more appropriate when the film is designed to have three minima, i.e. is "corrected" at three wavelengths.

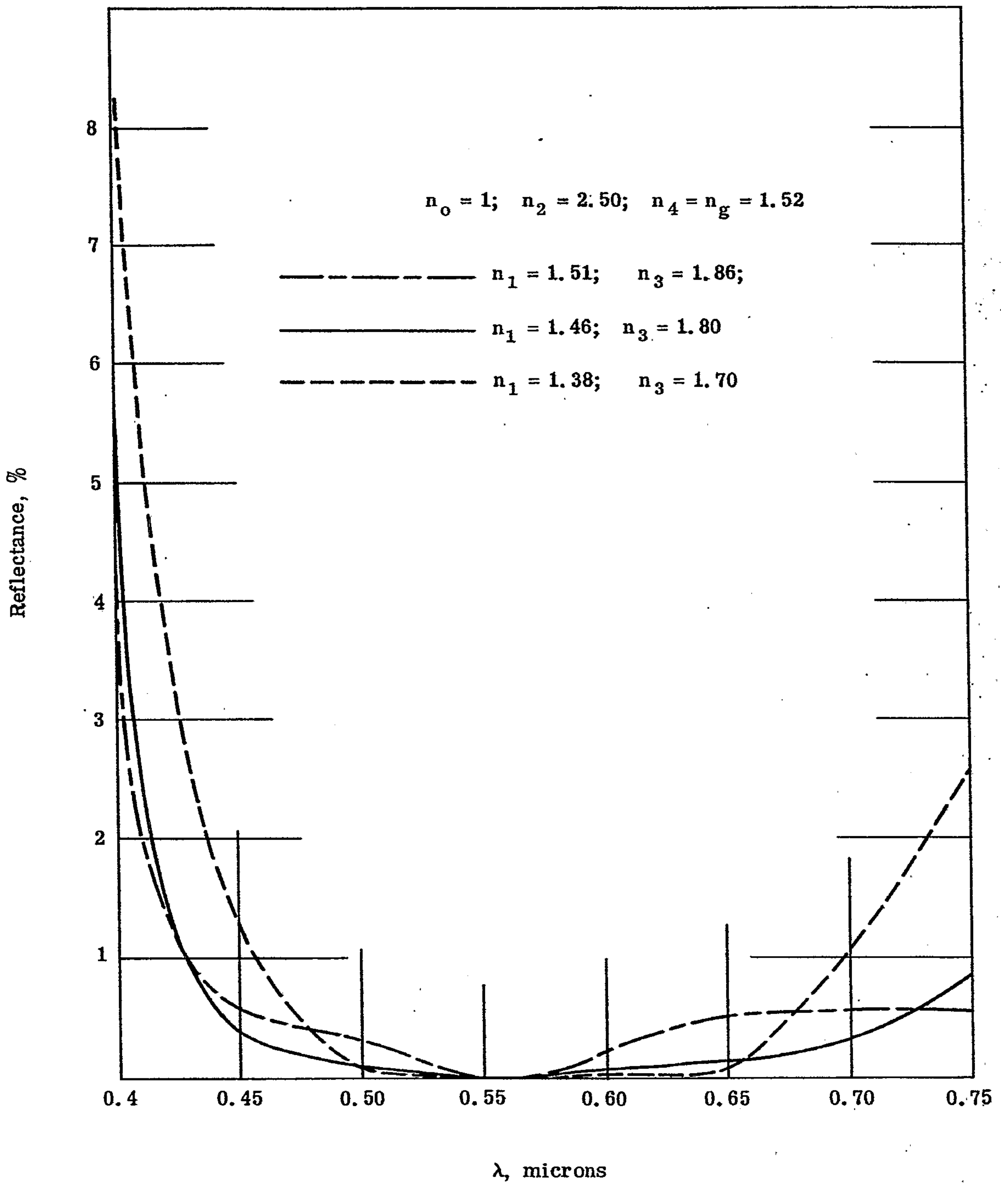


Figure 21.30- Spectral reflectances of quarter-half-quarter-wave trilayers on a substrate of refractive index 1.52. The high index $n_2 = 2.50$ belongs to the central, half-wave layer. The curves illustrate the effects of altering the refractive indices n_1 and n_3 of the quarter wave layers. The curves of this figure have been computed by the approximate method of section 21.2.11.

are properly identified with those of the trilayer. The condition for zero reflectance at $\lambda = \lambda_o$ is therefore

$$n_3 = \frac{n_1}{n_o} \sqrt{n_4} \,. \tag{199}$$

The refractive indices n_1 and n_3 of Figure 21.30 have been chosen in accordance with equation (199). The curves of Figure 21.30 are quite flat and low from 0.45 to 0.65 microns. They tend toward achromatization but do not succeed. The performances of the quarter-half-quarter and quarter-half-three quarter wave trilayers as anti-reflection films are not far different.

21.8.3 Band pass trilayers. A trilayer consisting of a dielectric layer silvered on both major surfaces is essentially a Fabry-Perot interferometer. Like Fabry-Perot interferometers, such trilayers are readily designed to transmit narrow bands of wavelengths after the manner described and illustrated in Section 16.16. These trilayers are utilized as narrow pass band filters. By forming the dielectric layer as a wedge of gradual taper, a convenient and effective monochromator is achieved. Thin film theory involves the assumption that the number of interreflections within each film is infinite. When regarded as Fabry-Perot interferometers, the trilayers discussed here require the choice of equation 16-(107) rather than 16-(106).

21.9 QUADRILAYERS

21.9.1 A low reflecting quadrilayer. The quarter-half-quarter trilayer discussed in Section 21.8.2 has been modified in a significant manner by Dr. Helen Jupnik so as to achieve a more practical anti-reflection film that has excellent performance. Difficulties occur in making the trilayers of Figure 21.30 because the refractive indices n_1 or n_3 or both are not available as durable, non-absorbing materials. To overcome difficulty due to availability of a material having the most desired refractive index n_3, Dr. Jupnik replaces the corresponding quarter wave layer by an "equivalent" quarter wave bilayer having refractive indices n'_3 and n'_4 as illustrated in Figure 21.31.

21.9.2 The principle of equivalence. The principle of equivalence is so important to the theory of thin films that its application to Jupnik's quadrilayer will be considered in detail. The substituted bilayer shall be a

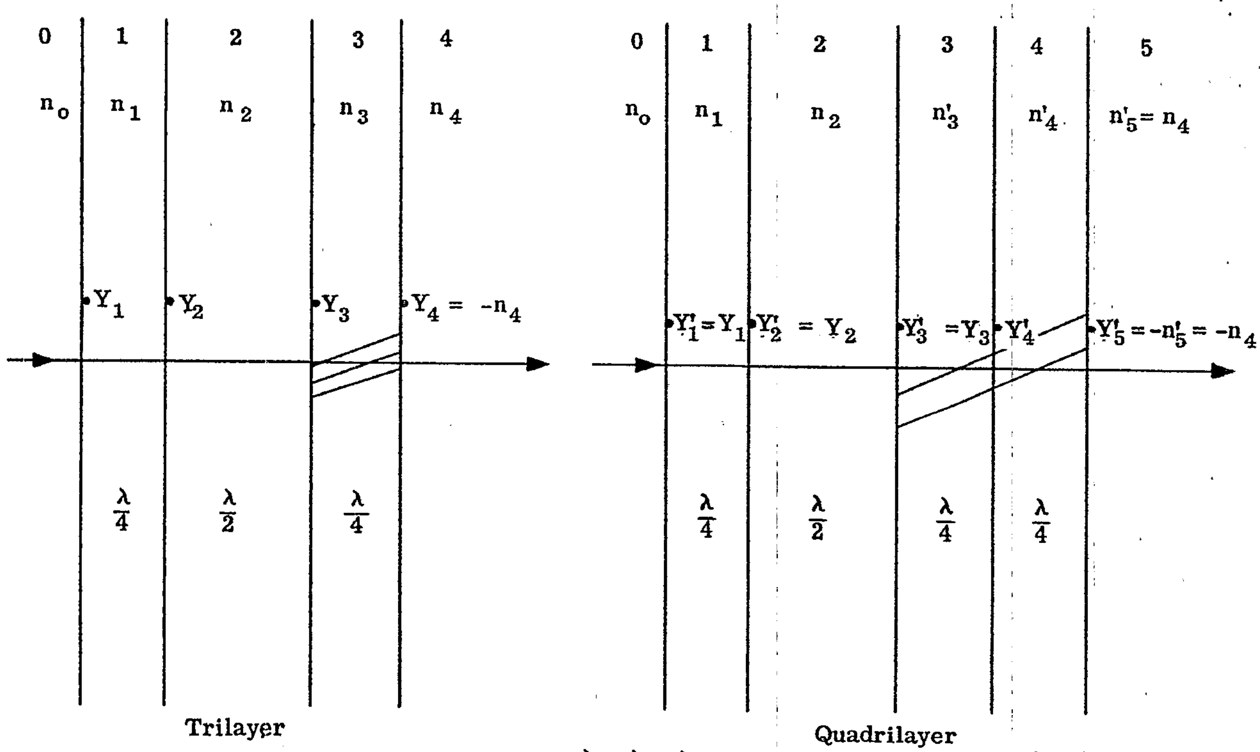


Figure 21.31- Notation with respect to the $\frac{\lambda}{4}-\frac{\lambda}{2}-\frac{\lambda}{4}$ trilayer and H. Jupnik's $\frac{\lambda}{4}-\frac{\lambda}{2}-\frac{\lambda}{4}-\frac{\lambda}{4}$ quadrilayer. The last quarter-wave layer is replaced by an equivalent bilayer. Y_ν denote admittances.

quarter wave bilayer and shall be equivalent to a quarter wave layer of refractive index n_3 at the wavelength λ_o at which the optical path is in fact one-quarter wavelength. The argument is simple on the basis of the admittances Y_ν. Suppose that it is possible to choose n'_3 and n'_4, Figure 21.31, such that $Y'_3 = Y_3$. Because layers number 1 and 2 have not been altered, it must now be true that $Y'_2 = Y_2$ and $Y'_1 = Y_1$, facts that can be checked, if desired, from equation (81a). From equation (79a) the complex reflectance ρ_o of the multilayer is $\rho_o = (n_o + Y_1)/(n_o - Y_1)$ at normal incidence. Hence ρ_o is left unaltered when $Y'_1 = Y_1$ or $Y'_3 = Y_3$. With respect to the trilayer (N = 3), equation (81b) yields

$$Y_4 = -n_4. \qquad (200)$$

Since $\beta = \pi$ for quarter wave layers, equation (81a) yields

$$Y_3 = n_3^2/Y_4 = -n_3^2/n_4. \qquad (200a)$$

Similarly, with respect to the last two proposed elements of the quadrilayer, Figure 21.31,

$$Y'_5 = -n'_5 = -n_4; \qquad (200b)$$

$$Y'_4 = (n'_4)^2/Y'_5 = -(n'_4)^2/n_4; \qquad (200c)$$

$$Y'_3 = (n'_3)^2/Y'_4 = -n_4(n'_3)^2/(n'_4)^2. \qquad (200d)$$

By setting $Y_3 = Y'_3$ from equations (200a and d), one finds almost directly that

$$n'_3/n'_4 = n_3/n_4. \qquad (201)$$

The quarter wave bilayer having refractive indices n'_3 and n'_4 that satisfy equation (201) is equivalent to the quarter wave layer of refractive index n_3 for all values of n_3.

21.9.3 Selection of values.

21.9.3.1 We have seen that choosing n_3 in accordance with equation (199) makes $\rho_o = 0$ at $\lambda = \lambda_o$. By introducing n_3 from equation (199) into equation (201), we obtain Dr. Jupnik's selection for n'_3 and n'_4 in the form

$$\frac{n'_3}{n'_4} = \frac{n_1}{n_o\sqrt{n_4}}; \quad n_4 = n'_5 = n_g. \qquad (202)$$

One is free to assign values to n'_3 or n'_4. In the example of Figure 21.32, n_1 and n'_4 are assigned the value 1.384 corresponding with the choice of $M_g F_2$. Then, with $n_o = 1$ and $n'_5 = 1.52$, one computes $n'_3 = 1.55$* from equation (202).

21.9.3.2 Comparison of Figures 21.30 and 21.32 reveals a number of interesting points. First, the substitution of the quarter wave bilayer serves also to achromatize the multilayer. Secondly, for the choice $n_1 = 1.38$ the spectral reflectance curve of Figure 21.32 is low and flat over a greater range of wavelengths. Thirdly, reflectances less than 0.1% are exhibited over a remarkably long range of wavelengths.

21.9.3.3 The effect of reducing the refractive index, n_2, of the half-wave member of the quadrilayer is illustrated in Figure 21.33. The achromatic points have moved outward to 0.45 and 0.70 microns to produce low reflectances over a greater portion of the visible spectrum than in Figure 21.32. This gain in spectral range is obtained at the cost of a slight increase of the reflectances at the points marked A and B. Further analysis shows also that the effects of increasing the refractive index, n_5, of the substrate are slight even when the refractive indices n_1 to n_4 of Figures 21.32 and 21.33 are left unchanged. From equation (202), a change in the refractive index of the substrate requires that n'_3 be changed correspondingly if one insists that the reflectance shall be zero at $\lambda = \lambda_o$.

21.10 QUARTER WAVE MULTILAYERS

21.10.1 Introduction. The following discussion is restricted to normal incidence upon non-absorbing systems and to the wavelength λ_o at which the optical path of each layer is one quarter wavelength. Equation (81a) shows that at $\lambda = \lambda_o$, where $\beta_\nu = \beta = \pi$,

$$Y_{\nu-1} = n^2_{\nu-1}/Y_\nu. \qquad (203)$$

* The refractive index of thorium oxyfluoride falls near 1.55.

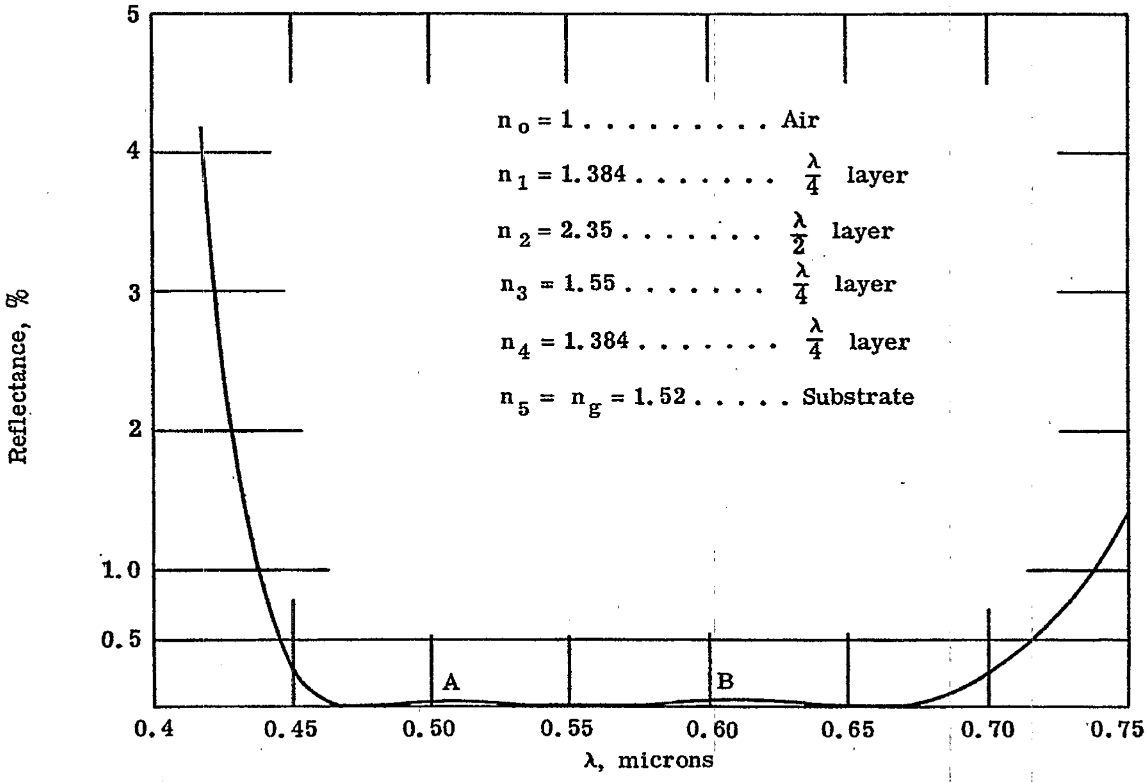


Figure 21.32- Curve of the computed spectral reflectances of a quadrilayer by Dr. H. Jupnik for the indicated refractive indices of the system.

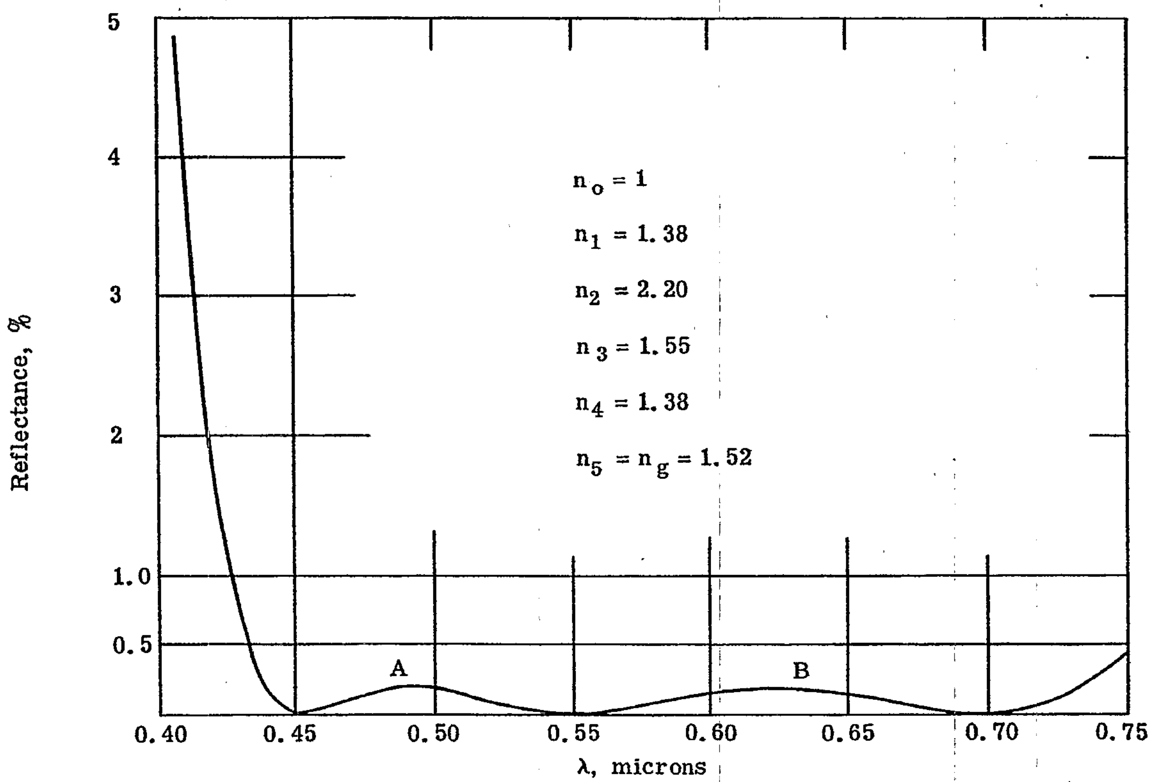


Figure 21.33- Curve of spectral reflectances illustrating the effect of decreasing the refractive index n_2 of the half-wave layer of the quadrilayer of Figure 21.32. The curves of both Figures 21.32 and 21.33 have been computed by an accurate method.

Closer examination of equation (81a) reveals that equation (203) holds whenever $\beta_\nu/2 = \beta/2 = \mu\,\pi/2$ where μ is an odd integer and $\beta/2$ is optical path. Increasing the optical path of any one or any number of the layers by an integral number of half-wavelengths will not alter the following conclusions.

21.10.2 The admittance, Y_1. From equation (203) one finds directly that $Y_2 = n_1^2/Y_1$; $Y_3 = n_2^2/Y_2 = Y_1\, n_2^2/n_1^2$; $Y_4 = n_3^2/Y_3 = n_3^2\, n_1^2/n_2^2\, Y_1$; etc. Hence one concludes by induction that

$$Y_N = \frac{1}{Y_1}\ \frac{n_{N-1}^2 n_{N-3}^2 n_{N-5}^2 \cdot \ldots\ldots n_1^2}{n_{N-2}^2 n_{N-4}^2 n_{N-6}^2 \cdot \ldots\ldots n_2^2}\ ;\ \text{N even}\ ; \tag{204}$$

$$Y_N = Y_1\ \frac{n_{N-1}^2 n_{N-3}^2 \cdot \ldots\ldots n_2^2}{n_{N-2}^2 n_{N-4}^2 \cdot \ldots\ldots n_1^2}\ ;\ \text{N odd}\ ; \tag{204a}$$

where N is the number of layers in the multilayer. However,

$$Y_N = n_N^2/Y_{N+1} = -\,n_N^2/n_{N+1} \tag{204b}$$

since $Y_{N+1} = -\,n_{N+1}$, see equation (81b). Hence, the admittance Y_1 of any quarter-wave multilayer is given by

$$Y_1 = -\,n_{N+1}\ \frac{n_{N-1}^2 n_{N-3}^2 \cdot \ldots\ldots n_1^2}{n_N^2\ n_{N-2}^2 \cdot \ldots\ldots n_2^2}\ ;\ \text{N even}\ ; \tag{205}$$

$$Y_1 = -\,\frac{1}{n_{N+1}}\ \frac{n_N^2\ n_{N-2}^2 \cdot \ldots\ldots n_1^2}{n_{N-1}^2 n_{N-3}^2 \cdot \ldots\ldots n_2^2}\ ;\ \text{N odd}\ . \tag{205a}$$

With Y_1 thus determined, the corresponding complex reflectance ρ_o of the multilayer can be computed from equation (79a). For normal incidence,

$$\rho_o = \frac{n_o + Y_1}{n_o - Y_1}\ . \tag{206}$$

21.10.3 The zero condition. According to equation (206), the reflectance will be zero at $\lambda = \lambda_o$, whenever $Y_1 = -\,n_o$, i.e. whenever

$$\frac{n_o}{n_{N+1}} = \frac{n_{N-1}^2 n_{N-3}^2 \cdot \ldots\ldots n_1^2}{n_N^2\ n_{N-2}^2 \cdot \ldots\ldots n_2^2}\ ;\ \text{N even}\ ; \tag{207}$$

$$n_o\, n_{N+1} = \frac{n_N^2\ n_{N-2}^2 \cdot \ldots\ldots n_1^2}{n_{N-1}^2 n_{N-3}^2 \cdot \ldots\ldots n_2^2}\ ;\ \text{N odd}\ . \tag{207a}$$

When N = 2, one obtains $n_o/n_3 = n_1^2/n_2^2$, the result of equation (194) for quarter wave bilayers. When N = 3, one obtains $n_o n_4 = n_1^2\, n_3^2/n_2^2$, the zero condition for trilayers. Equations (207 and 207a) give one much greater flexibility as regards the choice of materials for obtaining zero reflectance than does the highly specialized zero condition for a monolayer or for a bilayer.

21.10.4 High reflecting multilayers. Equation (206) shows that there are two different ways in which one can achieve the result $|\rho_o| \to 1$. Thus the energy reflectance approaches its highest value unity as

$$Y_1 \to 0\ ; \tag{208}$$

$$|Y_1| \to \infty\ . \tag{208a}$$

Suppose with respect to equation (205) that $n_{N-1}/n_N < 1$, $n_{N-3}/n_{N-2} < 1$, ... $n_1/n_2 < 1$. Then $|\rho_o| \to 1$ as $N \to \infty$ on account of equation (208). On the other hand, if one chooses $n_{N-1}/n_N > 1$, ... $n_1/n_2 > 1$, then $Y_1 \to \infty$ and $|\rho_o| \to 1$ as the number N of the layers in the multilayer approaches infinity. Similar observations apply when N is odd as in equation (205a). Therefore many possibilities are open for achieving high energy reflectance by increasing the number of layers in the multilayer.

21.10.5 The periodic system of repeated bilayers. The production of a multilayer is simplified by alternating layers of high and low refractive indices n_h and n_l. When the number N of layers is even, the resulting system of layers is periodic and forms an assembly of repeated bilayers as illustrated in Figure 21.34. There

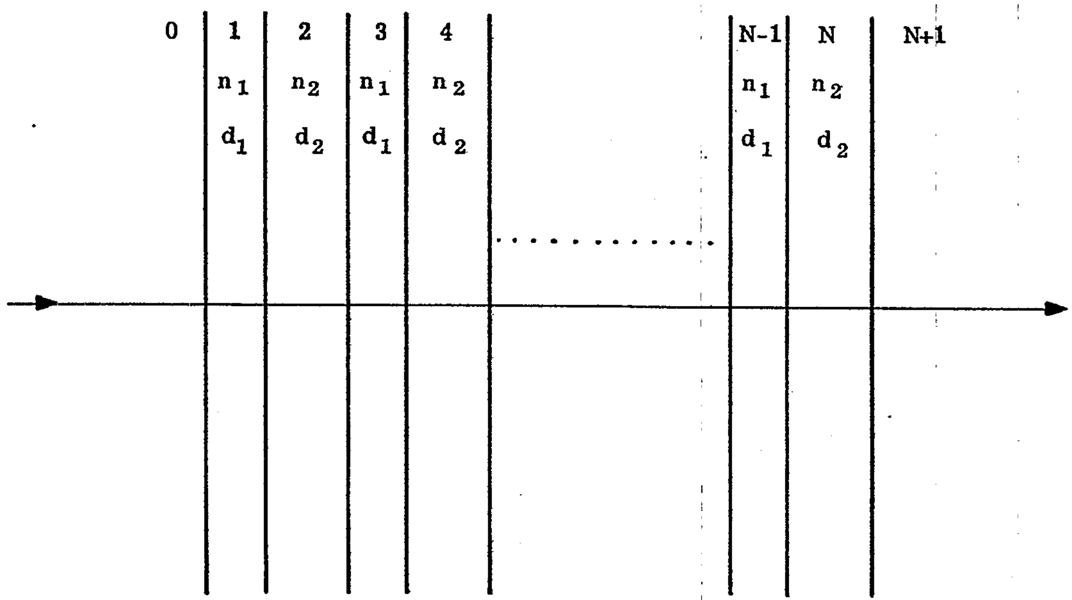


Figure 21.34- A multilayer consisting of repeated bilayers having the refractive indices n_1 and n_2. The thicknesses d_1 and d_2 are likewise repeated and are chosen so that the optical path is one quarter wavelength at an assigned wavelength λ_o. The number N of layers must now be even.

are N/2 even and N/2 odd integers between 0 and N when N is an even integer. Consequently, equation (205) simplifies to the result

$$Y_1 = - n_{N+1}\left(\frac{n_1}{n_2}\right)^N ; \tag{209}$$

and equation (206) assumes the explicit form

$$\rho_o = \frac{n_o - n_{N+1}\left(\frac{n_1}{n_2}\right)^N}{n_o + n_{N+1}\left(\frac{n_1}{n_2}\right)^N} = \frac{\frac{n_o}{n_{N+1}} - \left(\frac{n_1}{n_2}\right)^N}{\frac{n_o}{n_{N+1}} + \left(\frac{n_1}{n_2}\right)^N} . \tag{210}$$

The zero condition requires that

$$\frac{n_o}{n_{N+1}} = \left(\frac{n_1}{n_2}\right)^N . \tag{211}$$

If $n_o < n_{N+1}$, one must choose $n_1 < n_2$ in order to obtain zero reflectance, i.e. one must apply the layer of higher refractive index upon the substrate of refractive index n_{N+1}. On the other hand, equation (210) shows that $|\rho_o|$ can be made high whether one chooses $n_1 < n_2$ or $n_1 > n_2$ provided that N is taken sufficiently large; but that one should choose the alternative $n_1 > n_2$ in order to obtain the highest energy reflectance $|\rho_o|^2$ for a given number N of layers.

21.10.6 Odd number of alternating layers. An important group of quarter wave multilayers containing an odd number N of layers having alternating refractive indices n_1 and n_2 is illustrated by Figure 21.35. If, for example, N = 5 and $n_1 > n_2$, these facts are indicated by the notation HLHLH or $(HL)^2$ H in which H and L refer to the high and low refractive indices n_1 and n_2, respectively. As a second example, if N = 15 and $n_1 < n_2$, the multilayer is described by writing $(LH)^7$ L. As in section 24.9.4, the complex

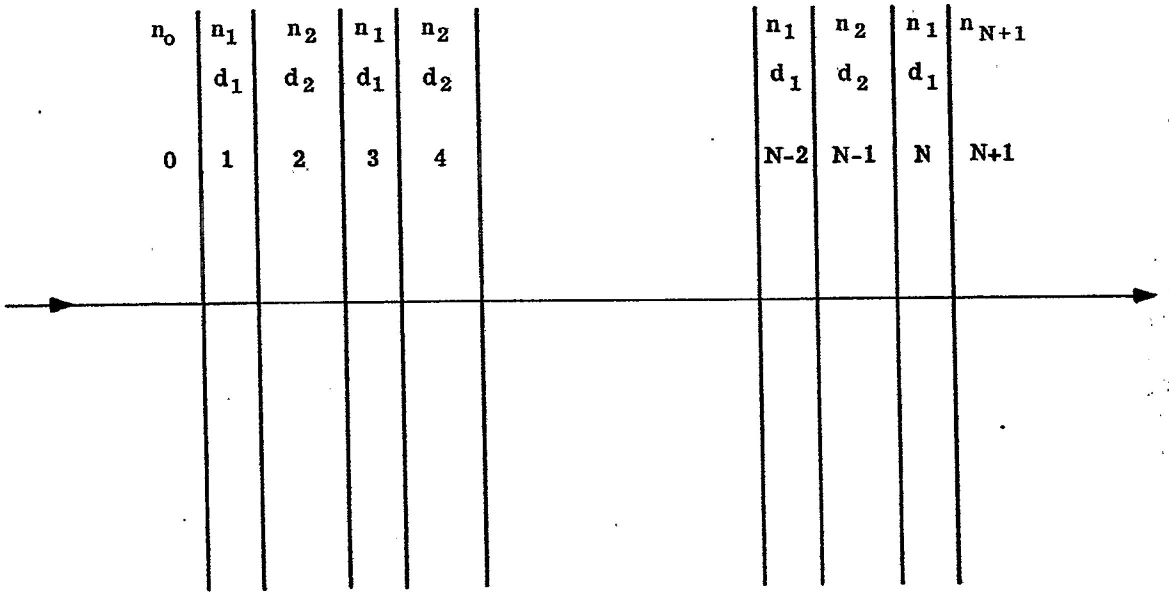


Figure 21.35- A multilayer consisting of an odd number N of quarter wave layers whose refractive indices n_1 and n_2 are alternated. A layer of refractive index n_1 occurs at each end of the multilayer. The optical paths of the layers are $\lambda/4$ at $\lambda = \lambda_o$.

reflectance ρ_o at $\lambda = \lambda_o$ can be found in a simple manner from equations (205a) and (206). The result is

$$\rho_o = \frac{n_o - \frac{n_1^2}{n_{N+1}}\left(\frac{n_1}{n_2}\right)^{N-1}}{n_o + \frac{n_1^2}{n_{N+1}}\left(\frac{n_1}{n_2}\right)^{N-1}}, \qquad (212)$$

with the energy reflectance $R = |\rho_o|^2$. To obtain the highest value of R with the fewest number of layers, one chooses n_1 as large as possible and n_2 as low as possible. The curve of computed spectral transmittance $T = 1 - |\rho_o(\lambda)|^2$ of the multilayer $(HL)^5 H$ is shown in Figure 21.36 for the indicated values of n_o, n_1, n_2 and $n_{N+1} = n_g$. The layers are quarter wave layers at $\lambda = \lambda_o = 0.75$ microns at which one computes from equation (212) that $|\rho_o|^2 = 0.9946$ whence $T = 0.0054$. Energy reflectances exceeding those from silver are obtained easily with multilayers.

21.10.7 Achromatization. The spectral transmittance curve of Figure 21.36 illustrates one of the more serious difficulties encountered in the design of thin films. The oscillations in the spectral transmittances exemplified by those occurring between 0.4 and 0.62 microns are usually undesirable. The term achromatization or achromatizing pertains to the minimization of the amplitudes of undesired, rapid oscillations of spectral transmittance or reflectance curves in such a manner that the curves are flattened. This usage of the term achromatization is not entirely consistent with that of Section 21.7.4. Achromatization of bilayers and multilayers differ slightly, we may say, as to the manner in which a curve of spectral reflectance or transmittance is "flattened". An example * of one method of achromatizing multilayers is illustrated in considerable detail by Figures 21.35, 21.36, 21.37, and 21.38. Flattening of the curve of spectral transmittances between 0.4 and 0.62 microns is accomplished without appreciable alteration of the curve between 0.62 and 0.75 microns. As with bilayers, achromatization of multilayers can be achieved also by adding half-wave layers at strategic locations within the multilayer.

* It will not be possible due to lack of time and space to do justice to the able work of many investigators who have contributed to methods of achromatization.

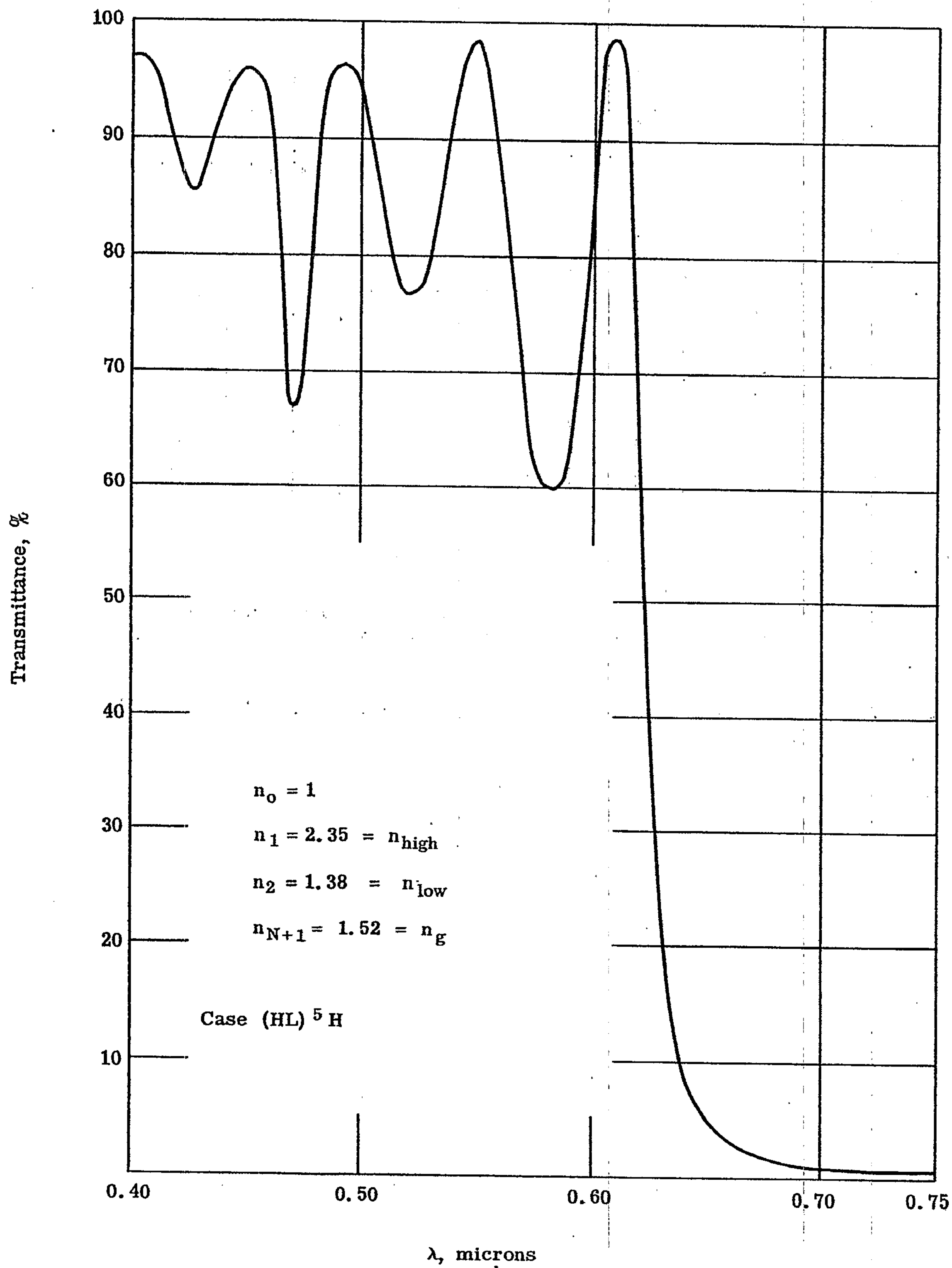


Figure 21.36 - Curve of computed spectral transmittances taken from the files of Dr. H. Jupnik for the alternating multilayer $(HL)^5$ H having layers that are quarter-wave layers at $\lambda_o = 0.75$ microns. The heights and locations of the numerous maxima and minima have not been determined with greatest possible care.

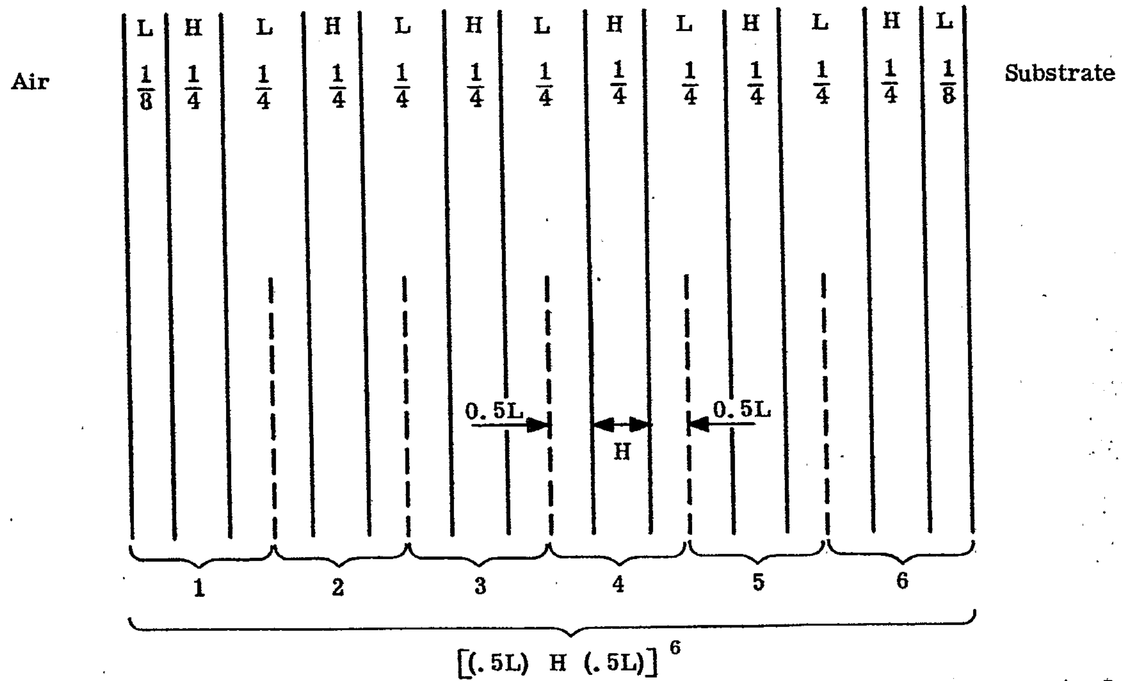


Figure 21.37- Explanation of the notation $[(.5L)\ H\ (.5L)]^6$ with respect to a multilayer $(HL)^5 H$ to which has been added at each end an achromatizing layer of low refractive index and of optical path $\lambda/8$ at $\lambda = \lambda_o$. The system consists of six repeated "trilayers," $(.5L)\ H\ (.5L)$ or $(\frac{L}{2})\ H\ (\frac{L}{2})$.

21.10.8 Narrow pass band filters.

21.10.8.1 The design of multilayers that are intended for use as narrow pass band filters is based upon the principles of the Fabry-Perot interferometer. The silver coatings on opposite surfaces of the plane parallel plate are replaced by high reflecting multilayers. In turn, the plane parallel plate is usually replaced by a layer that serves as the spacer. High reflecting multilayers are superior to silver coatings * when only small amounts of absorption can be tolerated and when durability becomes an important consideration. One arrangement is illustrated in Figure 21.39. When all the layers of multilayers, B_1 and B_2, are quarter wave layers at $\lambda = \lambda_o$, the optical path of the separating layer, S, should be an integral number, ν, of half waves at λ_o. It is not difficult to see that at normal incidence the transmittance is unity at $\lambda = \lambda_o$ for the idealized system that contains no absorption, scattering or departures from the rigid design of Figure 21.39. First, one notes that the spacer layer S is an absentee layer at λ_o. Layers 1 and 1' are then effectively in contact and comprise a half-wave or absentee layer. This now places layers 2 and 2' effectively in contact to form a third absentee layer. One concludes that all opposing pairs of layers form absentee layers at λ_o -- and, indeed, that the filter becomes an "absentee filter" that must have transmittance unity.

21.10.8.2 The behavior of the filter at $\lambda \neq \lambda_o$ can be appreciated and evaluated as follows from the theory of Fabry-Perot interferometers. With reference to equation 16-(103) we note that the parameter, A, is now given by

$$A = |\rho_o| \quad |\rho'_o| \tag{213}$$

in which ρ_o and ρ'_o are the complex reflectances of the "coated" surfaces of the spacer, S, Figure 21.39, since the spacer has been assumed to be non-absorbing. Let T denote the time averaged energy transmittance of the filter. Then T = W where W is the quantity given by equation 16-(107). From this equation we see that

$$2T = \frac{|\tau \ \tau'|^2}{1 - 2A\cos\alpha + A^2} \tag{214}$$

* Silver coatings having extremely small amounts of absorption can be produced by evaporation; but the required technique is not well known and the silver coatings are not likely to remain low-absorbing.

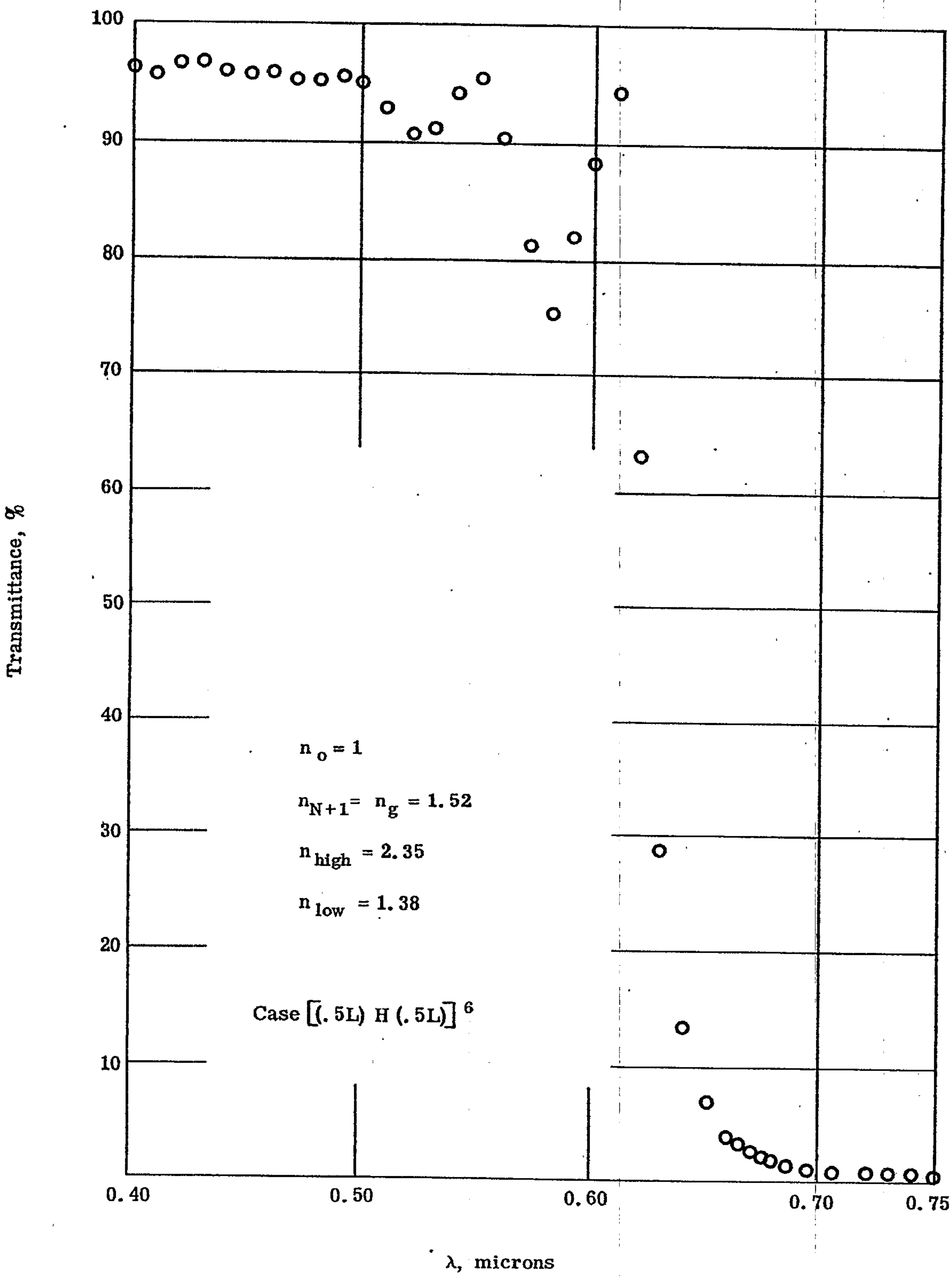


Figure 21.38 - Plot of the spectral transmittances obtained by achromatizing the system $(HL)^5 H$ of Figure 21.35. by the addition of a λ/8 layer of low refractive index at each end of the multilayer. The notation for the multilayer thus obtained is $[(.5L)\ H\ (.5L)]^6$. These plotted data have been taken from the files of Dr. H. Jupnik.

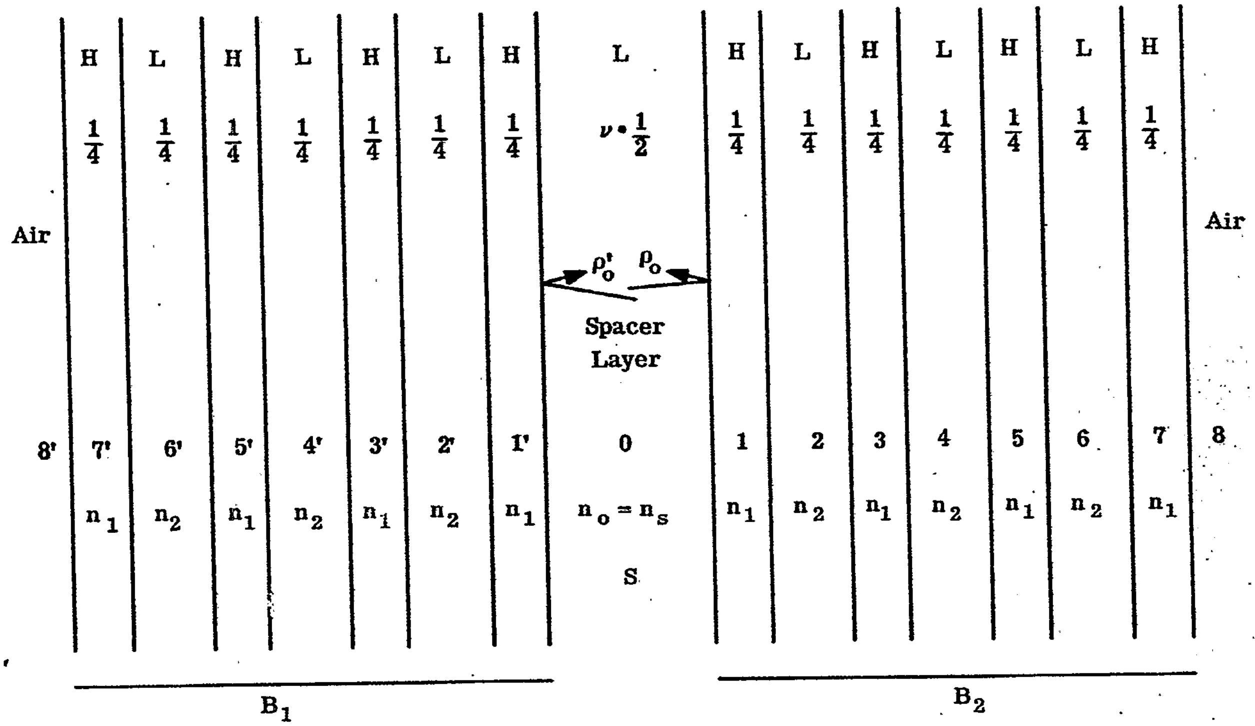


Figure 21.39- A narrow pass band, interference filter consisting of a spacer layer coated with high reflecting, quarter-wave multilayers B_1 and B_2 that consist of seven alternating layers.

in which τ' and τ are the complex transmittances of multilayers B_1 and B_2, Figure 21.39, and

$$\alpha = \frac{4\pi n_s d_s}{\lambda} + \arg(\rho_o) + \arg(\rho_o') \tag{215}$$

where n_s and d_s are the refractive index and thickness, respectively, of the spacer. The transmittance T is maximum when

$$\alpha = \nu 2\pi \tag{216}$$

where the integer ν is called the order number of the filter or interferometer. The corresponding wavelength is usually indicated by λ_ν in Fabry-Perot interferometry. By eliminating α from equation (215) with the aid of equation (216), one obtains the result

$$2n_s d_s + \frac{\lambda \nu}{2\pi}\left[\arg(\rho_o) + \arg(\rho_o')\right] = \nu \lambda_\nu . \tag{217}$$

Sharp peaks of transmittance $T = W$ are produced by equation (214), as illustrated by Figure 16.19, when the equivalent equations (216) and (217) are satisfied. These sharp peaks are the narrow pass bands.

21.10.8.3 Suppose now that the numerous conditions of Figure 21.39 are met by the interference filter at $\lambda = \lambda_o$. Then $\rho_o = \rho_o'$ and ρ_o is determined from equation (212) with $N = 7$. Because we have chosen $n_1 > n_2$, ρ_o turns out to be real and negative. We may write

$$\rho_o = |\rho_o| e^{\pm i\pi} . \tag{218}$$

The phase change on reflection can be taken as either of the two physically indistinguishable values, $\pm\pi$. The simplest way of interpreting equation (217) at $\lambda = \lambda_o$ is now to take $\rho_o = |\rho_o| e^{i\pi}$ and $\rho_o' = |\rho_o| e^{-i\pi}$ since these are physically indistinguishable. Correspondingly,

$$\arg(\rho_o) + \arg(\rho_o') = 0 ; \quad \lambda = \lambda_o . \tag{219}$$

Then, simply,

$$2n_s d_s = \nu \lambda_\nu \quad \text{at } \lambda = \lambda_o . \tag{220}$$

Hence we may regard the integers ν of Figure 21.39 and equation (220) as the same integer, i.e. as the spectral order number of the filter. Equation (220) explains why an interference filter can exhibit two pass bands in the visible region. Suppose that λ_o is chosen at the red end of the spectrum. Here, $\lambda_o = \lambda_\nu$. If ν is high enough, it can happen that at a shorter wavelength $\lambda_{\nu+1}$ the order number is increased to $\nu + 1$. In other words, it can happen that

$$2n_s d_s = (\nu + 1) \lambda_{\nu+1} , \tag{221}$$

where $\lambda_{\nu+1} < \lambda_o$. Because equation (217) and its simplifications are conditions for maxima of T, the wavelength $\lambda_{\nu+1}$ defines the center of a subsidiary pass band. By increasing ν (increasing d_s), one obtains additional pass bands in a specified spectral range such as the visible region.

21.10.8.4 The widths of the narrow pass bands are important properties of the interference filter. These widths can be evaluated as follows. From equation 16-(113),

$$|\Delta \alpha| = \frac{1 - A}{\sqrt{A}} , \tag{222}$$

where $\Delta\alpha$ is a particular departure of α from its value α_ν at the wavelength λ_ν at which T is maximum. This departure of α from α_ν changes T from $T = T_{maximum}$ to $T_{maximum}/2$. Let us assume for simplicity that the variation of the phase changes on reflection arg (ρ_o) and arg (ρ'_o) with wavelength can be ignored.* By differentiating α with respect to λ in equation (215), one obtains

$$|\Delta \alpha| = \frac{4\pi n_s d_s}{\lambda^2} |\Delta \lambda| . \tag{223}$$

We evaluate the derivative at the center of the pass band under consideration where $\lambda = \lambda_\nu$. Let $2n_s d_s$ be eliminated from equation (223) with the aid of equation (220). Then

$$|\Delta \alpha| = \nu 2\pi \frac{|\Delta\lambda|}{\lambda_\nu} . \tag{224}$$

By eliminating $\Delta\alpha$ from equation (224) with the aid of equation (222), we obtain the formulae

$$2|\Delta\lambda| = \frac{\lambda_\nu}{\nu} \frac{1 - A}{\pi\sqrt{A}} ; \tag{225}$$

$$= \frac{\lambda_\nu}{\nu} \cdot \frac{1 - |\rho_o| \; |\rho'_o|}{\pi\sqrt{|\rho_o| \; |\rho'_o|}} ; \tag{225a}$$

$$= \frac{\lambda_\nu}{\nu f} \tag{225b}$$

where the finesse (16) f is defined as

$$f = \pi\sqrt{A}/(1 - A) . \tag{226}$$

$|\Delta\lambda|$ is the half-width of the pass band of the filter at the selected order number ν. By definition, $|\Delta\lambda|$ is the departure of λ from λ_ν that causes the transmittance of the filter to drop from $T_{max.}$ at λ_ν to $T_{max.}/2$ at $\lambda = \lambda_\nu \pm |\Delta\lambda|$. When $|\rho_o| = |\rho'_o|$,

$$2|\Delta\lambda| = \frac{\lambda_\nu}{\nu} \frac{1 - |\rho_o|^2}{\pi |\rho_o|} , \tag{227}$$

where $|\rho_o|$ is evaluated at the wavelength λ_ν. At the wavelength $\lambda = \lambda_o$, ρ_o can be calculated from equation (212) provided that the high reflecting multilayer falls in the class governed by equation (212). Equation (227) shows that the half-width is decreased by increasing $|\rho_o|$ or the order number ν. At fixed λ_ν, the order number is increased by increasing the optical path $n_s d_s$ of the spacer.

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* In some applications, arg (ρ_o) is deliberately made a rapid function of wavelength in order to utilize the dispersion of the phase changes on reflection for obtaining narrow pass bands.

(16) P. Giacomo, Rev. D'optique, 35, 317 (1956).

21.10.8.5 As an example of the evaluation of a half-width from the theory, let us now consider the type of interference filter of Figure 21.39 with $n_1 = 2.35$, $n_2 = 1.38$ and $n_s = 1.38$. This choice corresponds to the use of ZnS and $M_g F_2$ as the material of high and low refractive index, respectively. We suppose that the multilayers B_1 and B_2 are quarter-wave systems at $\lambda_o = 5550$ Angstroms and that the spacer is a half-wave spacer at λ_o. We compute the half-width at the main transmittance peak located at $\lambda_\nu = \lambda_o$ where $\nu = 1$. In applying equation (212), we may take $n_o = n_2 = 1.38$, $n_{N+1} = 1$, $n_1 = 2.35$ and $N = 7$. Then $|\rho_o| = 0.9797$ and $|\rho_o|^2 = 0.9598$. Then from equation (227), $|\Delta\lambda| = 36.2$ Angstroms. The half-width may be decreased to 12.07 Angstroms by choosing $\nu = 3$. In practice, further reduction of $|\Delta\lambda|$ is accomplished by increasing $|\rho_o|$, i.e. by increasing N or by choosing a pair of dielectric materials for which the ratio n_1/n_2 is higher. We see that half-widths of one Angstrom or less become difficult to attain.

21.11 MATERIALS AND TEXTS

Whereas many scattered publications deal with the optical properties of substances that are suitable for use as thin films, the writer is unaware of a publication that contains an exhaustive summary of the optical properties of the many possible materials from the ultraviolet region into the region of the infrared.

One of the longest tables of the optical and mechanical properties of materials that are used in making thin films will be found in L. Holland, "Vacuum Deposition of Thin Films," John Wiley & Sons, Inc. (1956).

Quite detailed discussion of the optical constants and properties of metallic films is included in O. S. Heavens, "The Optical Properties of Thin Films," Butterworths Scientific Publications, London (1955). This book includes scattered information about the optical constants of other materials such as ZnS, Sb_2O_3, Ge and Te.

A useful list of the optical constants of metals and inorganic compounds appears in most editions of "Handbook of Chemistry and Physics," Chemical Rubber Publishing Co.

A book by W. Lewis, "Thin Films and Surfaces," Temple Press Ltd, London, First Edition is devoted to the structure, properties and production of various thin films. Emphasis is placed upon aluminum and alloys containing aluminum.

The excellent work of Dr. Georg Hass and his associates has contributed information about the optical properties and formation of thin films -- especially the oxides of titanium, silicon, aluminum and rare earths. As one example, see Georg Hass, Vacuum, 2, 331-345 (1952). This publication contains a substantial list of references.

The following texts may be consulted for much additional, valuable discussion relative to thin films.

Auwarter, Max, ed. -- "Ergebnisse der Hochvakuum technik und der Physik dunner Schichten," Stuttgart, Wissen schaftliche Verlagsgessellschaft (1957).

Mayer, Herbert -- "Physik dünner Schichten," Stuttgart, Wissenschaftliche Verlagsgesellschaft, 1950, Volume 1 and 2.

Vasicek, Antonin -- "Optics of thin films," Amsterdam, North-Holland Publishing Co., 1959.