13 MIRROR AND PRISM SYSTEMS

13.1 INTRODUCTION

13.1.1 Uses of mirrors and prisms. Mirrors and prisms are widely used in optical systems. Among the principal uses are the following:

(1) To bend light around corners.

(2) To fold an optical system into a smaller space.

(3) To provide proper image orientation.

(4) To combine or split optical beams with partial reflecting surfaces.

(5) To disperse light, as in refractometers and spectrographic equipment.

13.1.2 Design application. The principles discussed in this section are intended to develop an understanding of concepts, and to provide computational tools for use in designing optical systems for all the above applications with the exception of spectrographic equipment. Thus, since dispersion is not one of our primary aims, the problem can best be approached by the study of reflection.

13.2 REFLECTION

13.2.1 Reflection from a single surface.

13.2.1.1 The first problem involved in the study of reflecting surfaces is illustrated in Figure 13.1. An object point P is given. A mirror reflects the incident rays of light from P in a new direction so that the reflected rays appear to emerge from an image P'. The actual reflection problem might involve a number of possible variations from a design standpoint. For example, the problem might be to orient the mirror to send the reflected light in a given direction. This might then raise the question of image orientation at P'.

13.2.1.2 The simpler problems of this nature can be readily solved by elementary concepts known to most technical people. The discussion below is designed to provide the tools to handle more complex problems.

13.2.2 Multiple reflection.

13.2.2.1 Equations 2-(3) and 2-(4) provide a vector form for the law of refraction and the law of reflection. The same equations can be used to treat reflection problems by assuming that, $-n_1 = n_o = 1$. From equation 2-(4),

$$\Gamma = -\cos I - \sqrt{\cos^2 I}\ , \text{ or}$$

$$\Gamma = -2\cos I. \tag{1}$$

Cos I is given by the dot product, $\vec{S}\cdot\vec{M}$; therefore

$$\Gamma_i = -2(\vec{S}_{i-1}\cdot\vec{M}_i) = -2\rho_i\ . \tag{2}$$

Equation 2-(3) and Equation (2) above make it possible to handle reflection problems for any number of surfaces. For example, assume a system of mirrors as in Figure 13.2, with rays reflected as illustrated. If $\vec{S}$ is a unit vector along any ray, thereby indicating its direction, it is possible to write the following equations.

$$\vec{S}_1 = \vec{S}_o + \Gamma_1 \vec{M}_1, \tag{3a}$$

$$\rho_1 = \vec{S}_o\cdot\vec{M}_1, \tag{3b}$$

and

$$\vec{S}_2 = \vec{S}_1 + \Gamma_2 \vec{M}_2, \tag{4a}$$

$$\rho_2 = \vec{S}_1\cdot\vec{M}_2 = \vec{S}_o\cdot\vec{M}_2 + \Gamma_1 \vec{M}_1\cdot\vec{M}_2, \tag{4b}$$

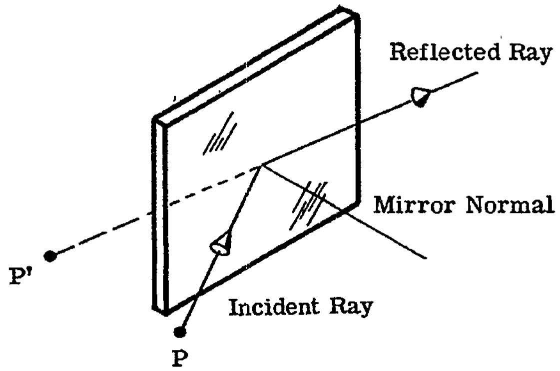


Figure 13.1-Reflection from a single surface mirror.

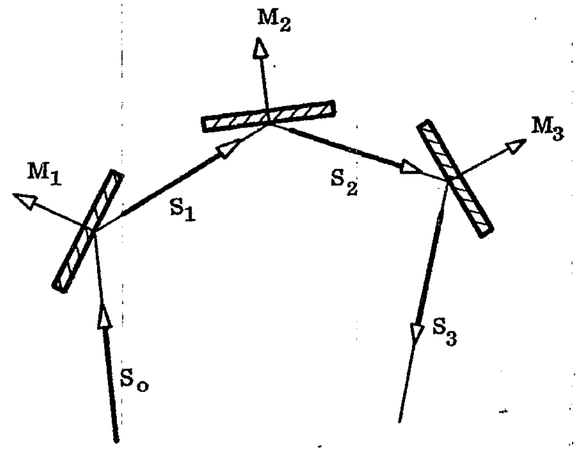


Figure 13.2-Reflection from multiple mirrors.

and

$$\vec{S}_3 = \vec{S}_2 + \Gamma_3 \vec{M}_3, \tag{5a}$$

$$\rho_3 = \vec{S}_2 \cdot \vec{M}_3 = \vec{S}_o \cdot \vec{M}_3 + \Gamma_1 \vec{M}_1 \cdot \vec{M}_3 + \Gamma_2 \vec{M}_2 \cdot \vec{M}_3, \tag{5b}$$

from which one can readily see the pattern that follows as more surfaces are added.

13.2.2.2 Let us examine an example of a problem involving a single reflection. Suppose it is desired to have a ray of light pass along the Z axis and reflect from a mirror in the XY plane at an angle of 45° to the X axis as in Figure 13.3. What are the coordinates of the normal to the mirror? By writing the incoming and outgoing vectors in component form, we have

$$\vec{S}_o = \vec{k},$$

and

$\vec{S}_1 = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the X, Y and Z axes, respectively. The unit vector for the mirror normal may then be written as

$$\vec{M} = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}.$$

Therefore

$$\rho_1 = \vec{S}_o \cdot \vec{M}_1 = M_z$$

and

$$\Gamma_1 = -2\,M_z.$$

Then, from equation (3a),

$$\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} = \vec{k} - 2M_z\,\vec{M}.$$

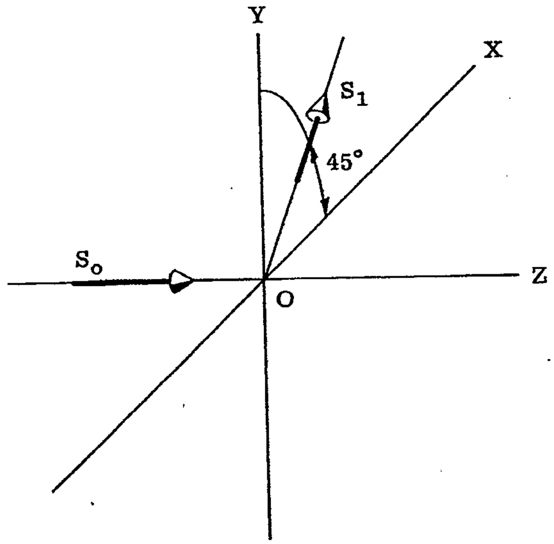


Figure 13.3-A single reflection problem.

It follows that

$$\vec{M} = -\frac{1}{2\sqrt{2}\,M_z}\vec{i} - \frac{1}{2\sqrt{2}\,M_z}\vec{j} + \frac{1}{2\,M_z}\vec{k}.$$

Since $\vec{M}$ is a unit vector, the sum of the squares of its components is equal to one. Therefore,

$$\left(\frac{1}{2\sqrt{2}\,M_z}\right)^2 + \left(\frac{1}{2\sqrt{2}\,M_z}\right)^2 + \left(\frac{1}{2\,M_z}\right)^2 = 1$$

since $\vec{i}$, $\vec{j}$ and $\vec{k}$ are also unit vectors.

Solving for M_z,

$$M_z^2 = \frac{1}{8} + \frac{1}{8} + \frac{1}{4},$$

$$M_z = \frac{1}{\sqrt{2}}.$$

Finally,

$$\vec{M} = -\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

From this we can see that

$$M_x = -\frac{1}{2}, \quad M_y = -\frac{1}{2} \quad \text{and} \quad M_z = \frac{1}{\sqrt{2}}$$

13.2.2.3 Consider the above solution. $\vec{M}$ is the vector for the mirror normal, but what is the significance of describing it thusly? We will find it very convenient to be able to describe the equations of a plane in terms of the components of a unit vector normal to the plane. The equation of a plane may be written as

$$Ax + By + Cz + D = 0. \qquad (6)$$

Taking the numerical value of D as negative, if P is the distance from the origin to the plane along the normal,

$$P = \frac{-D}{\sqrt{A^2 + B^2 + C^2}} = \frac{-D}{F},$$

where

$$F = \sqrt{A^2 + B^2 + C^2}. \tag{7}$$

The components of P on the X, Y, Z axes are,

$$P_x = -\frac{DA}{F^2}, \tag{8a}$$

$$P_y = -\frac{DB}{F^2}, \tag{8b}$$

and

$$P_z = \frac{DC}{F^2}. \tag{8c}$$

The coordinates of the unit vector along P are, therefore,

$$M_x = -\frac{A}{F}, \tag{9a}$$

$$M_y = \frac{B}{F}, \tag{9b}$$

and

$$M_z = \frac{C}{F}. \tag{9c}$$

These equations enable us to visualize the spatial position of the mirror discussed above. If $P = 1$, then $F = -D$ and the intercepts of the mirror on the X, Y, Z axes are equal to $\frac{1}{M_x}$, $\frac{1}{M_y}$ and $\frac{1}{M_z}$ because,

$$-\frac{D}{A} = \frac{1}{M_x}, \quad -\frac{D}{B} = \frac{1}{M_y}, \quad \text{and} \quad -\frac{D}{C} = \frac{1}{M_z}.$$

In the above example, then, the intercepts of the plane of the mirror are,

$$\frac{1}{M_x} = -2, \quad \frac{1}{M_y} = -2, \quad \text{and} \quad \frac{1}{M_z} = \sqrt{2}.$$

A plane mirror located with these intercepts will be parallel to the mirror specified in the problem, and at a distance $P = 1$ from it as shown in Figure 13.4. (The intercepts of the desired plane, of course, are 0, 0, 0.) The components of the mirror normal vector for the mirror at the origin will be equal to the components of mirror normal vector for the mirror at P since the mirrors describe two parallel planes.

13.3 LOCATION OF THE IMAGE

13.3.1 The plane of incidence. One of the conditions of the law of reflection is that the incident ray, a normal to the surface at the point of incidence, and the reflected ray all lie in a single plane. It is possible therefore to draw the plane containing the incident ray, the normal to the surface, and the reflected ray. This is illustrated in Figure 13.5. The plane containing this ray is called the plane of incidence.

13.3.2 Image location.

13.3.2.1 The next problem of interest is the following. If point P represents an object point, where will its image be located? In order to locate an image it is necessary to take at least two rays from the object point and reflect them from the mirror. These are indicated by R_1 and R_2 in Figure 13.5. One can readily determine that the second ray, when extended back, intersects the first ray at P'. P' is therefore the image of P; it is located on the line from P perpendicular to the mirror and lies behind the mirror the same distance that P is in front of the mirror.

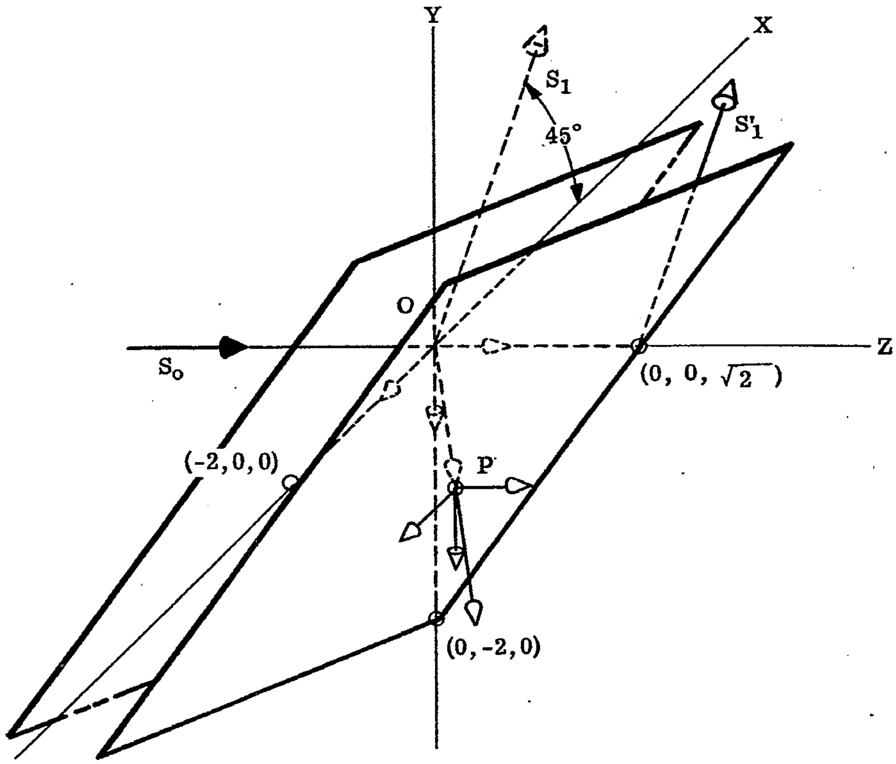


Figure 13.4-Solution to problem of Figure 13.3.

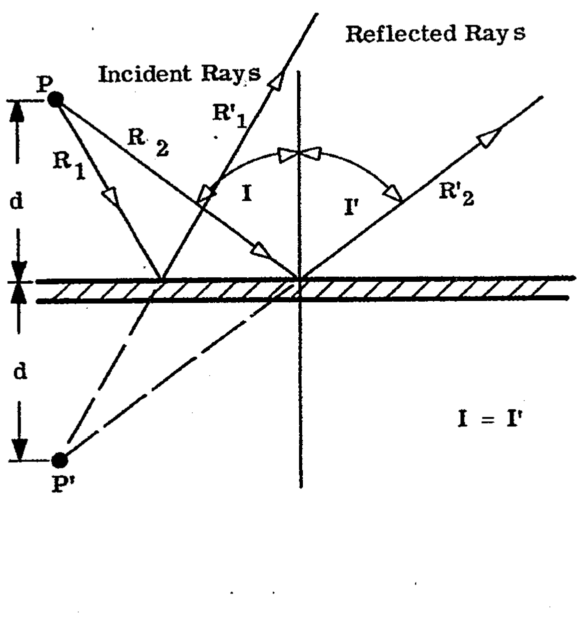


Figure 13.5-Plane of incidence.

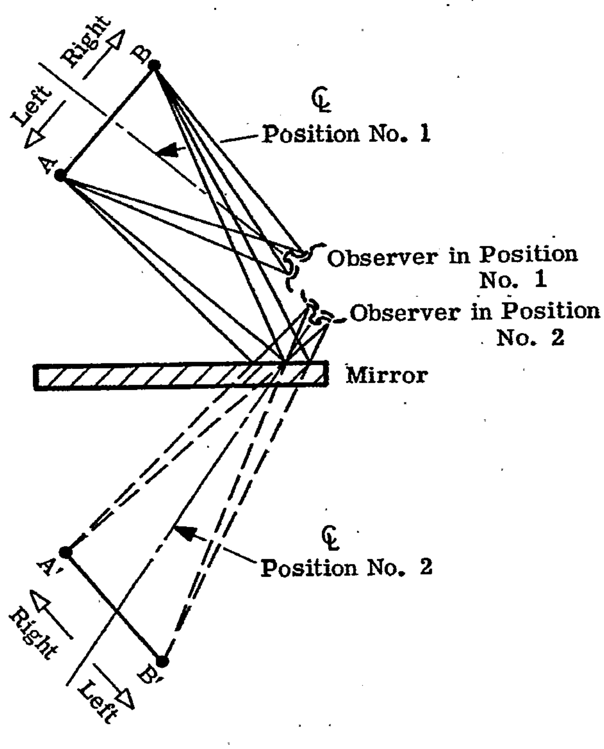


Figure 13.6-Observer, image, and object positions.

13.3.2.2 This means that the image of a point P in a mirror may be located immediately by drawing a line from P perpendicular to the mirror. If the distance along this line from P to the mirror is d, then the image P' will be located on this same line in back of the mirror at a distance d from the mirror. Alternately the image of a point P in a mirror may be found by rotating the object point around the axis formed by the intersection of the plane of the mirror and the plane of incidence.

13.4 ORIENTATION OF THE IMAGE

13.4.1 Single mirror imagery. Suppose we look at the image of two points A and B. See Figure 13.6. The images A' and B' are located readily by drawing normals through the mirror and laying off equal distances. Now suppose that an observer looks at the AB from position 1, shown. To the observer B lies to the right of A. Now if the observer wishes to see the image he must turn around and look into the mirror as in position 2. Then B' appears to lie to the left of A'. This means the mirror image appears to be "left handed". An object imaged by a single mirror always appears "left handed". One source of confusion in this field stems from the fact that one may not always look at the object from the same side. Figure 13.6 shows that A' and B' are actually in the same spatial orientation as A and B. It is because the observer has to change his point of view that makes the image appear left handed.

13.4.2 Mathematical formulae for locating the image of a point P in a mirror.*

13.4.2.1 It is possible to readily compute the image position of an object point P as reflected in a mirror. Referring to Figure 13.5, one may write the expression for a plane parallel to the mirror passing through P. The equation is

$$A(x_1) + B(y_1) + C(z_1) + D_1 = 0 \ . \tag{10}$$

This represents a plane through P which is located at coordinates x_1, y_1, z_1. The equation for the mirror is

$$A(x) + B(y) + C(z) + D = 0 \tag{11}$$

The perpendicular distance between the two planes is therefore

$$d = \frac{D-D_1}{F} = \frac{A(x_1) + B(y_1) + C(z_1) + D}{F} \ . \tag{12}$$

The image will lie at a distance d on the other side of the mirror from the point P on the normal to the mirror. Equation (9) gives the components for the unit vector perpendicular to the mirror, so if these are multiplied by 2d, one obtains the differences in the position coordinates for the object and image. The position coordinates of the image P' (x_1', y_1', z_1') are then given by

$$x_1' = x_1 - 2d\frac{A}{F} \ , \tag{13}$$

$$y_1' = y_1 - 2d\frac{B}{F} \ , \tag{14}$$

and

$$z_1' = z_1 - 2d\frac{C}{F} \ . \tag{15}$$

By inserting the value of d from Equation (12), it is possible to compute x_1', y_1' and z_1'.

13.4.2.2 It is convenient to use matrix notation for Equations (13), (14), and (15). These equations may be written in matrix form as follows,

$$\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2AD/F^2 & 1-2A^2/F^2 & -2BA/F^2 & -2AC/F^2 \\ -2BD/F^2 & -2AB/F^2 & 1-2B^2/F^2 & -2BC/F^2 \\ -2CD/F^2 & -2AC/F^2 & -2BC/F^2 & 1-2C^2/F^2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} \tag{16}$$

*J. S. Beggs, J. Opt. Soc. Am. 50, 388 (1960).

In abbreviated form, then, one can say,

$$[P] = [M] \; [P'], \tag{17}$$

in which P represents the column matrix, $\begin{bmatrix} 1 \\ x' \\ y' \\ z' \end{bmatrix}$,

and P' represents the column matrix, $\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$.

The M matrix is the large matrix made up of the constants of the mirror. Now if there are several mirrors involved, the image P' will be transformed to another image P'' and P'' to P''' etc. It follows that

$$[P'] = [M_1] \; [P] \quad \text{and} \quad [P''] = [M_2] \; [P'] \text{ etc.},$$

$$\therefore [P^n] = [M_n] \cdots [M_2] \; [M_1] \; [P]. \tag{18}$$

13.4.3 The vector ray tracing equation in matrix form.

13.4.3.1 Equations (3a) and 3b) may also be written in matrix form. First combine (3a) and (3b),

$$\vec{S}_1 = \vec{S}_0 - 2 \; (\vec{S}_0 \cdot \vec{M}_1) \; \vec{M}_1.$$

In component form this equatiom may be written

$$S_{1x} = S_{0x} - 2 \; M_x \; (S_{0x} \; M_x + S_{0y} \; M_y + S_{0z} \; M_z) ,$$

$$S_{1y} = S_{0y} - 2 \; M_y \; (S_{0x} \; M_x + S_{0y} \; M_y + S_{0z} \; M_z) ;$$

and

$$S_{1z} = S_{0z} - 2 \; M_z \; (S_{0x} \; M_x + S_{0y} \; M_y + S_{0z} \; M_z) .$$

These equations may also be reduced to matrix form,

$$\begin{bmatrix} S_{1x} \\ S_{1y} \\ S_{1z} \end{bmatrix} = \begin{bmatrix} (1-2M_x^2) & -2M_x M_y & -2M_x M_z \\ -2M_x \; M_y & (1-2M_y^2) & -2M_y M_z \\ -2M_x \; M_z & -2M_y M_z & (1-2M_z^2) \end{bmatrix} \begin{bmatrix} S_{0x} \\ S_{0y} \\ S_{0z} \end{bmatrix}. \tag{19}$$

By substituting Equations (9a), (9b) and (9c) into Equation (16), we see that this new matrix is a minor of the M matrix. Let us call this the R matrix. For several reflections, then, it is possible to write

$$[S_n] = [R_n] \; [R_{n-1}] \cdots [R_1] \; [S_0]. \tag{20}$$

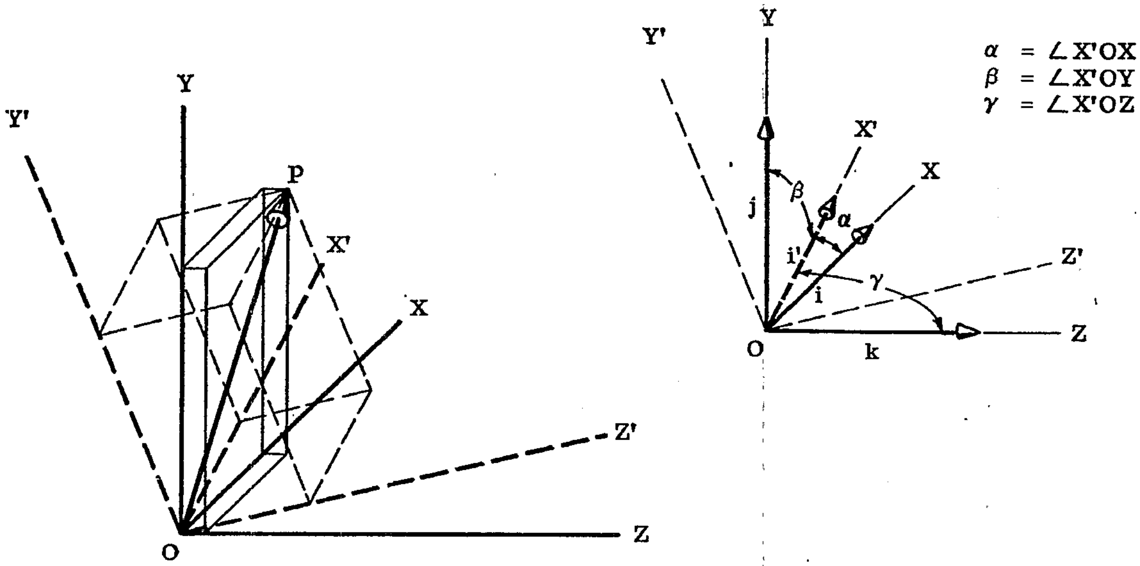
13.4.3.2 The matrix notation is conceptionally convenient because the matrix equation (19) represents a rotation of coordinate axes. To illustrate, consider the rectangular coordinate axes X, Y, Z and their respective unit vectors i, j, k and the rotated coordinate axes X', Y', Z' and their unit vectors i', j', k'. The vector $\overline{OP}$ shown in Figure 13.7 may be written in component form for either system of coordinates as,

$$\overrightarrow{OP} = xi + yj + zk = x'i' + y'j' + z'k'.$$

Performing scalar multiplication by i yields

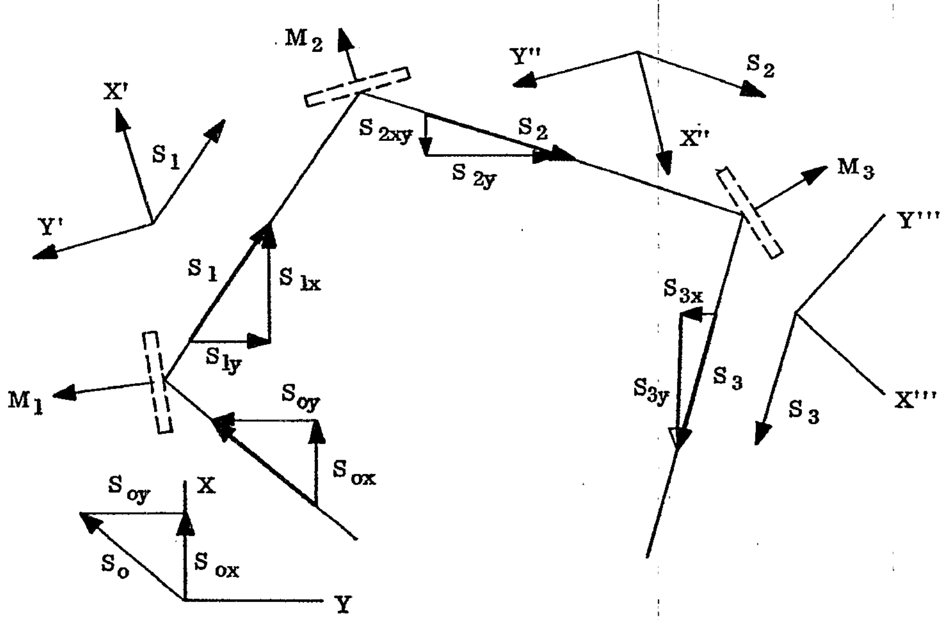
$$x\,(i \cdot i) + y\,(i . j) + z\,(i \cdot k) = x'\,(i \cdot i') + y'\,(i \cdot j') + z'\,(i \cdot k'). \tag{21}$$

If we let l_1, m_1, and n_1, be the direction cosines for the X' axis in the XYZ coordinate system, where the



Position coordinates of the point P are
P (x, y, z) in the XYZ system, and
P (x', y', z') in the X'Y'Z' system

Figure 13.7-Rotation of the coordinate axes.



Ray Unit Vector	Components in XYZ System	In Mirror Image System	
$\vec{S_o}$	S_{ox} and S_{oy}	$S_{ox} = S_{ox}$	$S_{oy} = S_{oy}$
$\vec{S_1}$	S_{1x} and S_{1y}	$S_{1x'} = S_{ox}$	$S_{1y'} = S_{oy}$
$\vec{S_2}$	S_{2x} and S_{2y}	$S_{2x''} = S_{ox}$	$S_{2y''} = S_{oy}$
$\vec{S_3}$	S_{3x} and S_{3y}	$S_{3x'''} = S_{ox}$	$S_{3y'''} = S_{oy}$

Figure 13. 8 - Diagram showing how the mirrors cause rotation of the coordinate system.

direction angles are α_1, β_1 and γ_1, respectively, then, the dot product,

$$i' \cdot i = (i' \cos \alpha_1)\ i = \cos \alpha_1,$$

since i' and i are unit vectors, and similarly;

$$i' \cdot j = \cos \beta_1,$$

and

$$i' \cdot k = \cos \gamma_1.$$

We may then let

$$i' \cdot i = l_1,\quad i' \cdot j = m_1 \quad \text{and} \quad i' \cdot k = n_1,$$

and, similarly,

$$j' \cdot i = l_2,\quad j' \cdot j = m_2,\quad j' \cdot k = n_2,$$

$$k' \cdot i = l_3\quad k' \cdot j = m_3,\quad k' \cdot k = n_3,$$

where l_2, m_2 and n_2 are direction cosines of the Y'-axis and l_3, m_3 and n_3 are direction cosine of the Z'-axis respectively, in the XYZ coordinate system. We may now rewrite Equation (21):

$$x = x' l_1 + y' l_2 + z' l_3, \tag{22}$$

$$y = x' m_1 + y' m_2 + z' m_3, \tag{23}$$

and

$$z = x' n_1 + y' n_2 + z' n_3. \tag{24}$$

These three equations may be written in the matrix form,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \tag{25}$$

By similar reasoning, it can be shown that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{26}$$

13.4.4 Interpretation of the vector matrix.

13.4.4.1 Note that the above equations are exactly similar to Equation (19) which, therefore, can be thought of in the following way. The object ray has the direction cosines S_{0x}, S_{0y}, S_{0z} with respect to the $x_1 y_1 z_1$ coordinate axis. After reflection it has the direction cosines S_{1x}, S_{1y}, S_{1z} in the same coordinate system. See Figure 13.8. Another way to look at it is that reflection has caused a rotation of the coordinate system. The direction cosines of the new coordinate system with respect to the old are given by the terms in the reflection matrix R. This is a very convenient concept because it gives directly the rotation between the object and its image. There is a great deal known about rotation matrices. For example if the determinant of the matrix is -1, it means the image coordinate system is left-handed. One can check the determinant in the R matrix in Equation (19) and see that it is -1. This follows from the condition that M is a unit vector, and

$$M_x^2 + M_y^2 + M_z^2 = 1.$$

13.4.4.2 By Equation (20) it is evident that if there are an even number of reflections the determinant of the total reflection matrix is +1 while if there are an odd number of reflections the determinant of the matrix is -1.

This is stated in optics in the following way.

(1) An image seen by an even number of reflections is right-handed.

(2) An image seen by an odd number of reflections is left-handed.

A left-handed image of a readable page of print is not readable. A right-handed image of a readable page of print is readable. It may be turned at an odd angle, even upside down, but the observer can read it by standing on his head. A left-handed image is always backwards regardless of the orientation of the image. In Figure 13.9 the letter R is shown as left handed and right handed. The right-handed image may be made to appear normal by turning the paper around. The paper cannot be rotated into a position which will make the left-handed image readable.

Figure 13.9-The right and left-hand image.

13.5 THE IMAGE SPHERE

13.5.1 The external observer concept.

13.5.1.1 Some people find it helpful in understanding the imagery of a single mirror to make use of the image sphere shown in Figure 13.10.

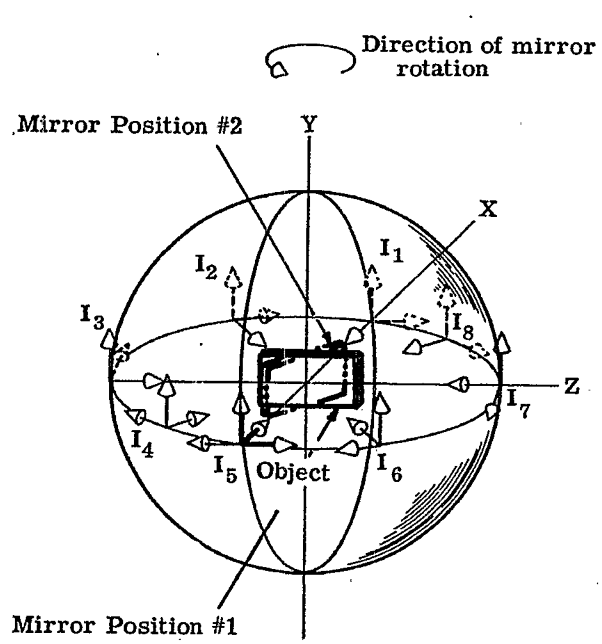


Figure 13.10-Image position and orientation in the Y-plane.

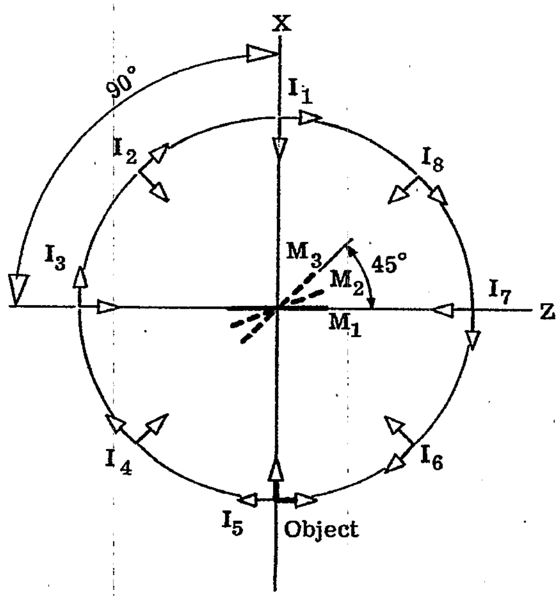


Figure 13.11-The Y-plane mirror rotation.

Suppose that an object represented by a small coordinate axis is located at 0° azimuth and 0° elevation as shown in Figure 13.10. Now imagine placing a mirror in the center of the sphere. By rotating this mirror the vertical images may be made to appear at any position of the surface of the sphere. For example, consider that we are looking directly down on the XZ plane of the sphere. Figure 13.11 shows this view. A plane mirror mounted in the center with its plane vertical, and facing the object as in position M_1, will produce a virtual image at I_1 as shown. This is very easily demonstrated by placing a small pocket mirror in position M_1 on Figure 13.11.

13.5.1.2 Now, as the mirror is rotated about the vertical axis (the Y axis) to position M_2, the image shifts to I_2, and similarly with M_3 and I_3 and so on until the image swings completely around in the horizontal plane. If you are using a pocket mirror, you will note that the image position and orientation coincides exactly with that drawn, regardless of the observer's position. Of course, the observer must place himself so that he can see the image to confirm this. The significance is that the image does have spatial position and orientation whether observed or not, and that this is related only to the object and mirror relationship.

13.5.1.3 Consider now, the image position shift in relation to the mirror. As the plane of the mirror was rotated through an angle of 45° from M_1 to M_3 the image position shifted through an angle of 90°.

13.5.1.4 Vertical relations are similar. If the mirror placed initially in the position shown in Figure 13.12, and then rotated about the horizontal axis (Z axis), the image will assume the positions and orientations shown. Figure 13.13 shows a projection of the XY plane. Experiments with a plane mirror will again confirm the accuracy of the illustrations, if the observer remembers that the Z axis is pointing "up" from the paper.

13.5.1.5 To make full use of this concept, Figure 13.14 illustrates the position and orientation of the image for compound angles. In each case, the mirror has been tipped 22 1/2° from vertical and rotated 22 1/2° from the Z axis in the XZ plane.

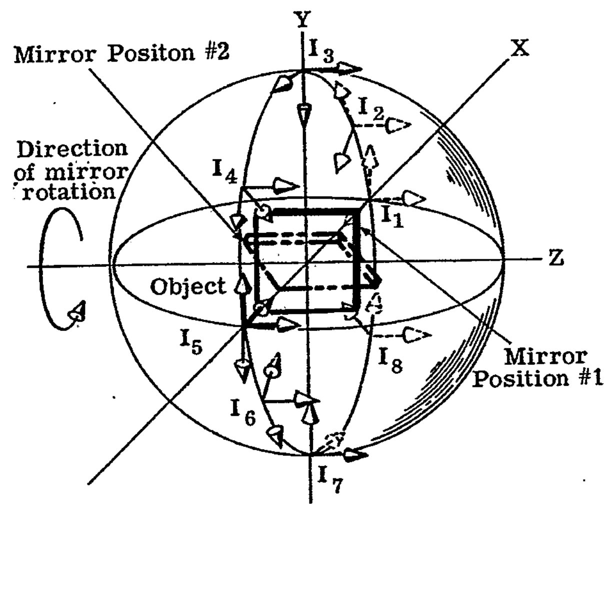


Figure 13.12-Image position and orientation in the Z-plane.

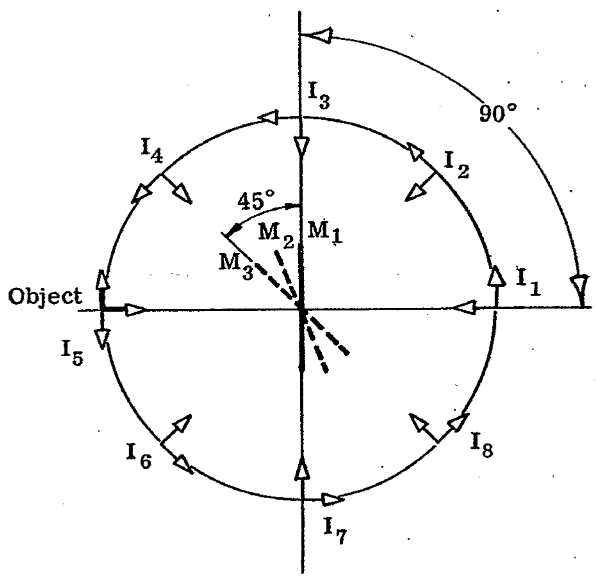


Figure 13.13-Projection of the XY-plane.

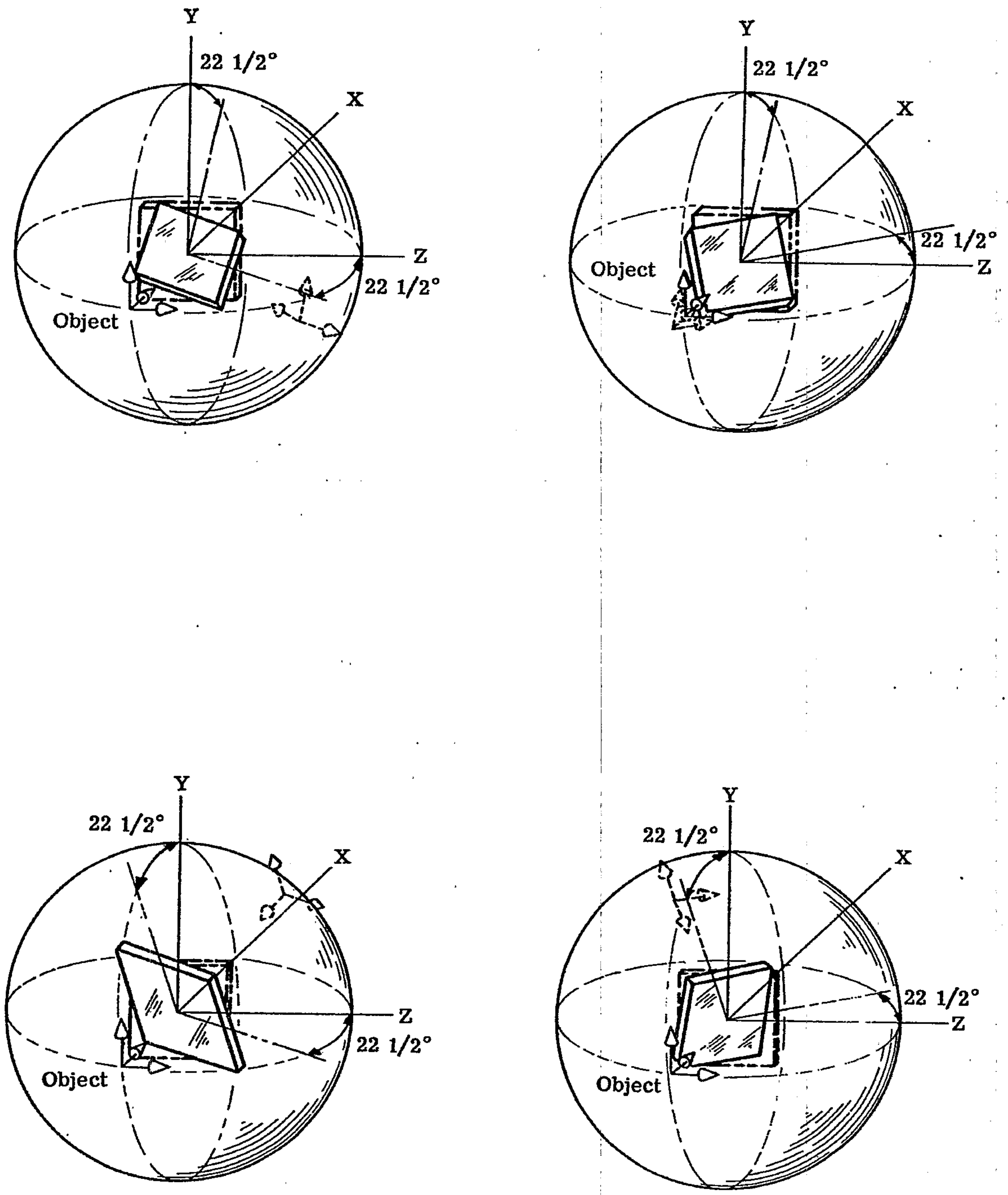


Figure 13.14-Image position and orientation for compound angles.

13.5.2 The internal observer concept.

13.5.2.1 It may be more convenient to visualize this in the following way. Imagine you are in a large sphere at the center. Assume that the X axis is due North and South. The Z axis is the East and West and the Y axis is straight up and down. See Figure 13.15(d). Along side of you is a projector, projecting an image on the inside of the sphere due south on the horizon. By placing a mirror in front of the projector the virtual images may be projected to any position on the sphere. See Figure 13.15.

13.5.2.2 First consider the case where the mirror reflects the light just east or west of due south at 0° elevation. It will not be possible to project it exactly where the original projected image is for then the plane of the mirror would be exactly parallel to the mirror, but it would be possible to reflect some light a few degrees to the east or west. The projected image would then appear as shown in Figure 13.15. As the mirror is rotated and the images are always located at an equal angular position around the object, they appear to be rotated. When the images are located in the horizontal plane it appears left handed but erect. (The y' axis is in the same direction as the y axis). When the image appears in the vertical plane it appears left handed but upside down. As the image is rotated through 90° its orientation turns 180°. Intermediate positions are linearly connected.

13.5.2.3 This concept enables us to predict the orientation for the position of the image at any position on the sphere. To do this one uses the following reasoning. Suppose one wishes to project an image on the inside of the sphere at a point with an azimuth angle of 45° and an elevation angle of 30°. If one images a cone with its apex at the center and its axis along the X axis, it will pierce the sphere at a circle. This circle is the one shown in Figure 13.12. This circle defines a plane. Images on this circle rotate twice as fast as the angle θ between the Y plane and a line drawn perpendicular from the X axis to the image point P. Therefore if θ can be calculated, the rotation of the image is known. Figure 13.15 illustrates the above case.

$$\tan \theta = \frac{\tan \phi}{\sin \omega}$$

where ω = azimuth angle and ϕ = elevation angle.

In the above cited example $\omega = 45°$ and $\phi = 30°$.

$$\tan \theta = \frac{.5774}{.7071} = 0.81657$$

$$\theta = 39.2°.$$

The image will therefore have been rotated by 2θ or 78.4°.

13.6 REFLECTION FROM TWO MIRRORS

13.6.1 Location of the image.

13.6.1.1 In the case of reflection from a single mirror, the image may always be located by projecting a line from the object perpendicular to the mirror and locating the image on the extension of this perpendicular at an equal distance behind the mirror as in paragraph 13.3. For the double mirror system the image is located in a plane perpendicular to the intersecting edges.

13.6.1.2 Figure 13.16 illustrates a special case of this. In the illustration, the two mirrors are perpendicular. The image points P' and P'' have been located by first constructing the perpendicular from P to mirror #1 and locating P' as above. Then, using P' as the object point for mirror #2, the same procedure was used to locate P''. It is, therefore, evident that the perpendiculars PP' and P'P'' lie in the plane PP'P''. Now, since mirrors #1 and #2 are perpendicular to PP' and P'P'' respectively, the intersection of their planes, LL' is perpendicular to the plane PP'P''. From the illustration one can see that the image P'' formed by the second reflector lies on the line PP'' and that this line intersects LL' and is perpendicular to it. In a more general case where the mirrors are not perpendicular, the plane PP'P'' will still be perpendicular to LL' but the line PP'' will not intersect LL'.

13.6.2 Axiom for locating the image. The location of the image in a perpendicular double-mirror system may be found by projecting a line from the object point through, and perpenducular to the line of intersection of the mirror surfaces. The image may lay on the extended perpendicular an equal distance behind the line of intersection and will be right handed since there are two reflections.

13.6.3 Invariant position of the image. Since the image in a double mirror lies in a plane normal to the intersecting edge of the two mirrors, the positioning of the image depends on the position in space of the intersecting edge. If the double mirror system is rotated around the intersecting edge the image does not move at all. If the intersecting edge is rotated or moved sidewise the image will move accordingly.

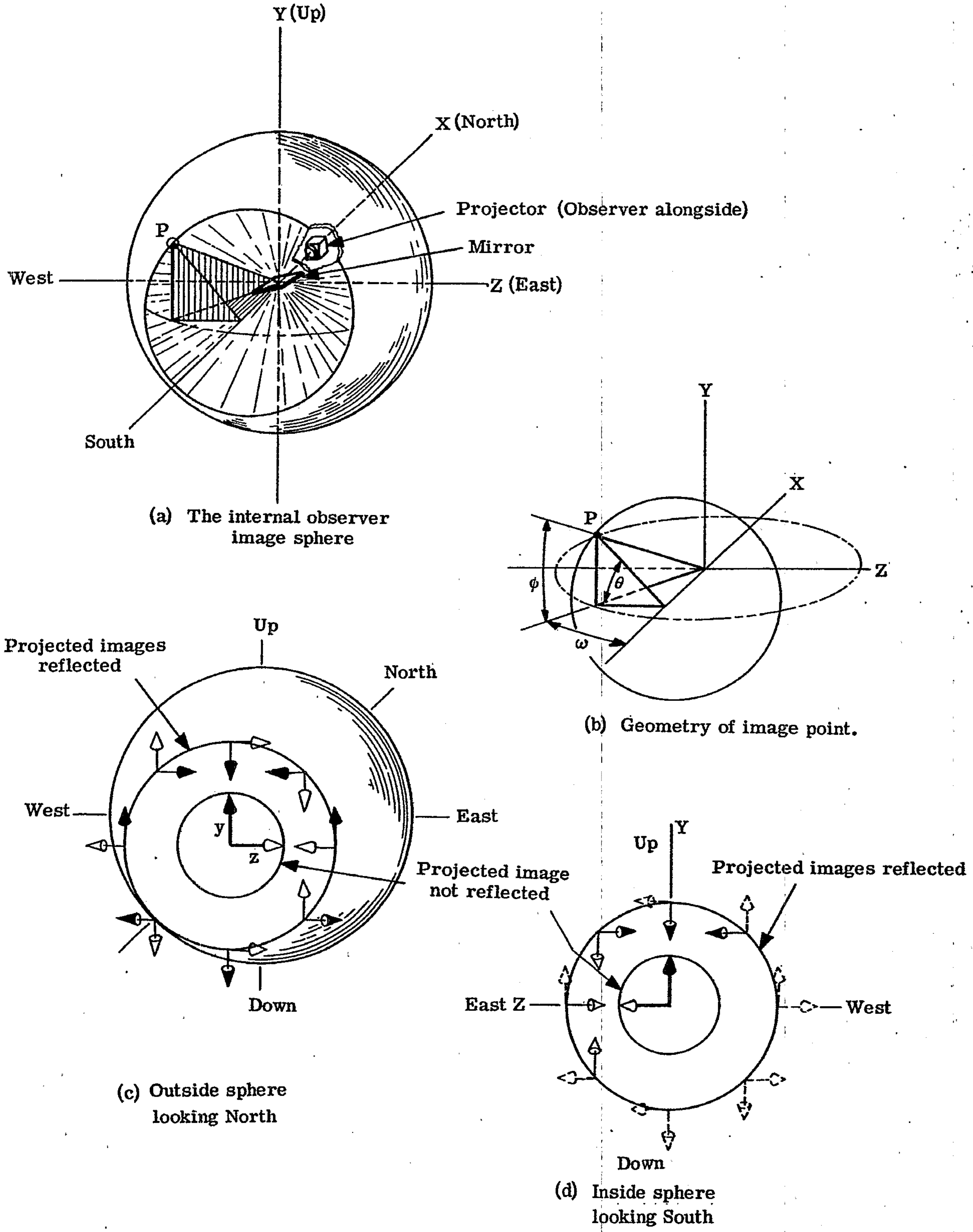
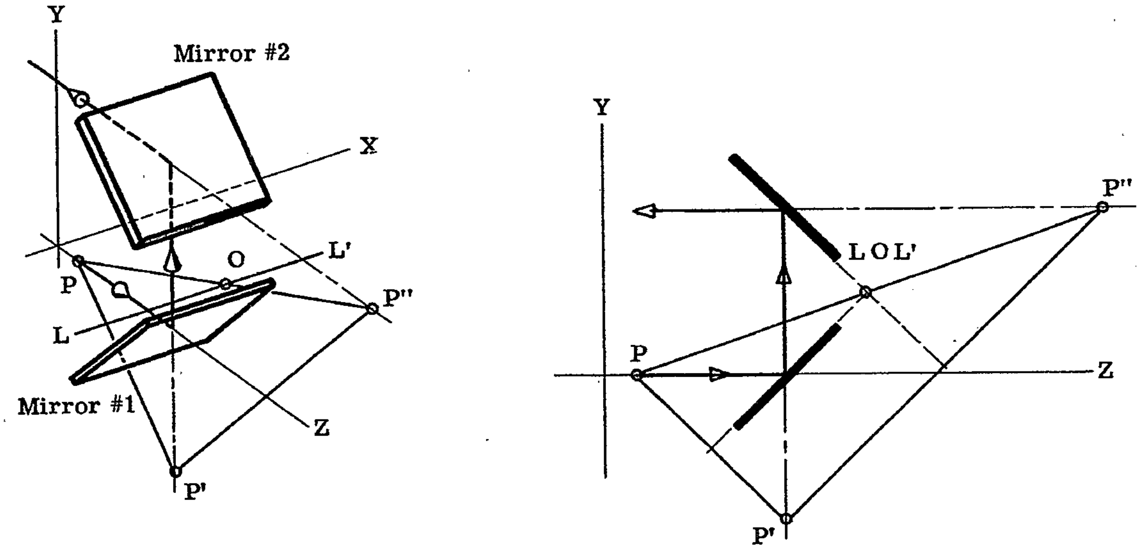


Figure 13.15-The solid-angle image.



(a) Three dimensional view. (b) Projection on YZ plane.

Figure 13.16-Reflection from two perpendicular mirrors.

13.7 TYPICAL PRISM SYSTEMS

13.7.1 Prisms and Mirrors. With the basic principles of mirror systems having been discussed, the analysis of some simple systems can be undertaken. In this analysis the reader should bear in mind that we are concerning ourselves principally with reflecting prisms. The reflecting faces of these prisms behave like mirrors rigidly mounted with respect to each other.

13.7.2 Illustration conventions.

13.7.2.1 In order to provide the reader with illustrations which require the minimum mental orientation to see both object and image correctly, we have portrayed the object as the letter R illuminated from behind by a collimated beam, the central ray of which is indicated by ⟶ . The image is illustrated by the appearance of the projected image that would be produced if a direct vision screen, such as frosted glass, were held normal to the emergent beam.

13.7.2.2 To observe either object or image the reader should view them as if the central ray from them were directed at his eye. When the limits of graphic art prohibit showing both object and image from the viewpoint of the observer, the projected image will be dashed to indicate it is shown from the wrong viewpoint. This enables illustration of the effect produced by multiple reflection systems without concern for the effect of each individual reflection. This does not permit indication of the apparent position of the virtual image (except for Figure 13.17 where both are shown) but does show left-or right-handedness.

13.7.3 The 45°-90°-45° Prism.

13.7.3.1 This simple prism can be used in many different ways. It can turn a beam through a 90 degree or 180 degree bend, or it can be used to invert an image.

13.7.3.2 To turn a beam through 90 degree, the prism is used as shown in Figure 13.17. Since there is only one reflection, the image is left-handed. The projected image is what the observer would see on a translucent back-lighted screen as described in paragraph 13.7.2, above. If the screen were removed, the virtual image would still be left-handed but located on the extended line of sight behind the reflecting surface as in the case of a single plane mirror. If the normal to the hypotenuse is in a horizontal plane the right hand object is swung around a vertical axis. The letter R will appear as Я. If the normal to the hypotenuse lies in the vertical plane the image will appear rotated around a horizontal axis. The letter R will appear as ᴚ.

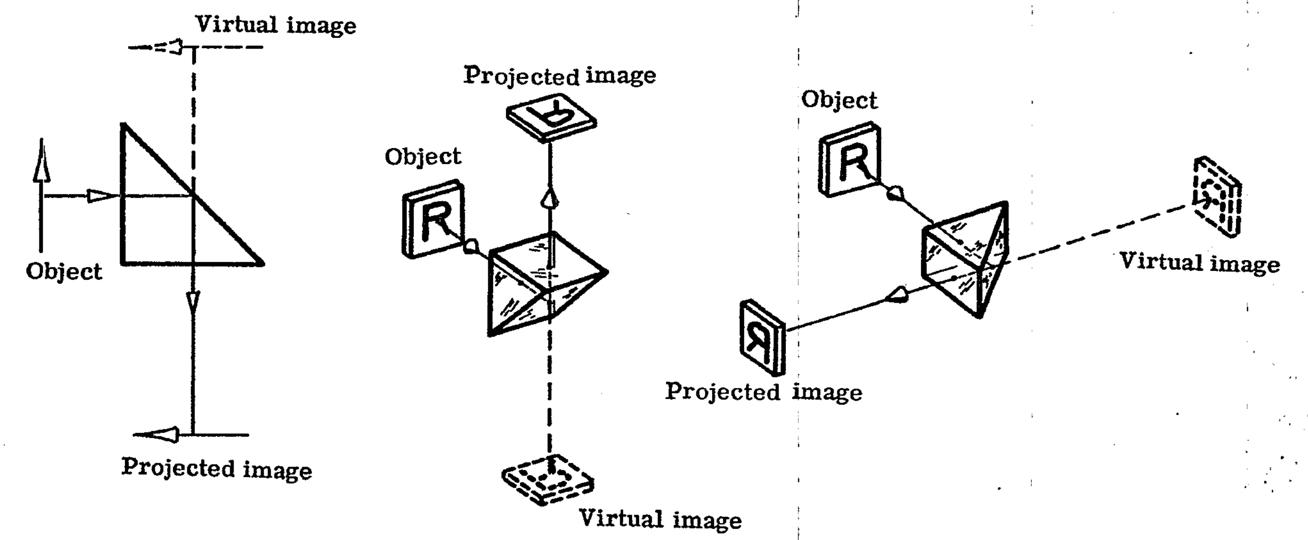


Figure 13.17-The 45° - 90° - 45° prism used as a right-angle prism.

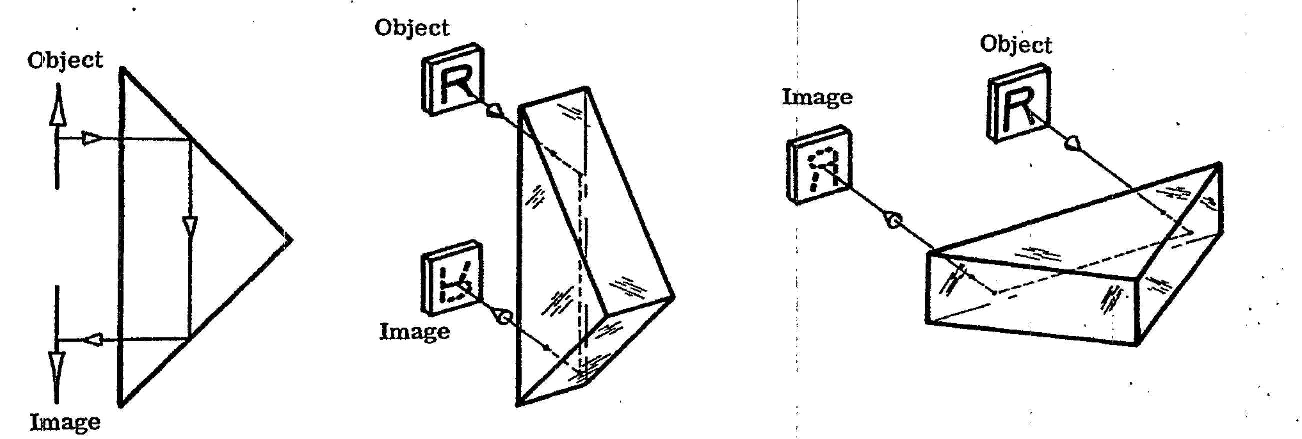


Figure 13.18-The 45° - 90° - 45° prism used as a Porro prism.

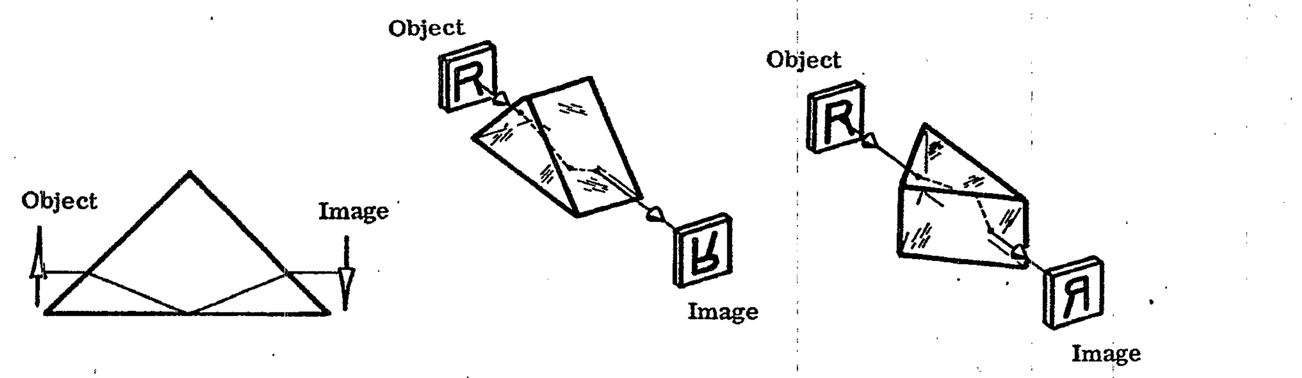


Figure 13.19-The 45° - 90° - 45° prism used as a Dove prism.

13.7.3.3 When used as a double mirror system, the prism is positioned as shown in Figure 13.18. Since there are two reflections, the image will be right-handed. In the illustration the projected image is shown in dashed lines indicating the observer would view it from the opposite side of the screen. To the observer so stationed it would appear as ꓤ if the roof edge of the prism (the edge formed by the intersection of the reflecting surfaces) is horizontal. If the roof edge is vertical, the image will appear as R. With this prism it is possible to rotate the image into any desired orientation and always have it right-handed. Used in this fashion, the 45° - 90° - 45° prism is called a Porro prism and will be discussed in detail later.

13.7.3.4 When used as shown in Figure 13.19, it is called a Dove prism and can be used to rotate an image. There is a single reflection so the image is left-handed. If the normal to the hypotenuse face lies in the vertical plane, the letter R appears as ᴚ . When the normal lies in the horizontal plane, the image appears as Я

13.7.4 <u>Use of prisms in telescope systems.</u>

13.7.4.1 One of the main uses of prisms is to provide the proper orientation of the image in telescopes. The image in a simple telescope, which consists of an objective and eyepiece, is right-handed but upside down. The image may be made erect by using two prisms. In order to keep the image right-handed the prism system must have an even number of reflections. The minimum number of reflecting surfaces is two. A prism system which does this is shown in Figure 13.20 as it may be used in a telescope.

13.7.4.2 The prism illustrated in Figure 13.20 is called an Amici prism and is described in more detail in Section 13.7.5. It is essentially a 45° - 45° - 90° prism with the hypotenuse face made into a roof. It is for that reason often called a <u>roof prism.</u> In Figure 13.21 a beam is drawn showing how it reflects a cylinder of light. This drawing shows plan and elevation views of the prism. A view looking along the roof edge and a pictorial three dimensional view are also shown. The selected rays traced through the prism show how the image is rotated 180 degrees. The dotted lines show that this prism is cut out of a large Amici prism. One can see that as the cylinder of light passes through the prism the complete cylinder strikes first one face of the roof and then crosses over to the other roof. If the roof angle is not exactly 90 degrees the only effect is that the exit and entrance angles no longer remain in parallel planes. While permitting easier manufacturing tolerances, this method is seldom used because it requires too large a block of glass for the space and

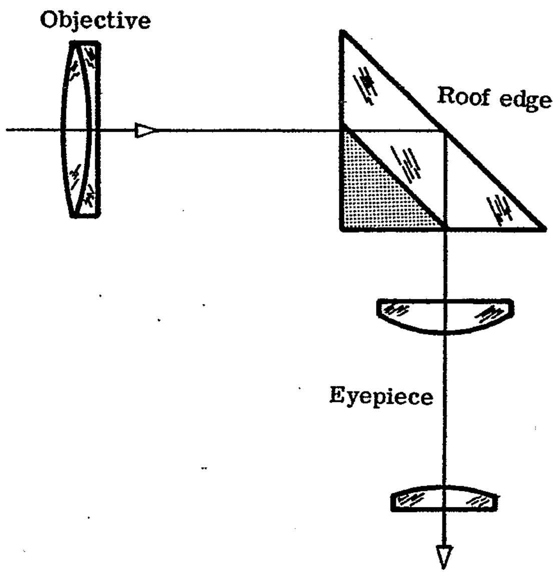


Figure 13.20-The Amici prism in a telescope.

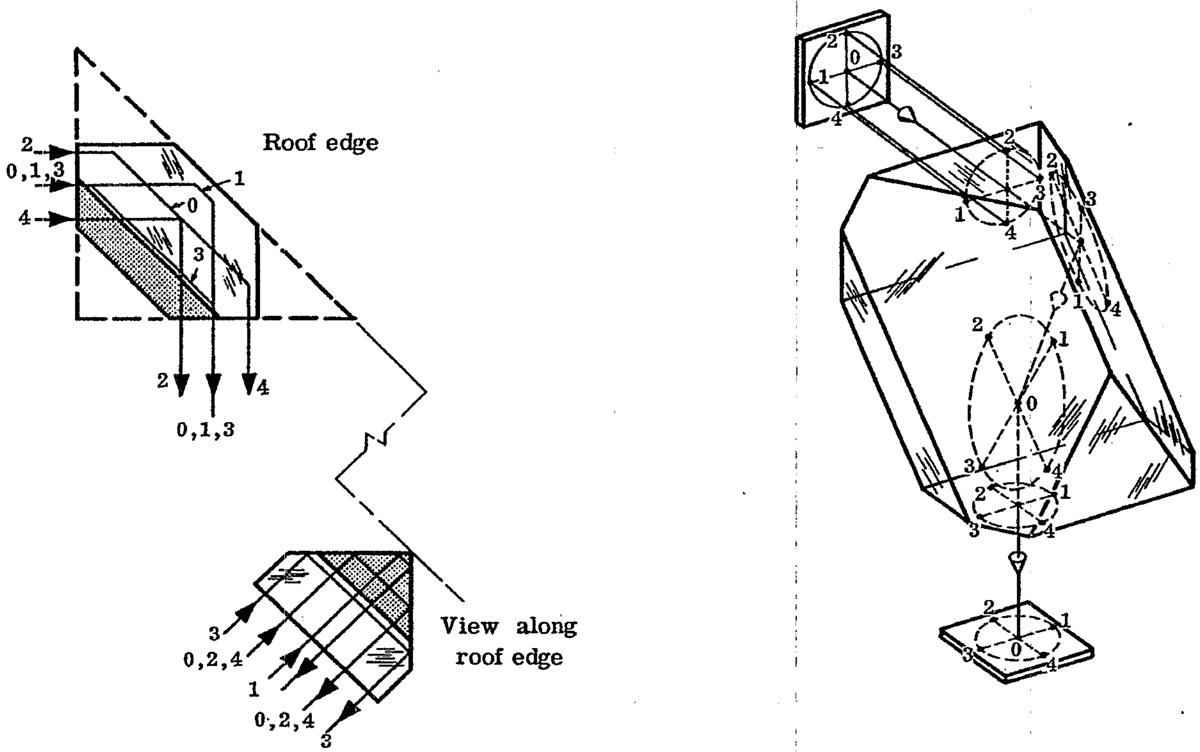


Figure 13.21-The Amici prism as a double reflector.

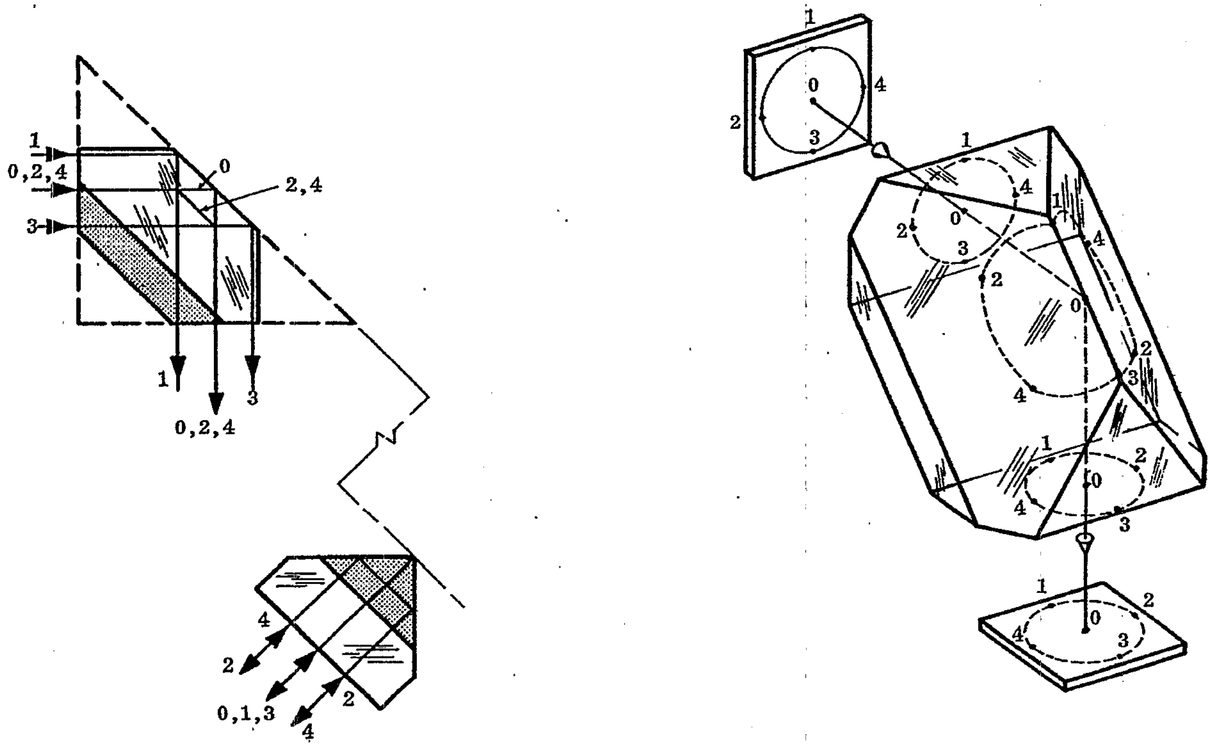


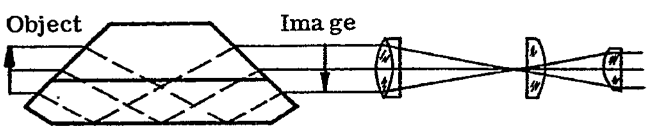
Figure 13.22-The Amici prism as a split reflector.

weight limitations of most applications.

13.7.4.3 A more common method of using the Amici prism is shown in Figure 13.22. This useage permits a much larger cylinder of light to pass through the same size prism, or conversely, to handle the same size cylinder of light with a much smaller prism than that of Figure 13.21. There is a fundamental difference between the two applications. In Figure 13.22 the beam is split by the prism's roof edge. If there is any error in the 90 degree roof edge angle the entering beam is split into two beams and a double image is formed. This means that if the Amici prism is used in this manner the 90 degree angle must be made to high degree of precision. In most applications this angle has to be held to 90 degree ± three or four seconds. Roof prisms as used in Figure 13.22 are very efficient as far as size goes but they are expensive to make because of the precision required. If the prism is used as shown in Figure 13.21 the accuracy required is not as high but the prism has to be much larger in order to pass the same size beam.

13.7.4.4 It is instructive to draw these views of roof prisms and show the path of rays passing through them. In Figure 13.21 the prism can be cut even further to reduce the weight of glass. How would one decide how it could be cut and not interfere with the beam passing through? In telescopes the objective is usually larger than the eyepiece field stop, so that the prism must pass a section of a cone rather than a cylinder. This means the entering circle and the exit circle are different sizes. It is a good exercise to try and lay out a prism of minimum size and then to determine how corners can be cut to further reduce the weight. This is prism design.

13.7.5 Prism rotation of the image through 180 degree. The Amici prism is the simplest method for erecting the image in a telescope, but it has the difficulty that one must look around a corner. A Dove prism with a roof on the hypotenuse face as shown in Figure 13.23 uses the double mirror principle of the Amici prism. This prism must be located in front of the objective in parallel light. If it is located in between the objective and the eyepiece it causes aberrations because of the refraction of the slanting surfaces. More will be said about this in the tunnel diagram Section 13.8. If it is necessary to have the optical axis of the telescope objective and eyepiece parallel, the Amici prism can be used with other prisms to bend the light through 90 degrees. It is necessary however to use two reflections in order to preserve the right-handed use of the image. Figure 13.24 shows a penta prism with two reflections which could be used with the Amici prism.



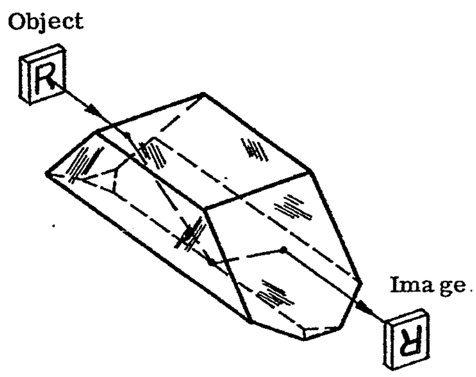


Figure 13.23-The Amici prism in telescope systems.

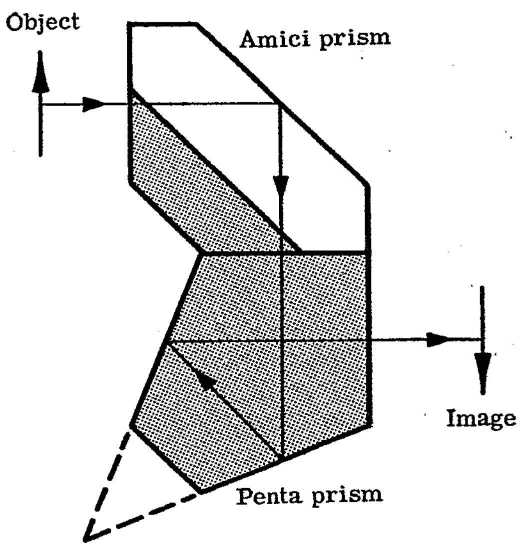


Figure 13.24-An Amici and penta prism system.

13.7.6 The Porro prism.

13.7.6.1 The most common method for erecting the image in a telescope is the Porro prism system. This is made up of two 45° - 45° - 90° prisms as indicated in Figure 13.25. The first prism is positioned so that the roof edge is perpendicular to the corresponding edge in the other prism. One can understand the action of the prism by considering the explanation illustrated in Figure 13.26.

13.7.6.2 This diagram illustrates one of the sources of confusion in understanding of prisms. It shows how the prism A rotates the image around the line of the intersecting edge. But note that as shown the R is left-handed. Why is this when it has been clearly stated that double reflection always provide a right-handed image? If the reader will recall Paragraph 13.7.2 on illustration conventions, it then will become apparent that in the drawing of Figure 13.26 the image of the object is not being presented from the viewpoint of the observer. If you imagine standing and looking at the original object, then it would not be possible to see the image after passing through prism A. It would be necessary to turn yourself completely around. The image shown in the drawing is the image as viewed with the light moving away from you. If you turn yourself around and look at this image from the back of the paper it will appear right-handed. Prism B in effect does this for us. It merely reflects the image from prism A around so that it can be seen from the same direction as the original object.

13.7.6.3 Figure 13.26 shows that the orientation of the final image depends only on the relative positions of the intersecting edges of the two prisms. As long as they are perpendicular to each other the final image is completely erected. If there is an error from perpendicularity of the amount ϵ, the image will be rotated by 2ϵ.

13.7.6.4 The Porro system is a popular design because the 45° - 45° - 90° prisms can be made with reasonably broad tolerances in the angles. The optical beam is not split as it is with the roof prism so prism angle errors do not cause any image doubling. Angle errors merely cause a deviation in the optical axis as it passes through the prism. The exit optical axis may not end exactly parallel to the entering axis.

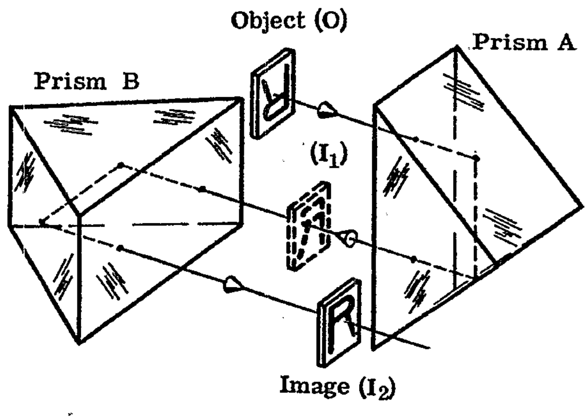


Figure 13.25-Reflections through the Porro prism.

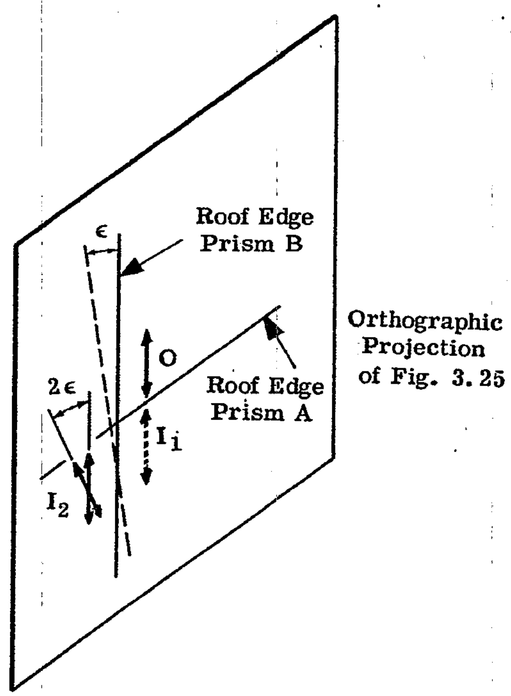


Figure 13.26-Image rotation in the Porro prism.

13.8 THE TUNNEL DIAGRAM

13.8.1 Right angle prism tunnel.

13.8.1.1 It is very convenient in laying out prisms to "fold" the prism around the reflecting surfaces. This generates a tunnel diagram. Consider the prism in Figure 13.27 The hypotenuse face BC can be considered as a mirror. The faces AB and AC may be considered as imaged in this mirror as shown dotted. The ray of light passing through the prism may also be considered imaged as shown. An observer looking into face AB therefore sees face AC at A'C. It appears as though he is looking straight through a block of glass of thickness BA'. One can check immediately that the angle ABC is equal to the angle ACB then the imaged face A'C is parallel to the face BA. Optically then the prism introduces a block of glass in the optical system. As far as design considerations are concerned the prism may be considered as merely the insertion of a thick block of glass and may be treated as two ordinary parallel plane surfaces where rays are traced as straight lines within the prism.

13.8.1.2 The tunnel diagram helps one to realize that any prism system used to erect images or turn light around corners should "fold" out in a tunnel diagram so that the entering and exit faces are parallel. If they end up nonparallel then the prism will cause chromatic dispersion.

13.8.2 The Porro prism tunnel.

13.8.2.1 Figure 13.28 is the tunnel diagram for the 45° - 45° -90° prism as used in a Porro system. The original Porro, ABC, Figure 13.29, has been folded around AB to image C as C' and around BC' to image A as A'. The tunnel diagram, Figure 13.28, is then a square with AC', A'C' and A'C as images of AC while A'B and BC' are images of AB and BC respectively. However, since the prism is now considered to be replaced by a glass block and since AB , BC, A'B and BC' all lie within the block, we can ignore them, as a little thought will soon show. We can now easily lay out rays entering the block through face AC by computing their refraction and extending the refracted ray on a straight line through the prism.

13.8.2.2 Let us consider the passage of several rays traced through the Porro prism Figure 13.29 and through the tunnel diagram Figure 13.28. Ray R_1 Figure 13.29 enters parallel to and above the optical axis of the prism and is reflected parallel to and an equal distance below the optical axis as R'_1. In the tunnel diagram,

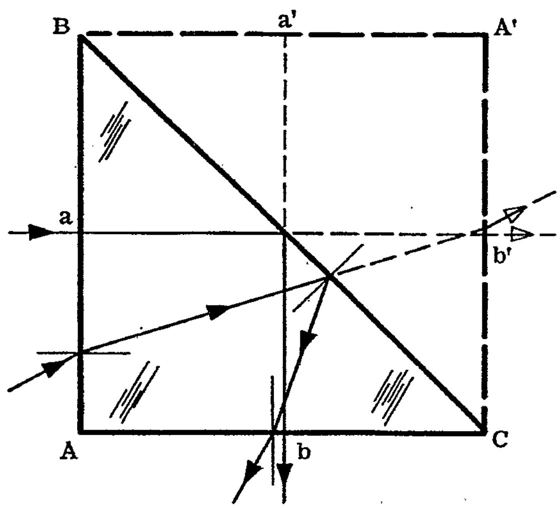


Figure 13.27-Right-angle prism tunnel diagram.

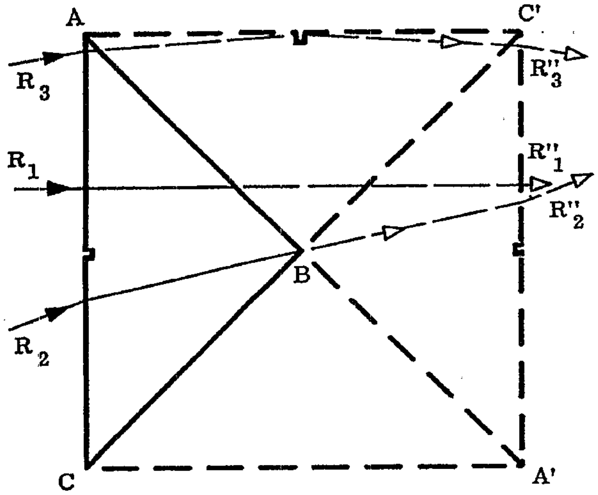


Figure 13.28-Porro prism tunnel diagram.

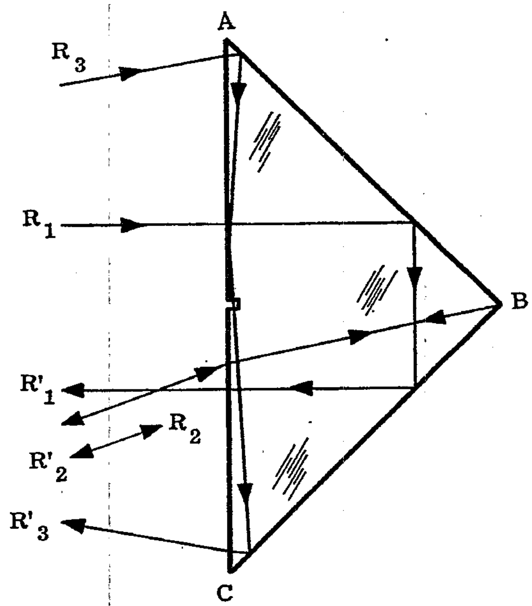


Figure 13.29-The Porro prism.

it emerges as R''_1, above the optical axis. However, note its relation to A' and C' as compared with the R'_1 relation to A and C. This tells us that the designer must interpret the tunnel diagram in the light of his knowledge of prism effect on the image orientation.

13.8.2.3 Consider further the ray R_2 in Figure 13.29 which enters the prism so as to strike the roof edge and be reflected back upon itself. Note the path of R''_2 in Figure 13.28 and again observe relation to A'C'.

13.8.2.4 The tunnel diagram is particularly useful in detecting the presence of unwanted reflections. In Figure 13.28 notice the ray R_3 entering the prism near A. It passes through the tunnel diagram very close to the hypotenuse face. A slight inclination of this ray and it could reflect off the hypotenuse surface as shown by the ray R''_3. This ray encounters three reflections in passing through the prism. This would cause a left-handed image. Since the prism is intended to be used with two reflections these rays with the extra reflection are called ghost rays. The ghost reflections may be eliminated by cutting a notch in the prism as shown in Figure 13.28. The tunnel diagrams for several prisms are shown in the data sheets on prisms at the end of this section.

13.8.3 <u>The reduced or apparent prism length.</u>

13.8.3.1 We have now satisfied ourselves that when prisms are introduced into an optical system they behave optically as would a block of solid glass with plane parallel faces; that rays may be easily traced through by refracting at entrance and exit faces, with the refracted ray travelling in a straight line within the prism; that the entering and exiting ray will be parallel. Consider then the point P on the surface of the block of glass shown in Figure 13.30. By using equations 6 - (2), 6 - (3), and 6 - (4), it may easily be determined that the image P' lies at a distance $t\,\frac{(n-1)}{n}$ from P. This means that from the right hand side of the block, P appears to be separated by t/n surface of the prism, or the prism appears to have a thickness of t/n. This is variously called the <u>reduced</u> or <u>apparent thickness</u> of the prism or the <u>air-equivalent prism.</u>

13.8.3.2 In drawing tunnel diagrams it is convenient to draw the reduced tunnel diagram. The actual and the reduced tunnel diagram for a penta prism are shown in Figure 13.31. The reduced prism is convenient for it is possible to trace rays directly through it without refracting them at the outside surfaces. This is of course an approximation since the effective thickness of a block was computed to be t/n with paraxial ray approximations.

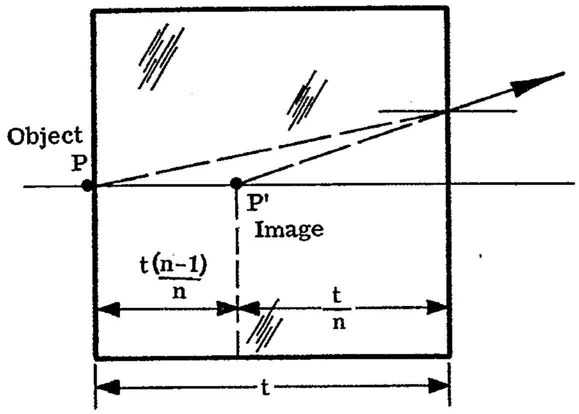


Figure 13.30-The apparent thickness of a glass block.

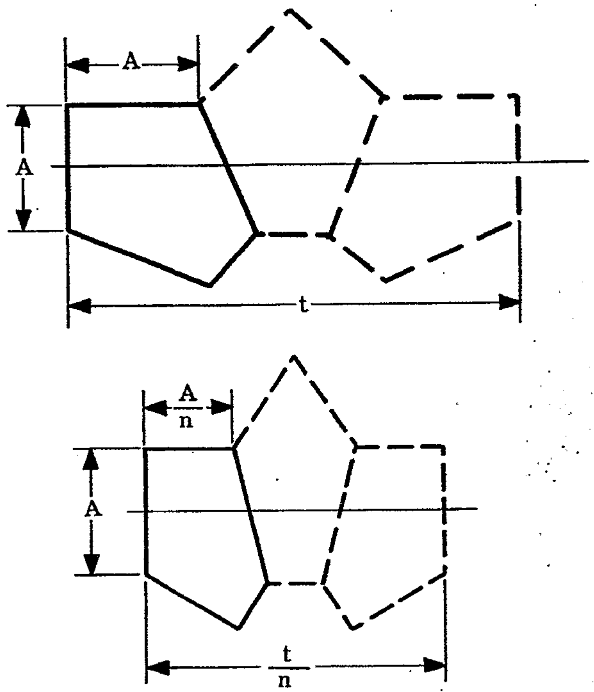


Figure 13.31-Actual and reduced tunnel diagram of a penta prism.

13.9 ABERRATIONS INTRODUCED BY PRISMS

13.9.1 Typical orientation. Reflecting prisms are generally designed so that the entering and exit faces are parallel and the entrance face is perpendicular to the optical axis. The aberrations introduced by the block of glass so oriented may be corrected by the normal centered lens system. The prism adds aberrations however only if it is located in a convergent or divergent beam of light. If the prism is in parallel light which is perpendicularly incident on the entrance or exit face, obviously no refraction, and therefore no aberrations will be introduced.

13.9.2 The third order aberrations introduced by a prism of thickness t and index n.

13.9.2.1 Figure 13.32 shows a block of glass in a convergent beam of light. The third order calculations for B, F and C are included in Table 13.1 . The contributions to E, a and b, are not included. They may be readily calculated as an exercise. One should notice that the total aberrations introduced do not depend on y_1 or $\bar{y}_1$. This shows that, as long as its faces are perpendicular to the optical axis of the system, the position of the prism has no influence on the aberrations. If the optical axis of the prism is parallel to but displaced from the system's axis, occlusion of part of the beam may occur with the resultant loss of imagery being comparable to the effect of an unsymmetrical stop being introduced. Angular misalignment however, will have the effect of changing the value of t and, further, will introduce assymmetry into the system.

13.9.2.2 The problem of prism design then, is not complete until the designer has computed manufacturing tolerances on the prism faces and provided for proper alignment within the system. Fortunately the latter is usually a problem in line with centering the instrument system, while the former is somewhat simplified by existence of design data on many commonly used prisms. This data is presented in the remaining pages of this section.

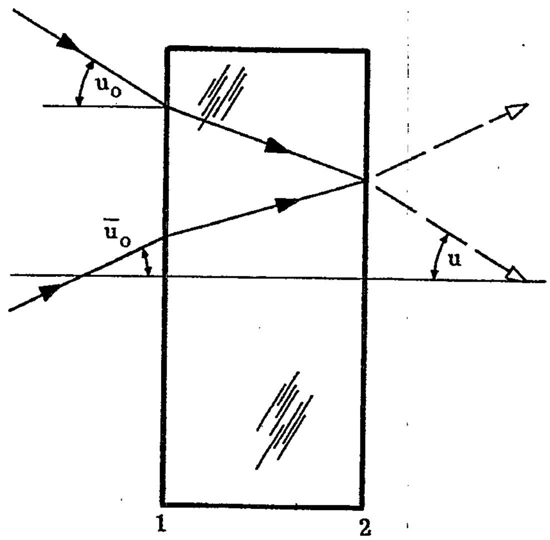


Figure 13.32-The glass block and the convergent beam of light.

Surface	0	1	2	Totals
c		0	0	
t	1	t	2	
n	1	n	1	
$\left(\frac{n_{-1}}{n} - 1\right)$		$\left(\frac{1}{n} - 1\right)$	$(n-1)$	
y		y_1	$(y_1 - \frac{t}{n} u_o)$	
u	u_o	$u_{o/n}$	u_o	
i		u_o	$u_{o/n}$	
$\bar{y}$		$\bar{y}_1$	$\bar{y}_2$	
$\bar{u}$	$\bar{u}_o$	$\bar{u}_{o/n}$	$\bar{u}_o$	
$\bar{i}$		$+\bar{u}_o$	$\bar{u}_{o/n}$	
S		$-y_1 u_o (n^2 - 1)/n^2$	$y_1 u_o (n - 1) - \frac{t}{n} u_o^2 (n^2 - 1)$	
B		$-y_1 u_o^3 \frac{(n^2-1)}{n^2}$	$y_1 u_o^3 \frac{(n^2-1)}{n^2} - \frac{t}{n^3} u \frac{(n^2-1)}{n^2}$	$\Sigma B = -t u_o^4 \frac{(n^2-1)}{n^3}$
F		$-y_1 u_o^2 \bar{u}_o \frac{(n-1)}{n}$	$y_1 u_o^2 \bar{u}_o \frac{(n^2-1)}{n^2} - t u_o^3 \bar{u}_o \frac{(n^2-1)}{n^3}$	$\Sigma F = -t u_o^3 \bar{u}_o \frac{(n^2-1)}{n^3}$
C		$-y_1 u_o \bar{u}_o^2 \frac{(n^2-1)}{n^2}$	$y_1 u_o \bar{u}_o^2 \frac{(n^2-1)}{n^2} - t u_o^2 \bar{u}_o^2 \frac{(n^2-1)}{n^3}$	$\Sigma C = -t u_o^2 \bar{u}_o^2 \frac{(n^2-1)}{n^3}$

Table 13.1-The third order aberrations introduced by a prism.

13.10 PRISM DATA SHEETS

13.10.1 Introduction.

13.10.1.1 The prism data sheets are presented as a guide to the designer and provide him with an orthographic projection, a tabular list of the dimensions, a tunnel diagram and a brief description of many different kinds of prisms.

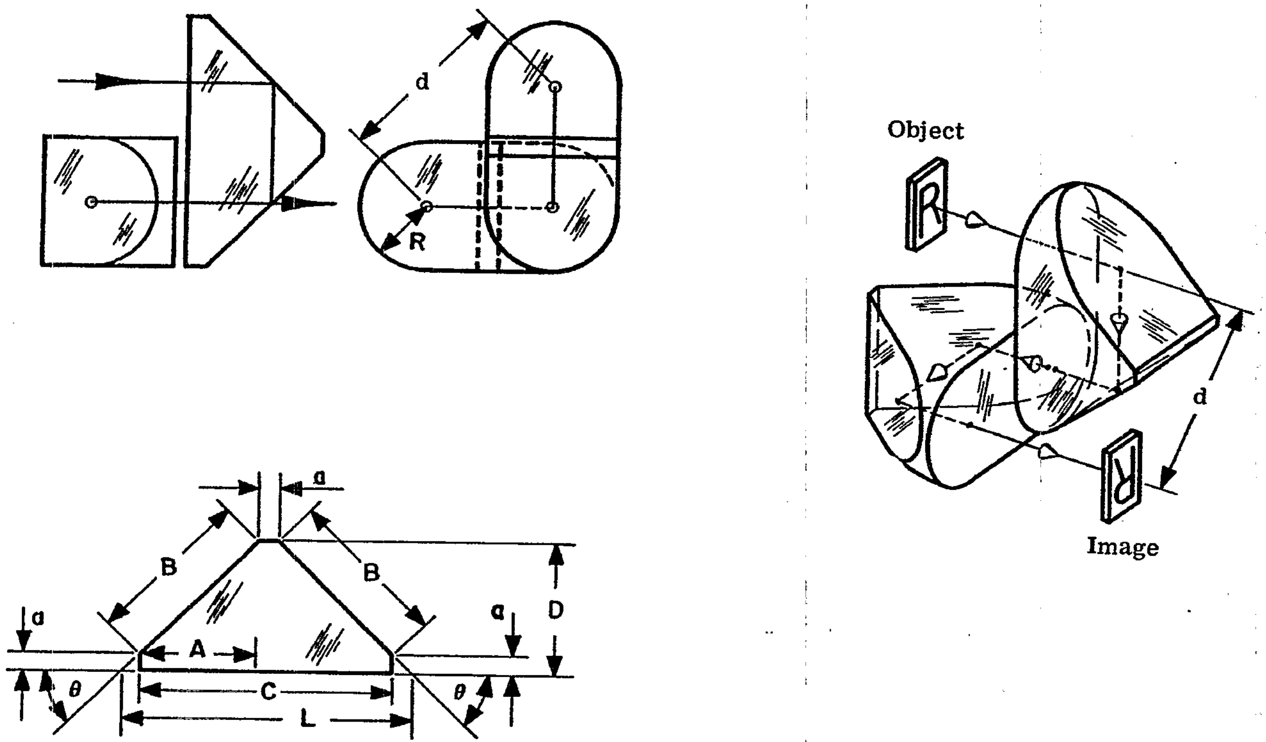
13.10.1.2 Notice that in the following data sheets, the terms invert and revert are used to describe the image. Invert means to rotate the object plane about a horizontal line in or parallel to the plane and produces the left-handed image one sees in a reflecting pool. Thus, for object R, the inverted image is ʁ. Revert means to rotate the object plane about a vertical line in or parallel to the plane and produces the left-handed image one sees in a shaving or dressing mirror. Thus, for the object R, Я is a reverted image. Obviously then for the object R, ᴚ is an inverted and reverted image.

13.10.1.3 The term "displace" refers to parallel separation of two lines. Thus we find that if an oblique ray strikes the entrance face of a plane parallel block, the ray leaving the exit face is parallel to but displaced from the entering ray. The word "deviate" refers to an angular relation between two lines. Thus in the foregoing example the line tracing the ray through the block is deviated by refraction at the surface.

13.10.1.4 The following symbols are used on the prism data sheets:

NOTATION	USE
Lettering guide capitals A, B, C,...	Linear dimensions of the geometric figure.
L	Over all length.
Lettering guide lower case a, b, c,...	Dimensions which are trigonometric functions of corresponding capital letters.
d	Displacement of the axial ray.
t	Optical path length of axial ray.
n	Index of refraction of the glass.
Greek letters	Angles.
α, β, γ	Direction angles.

13.10.2 Porro Prism System. In 1850 the Italian engineer Porro designed the prism system discussed here. This system consists of two right-angle prisms, usually identical in construction, placed at right angles to each other. It is a direct vision prism system but the axis is displaced by the amount d. This system will invert and revert the image.



$A = 1.00$ $n = 1.5170$ $\theta = 45°$ (These values are given) $\alpha = 0.10$ (chosen arbitrarily)
$R = A/2 = 0.50$ $B = 1.4142A = 1.4142$ $C = 2A + \alpha = 2.1$ $D = A + \alpha = 1.1$
$L = 2A + 3\alpha = 2.30$ $d = 1.4142\,(A + \alpha) = 1.5556$ $t = 2\,(2A + 3\alpha) = 4.60$ $t/n = 3.0324$

Figure 13.33-Porro prism system.

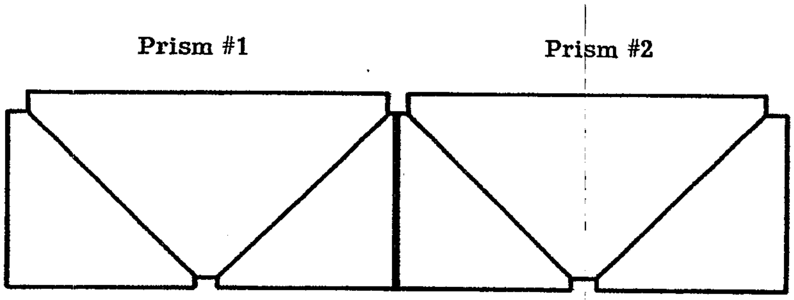
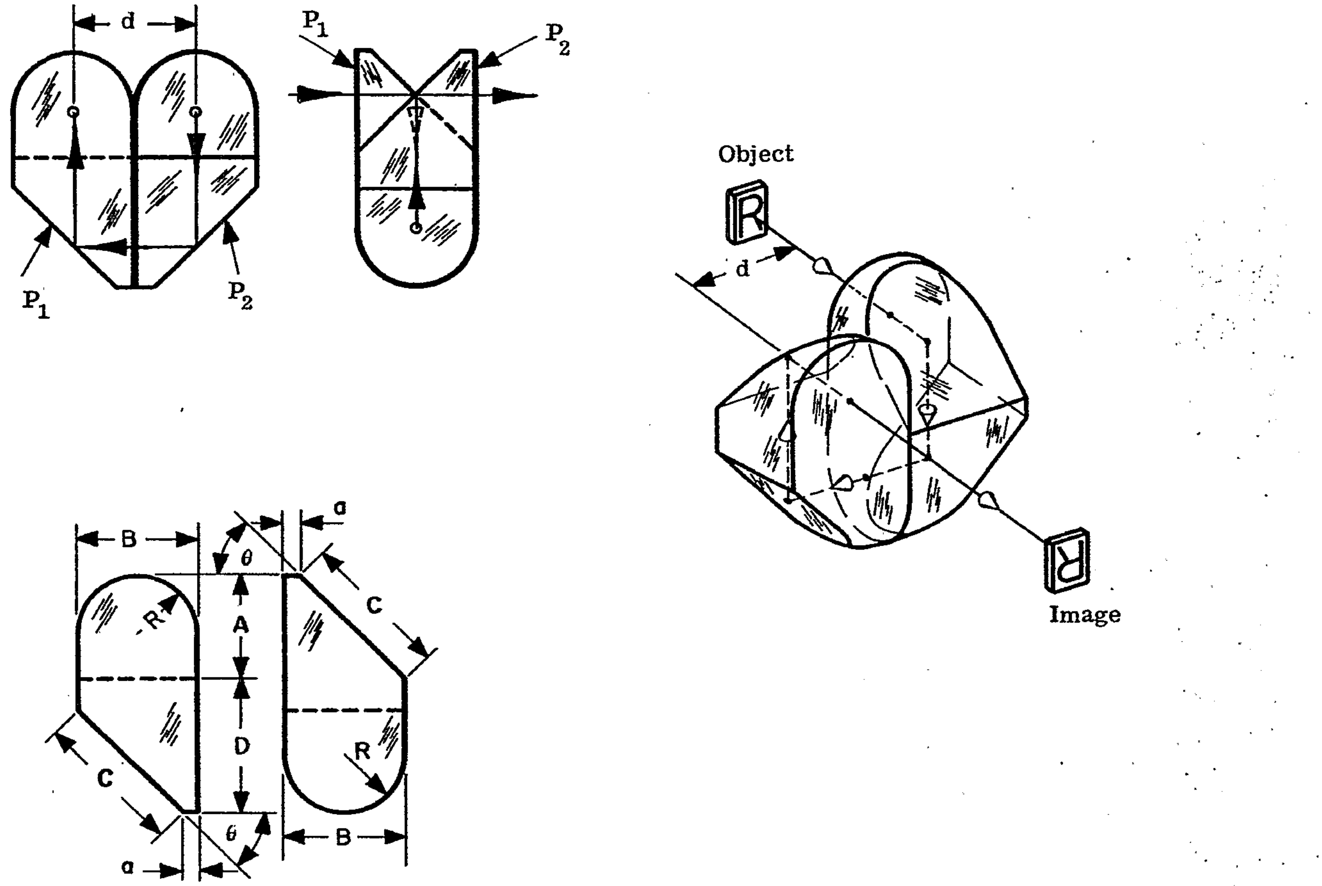


Figure 13.34-Porro prism tunnel diagram.

13.10.3 Abbe's Modification of the Porro Prism System. This prism system consists of two prisms cemented together. It will invert and revert the image. The system is a direct vision prism but the line of sight will be displaced by the amount d.



$A = 1.00$ $n = 1.5170$ $\theta = 45°$ $a = 0.10$ (chosen arbitrarily) $B = A + a = 1.10$
$C = 1.4142A = 1.4142$ $D = A + 2a = 1.20$ $R = B/2 = 0.55$ $d = B = 1.10$ $t/n = 3.0323$
$t = 2(2A + 3a) = 4.60$

Figure 13.35-Abbe prism system.

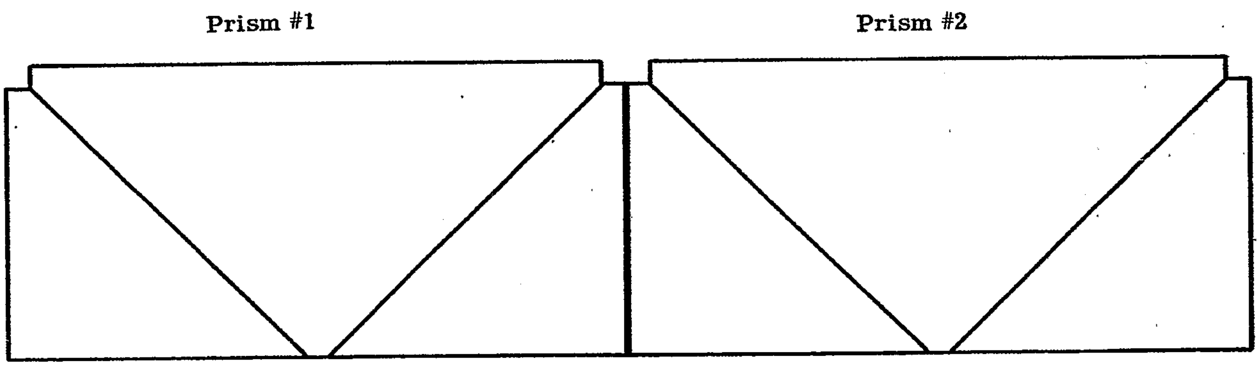
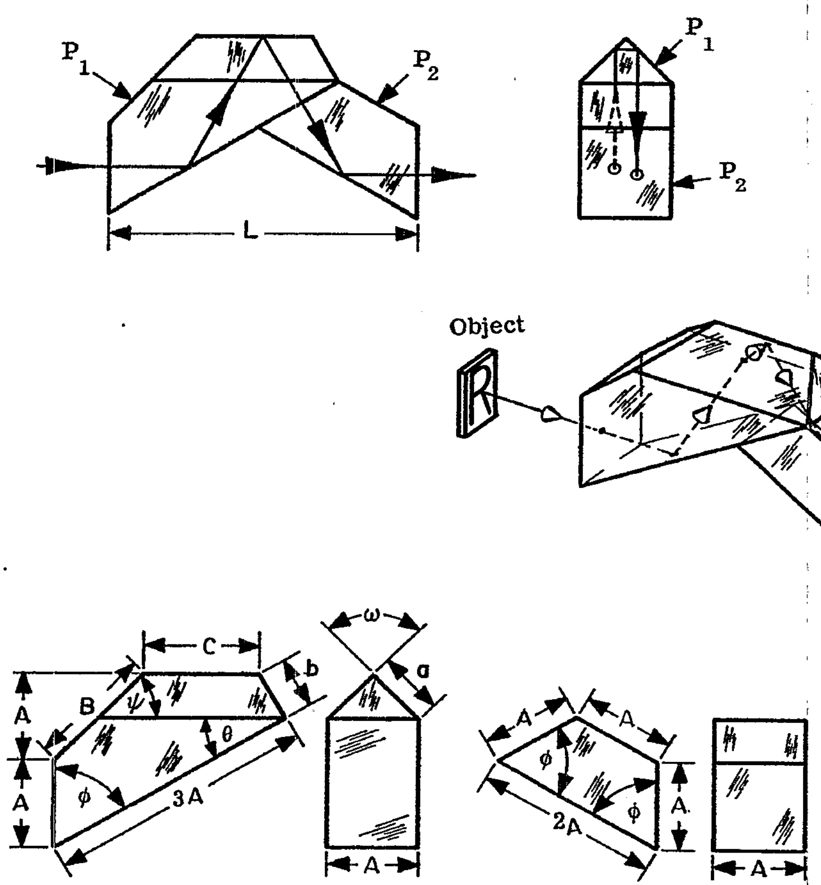


Figure 13.36-Abbe prism system tunnel diagram.

13.10.4 Abbe Prism, Type A. This prism inverts and reverts the image, but will not deviate the line of sight; hence, it is a "Direct Vision Prism." The prism is made in two pieces which are cemented together.



A = 1.00 θ = 30° ω = 90° n = 1.5170 ϕ = 60° ψ = 45° B = 1.4142 A = 1.4142
C = 1.3094A = 1.3094 a = 0.7071 A = 0.7071 b = 0.5774 A = 0.5774 L = 3.4644 A = 3.4644
t = 5.1962 A = 5.1962 t/n = 3.4253

Figure 13.37-Abbe prism, type A.

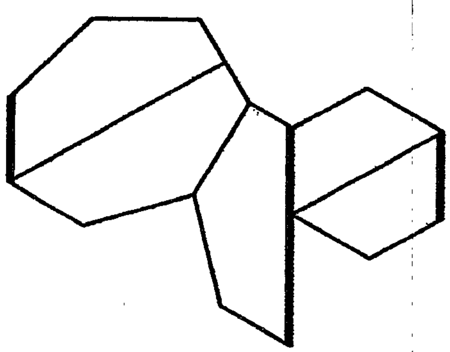
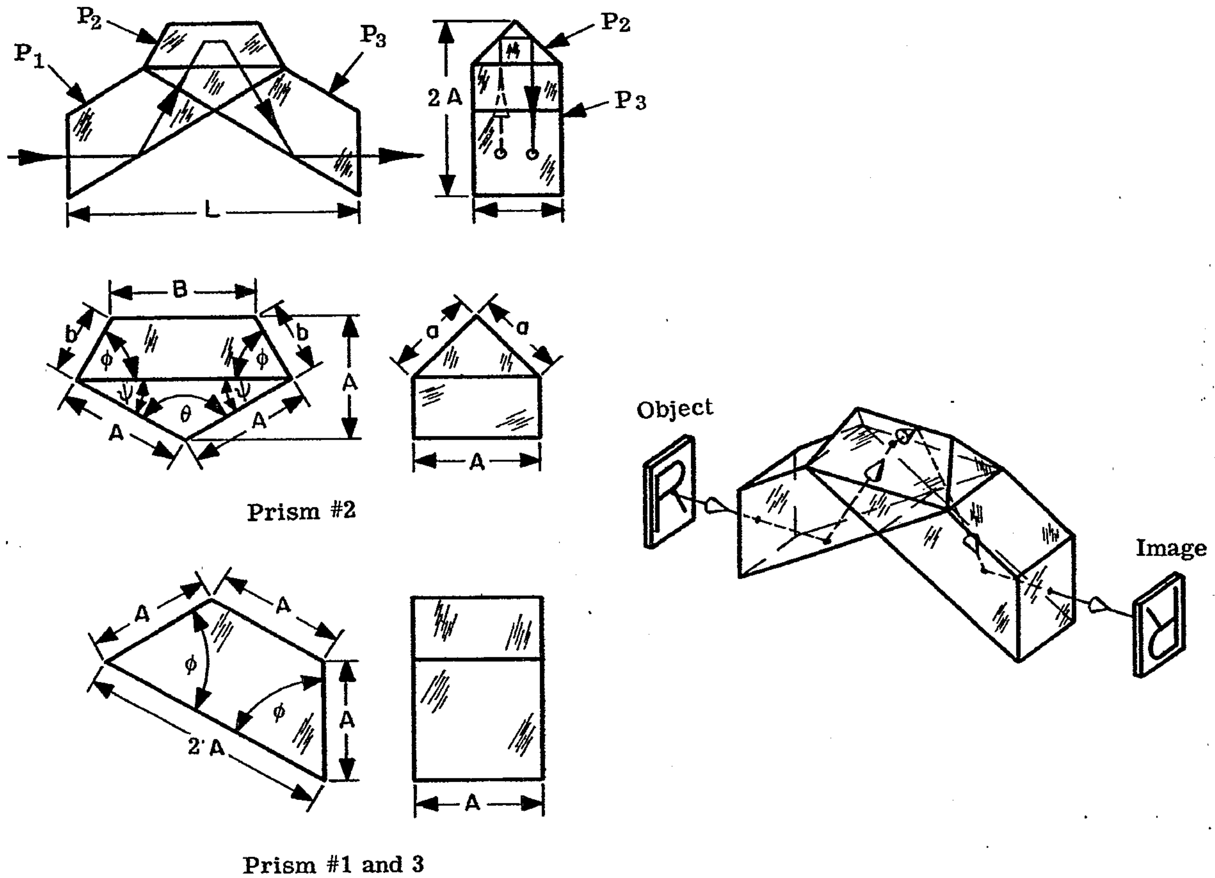


Figure 13.38-Abbe prism, type A, tunnel diagram.

13.10.5 Abbe Prism, Type B. This prism is made of three single units which are cemented together. This prism will invert and revert the image but will not deviate the line of sight. This also is a "Direct Vision Prism."



A = 1.00 θ = 135° ω = 45° ϕ = 60° ψ = 30° n = 1.5170 a = 0.7071A = 0.7071 t/n = 3.4253
b = 0.5773A = 0.5773 B = 1.1547A = 1.1547 L = 3.4641 A = 3.4641 t = 5.1962A = 5.1962

Figure 13.39-Abbe prism, type B.

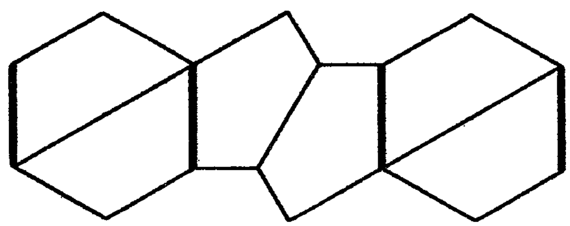
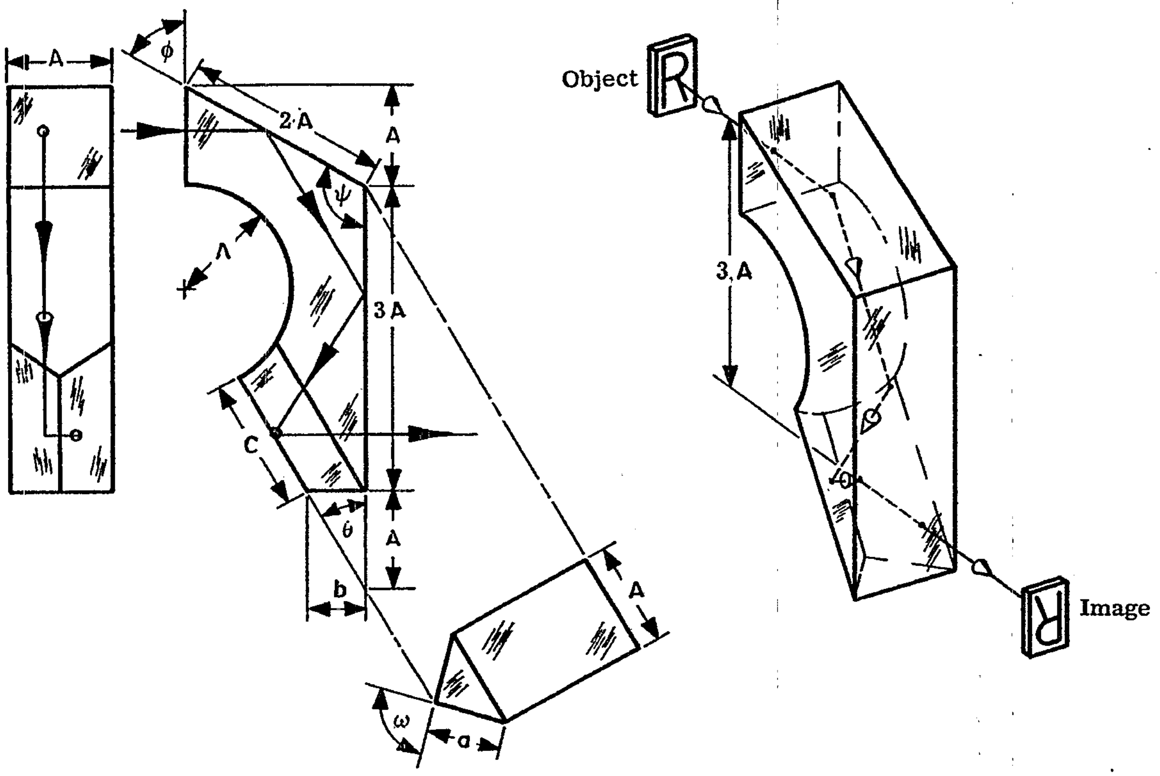


Figure 13.40-Abbe prism, type B, tunnel diagram.

13.10.6 Leman Prism. The Leman prism will revert and invert the image. The line of sight will be displaced laterally by an amount equal to 3A inches .



A = 1.00 B = 1.7321A = 1.7321 n = 1.5170 a = 0.7071A = 0.7071 θ = 30° C = 1.3099A = 1.3099
ϕ = 60° b = 0.5774A = 0.5774 ω = 90° ψ = 120° t = 5.1962A = 5.1962 t/n = 3.4253

Figure 13.41-Leman prism.

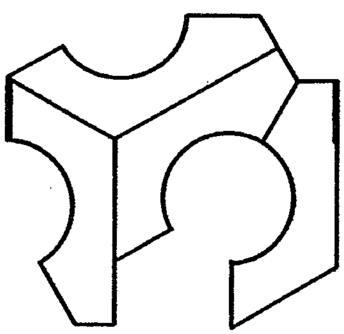
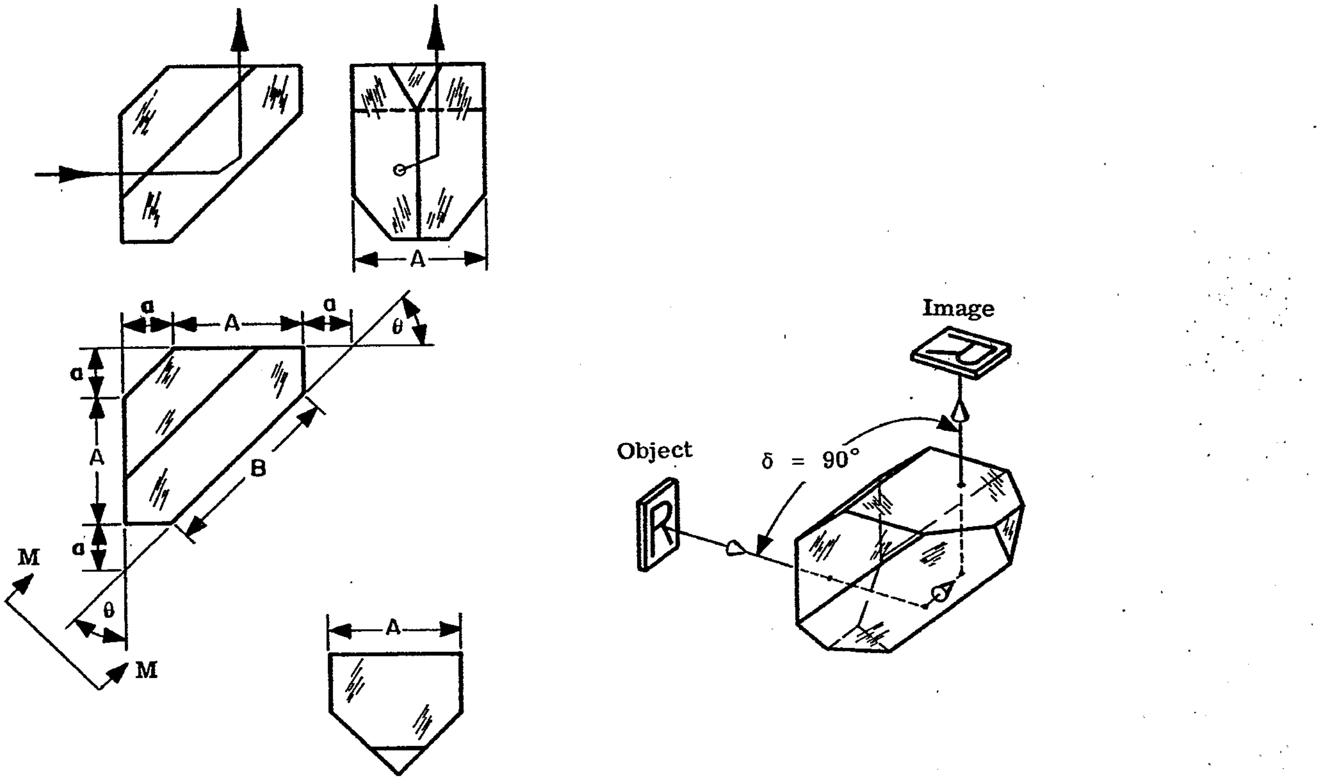


Figure 13.42-Leman prism tunnel diagram.

13.10.7 Amici Prism. During his life, 1784 to 1863, the Italian astronomer Amici designed many prisms. This is one of them. This prism will revert and invert the image and, at the same time, it will deviate the line of sight through an angle δ of 90°.



A = 1.00 n = 1.5170 θ = 45° B = 1.4142A = 1.4142 a = 0.3536A = 0.3536 t/n = 1.1253
t = 1.7071A = 1.7071

Figure 13.43-Amici prism.

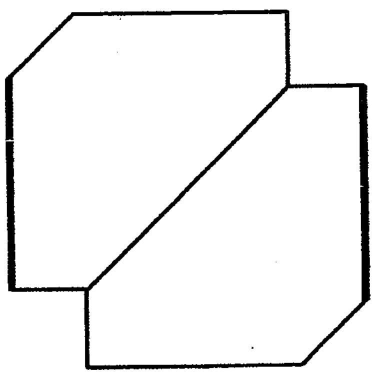
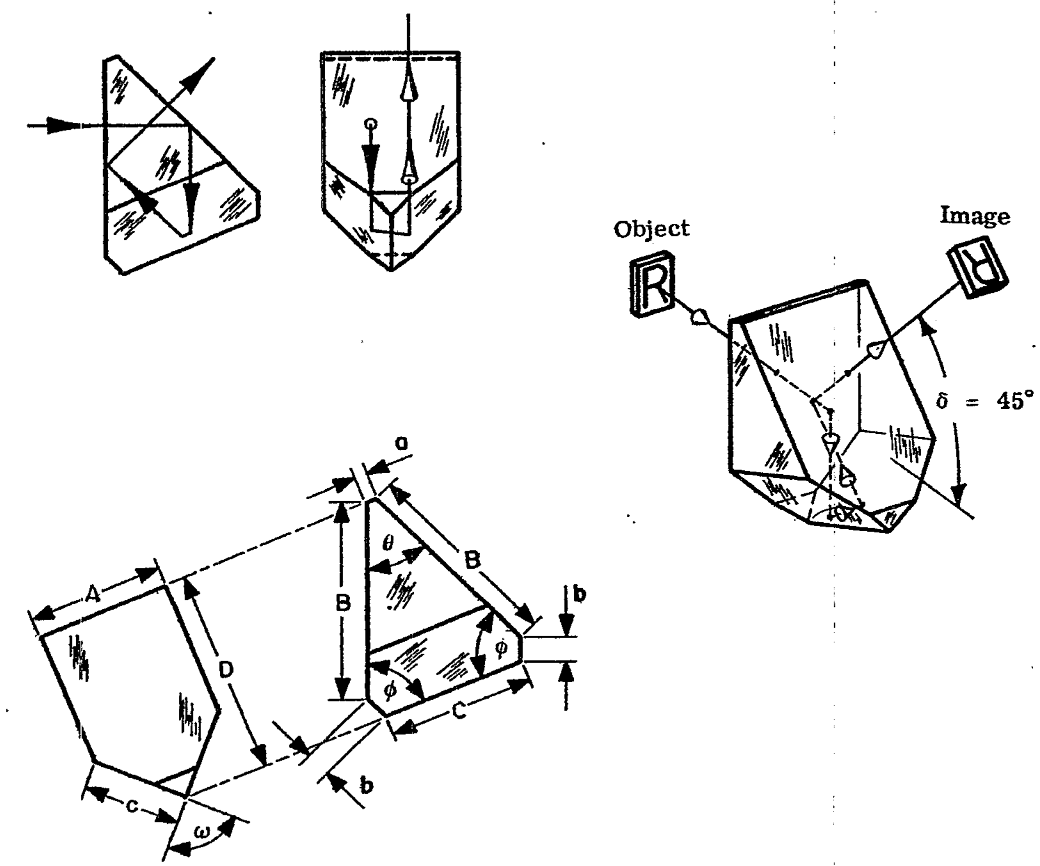


Figure 13.44-Amici prism tunnel diagram.

13.10.8 Schmidt Prism. This prism will revert and invert the image and, at the same time, it will deviate the line of sight through an angle $\delta = 45°$.



$A = 1.00$ $n = 1.5170$ $\theta = 45°$ $\omega = 90°$ $\phi = 67° 30'$ $a = 0.10$ (chosen at will) $t/n = 2.2506$
$B = 1.4142A + 0.5412a = 1.4683$ $C = 1.0824A = 1.082$ $D = 1.4142A + 2.3890a = 1.6531$
$t = 3.4142A = 3.4142$ $c = 0.7071A = 0.7071$ $b = 1.8478a = 0.1848$

Figure 13.45-Schmidt prism.

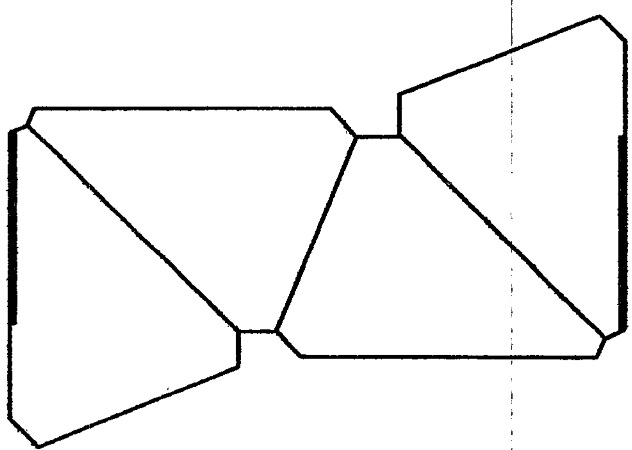
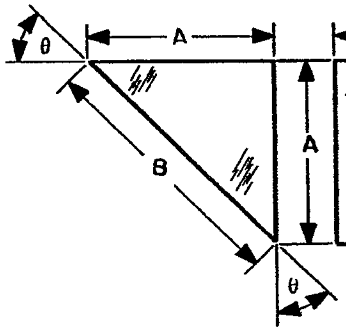
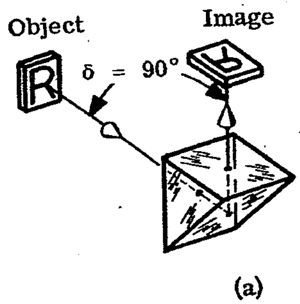
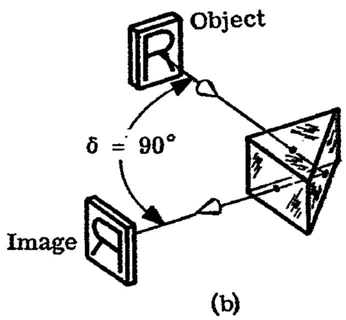


Figure 13.46-Schmidt prism tunnel diagram.

13.10.9 Right-Angle Prism. This single prism will deviate the line of sight through an angle $\delta = 90°$. The image will be inverted when the prism is held before the eye as shown in Figure 13.47(a), and it will appear reverted when the prism is turned through an angle of 90° as illustrated in Figure 13.47(b).







A = 1.00 n = 1.5170 θ = 45° B = 1.4142 A = 1.4142 t = A = 1.00 t/n = 0.6592

Figure 13.47-Right-angle prism.

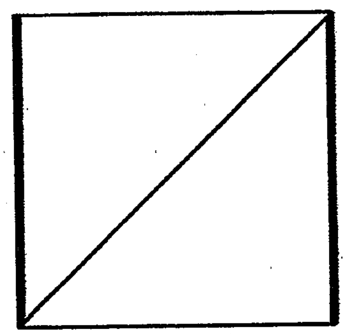
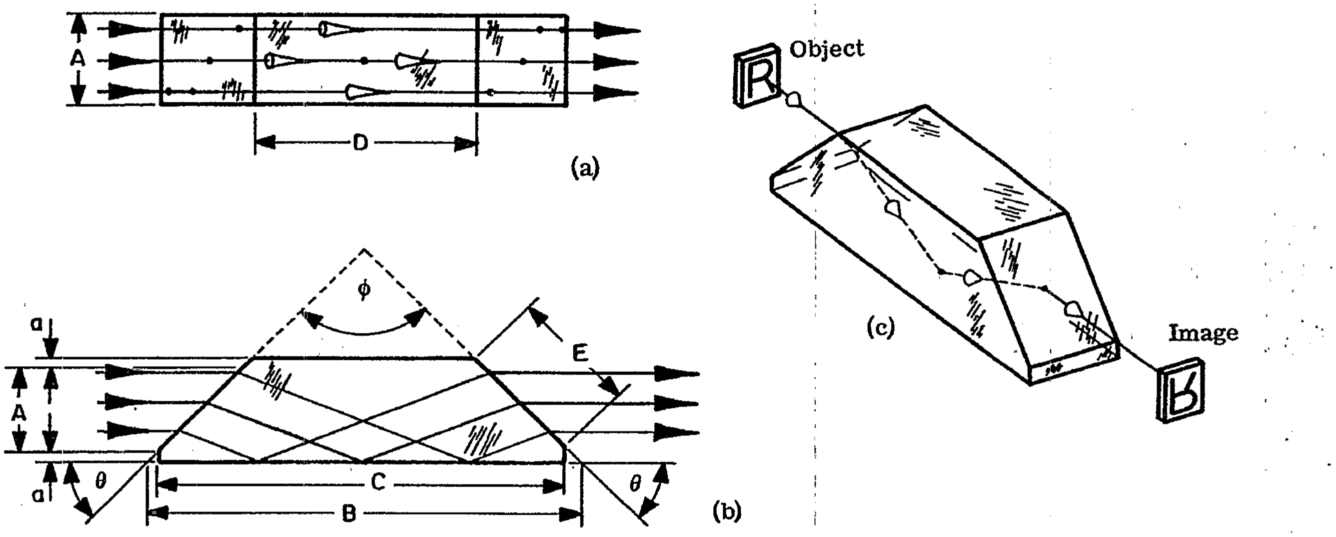


Figure 13.48-Right-angle prism tunnel diagram.

13.10.10 Harting-Dove Prism. This direct vision prism is made in one piece. The image will be inverted when the prism is held as shown in Figure 13.49(c), and it is inverted when the prism is turned about the axis through an angle of 90°. It can be used only in parallel light.



Effect on the Prism Constants When Different Types of Glass are Used

n =	1.5170	1.5725	1.6170	1.7200
B =	4.6498	4.4303	4.2822	4.0072
C =	4.5498	4.3303	4.1822	3.9072
D =	2.4498	2.2303	2.1822	1.9072
E =	1.4849	1.4849	1.4849	1.4849
t =	3.7165	3.5071	3.3637	3.1084
t/n =	2.4499	2.2303	2.0802	1.8072

$A = 1.00 \quad a = 0.05 \quad \phi = 90° \quad \theta = 45° \quad D = B - 2(A + 2a) = 2.4498 \quad n = 1.5170 \quad t/n = 2.4499$

$$B = (A + 2a)\left[\frac{\sqrt{n^2 - \sin^2\theta} + \sin\theta}{\sqrt{n^2 - \sin^2\theta} - \sin\theta} + 1\right] = 4.2271\,(A + 2a) = 4.6498 \qquad C = B - 2a = 4.5498$$

$$t = \frac{n\,(A + 2a)}{\sin\theta\,\sqrt{n^2 - \sin^2\theta} - \sin\theta} = 3.3787\,(A + 2a) = 3.7165 \qquad E = \frac{a + A}{\cos\theta} = 1.4142\,(A + 2a) = 1.4849$$

Figure 13.49-Dove prism.

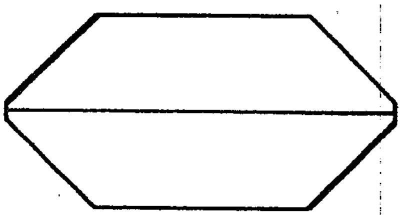
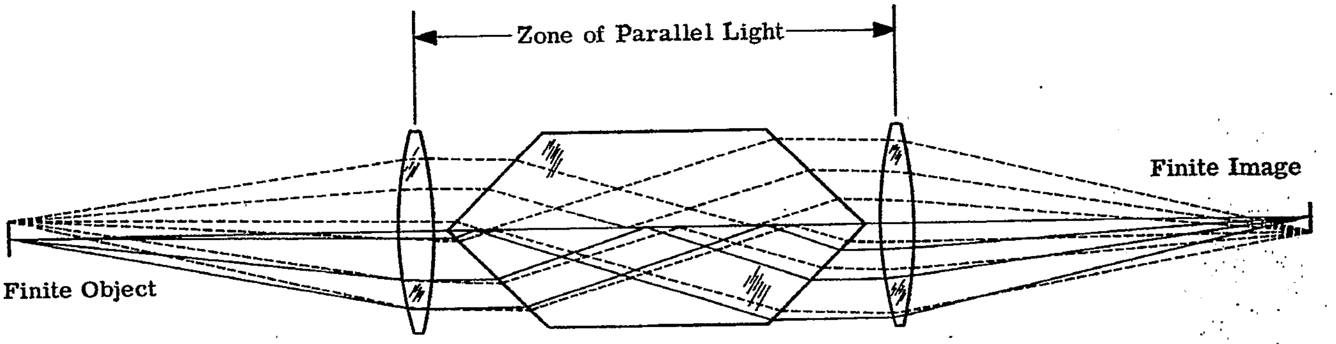
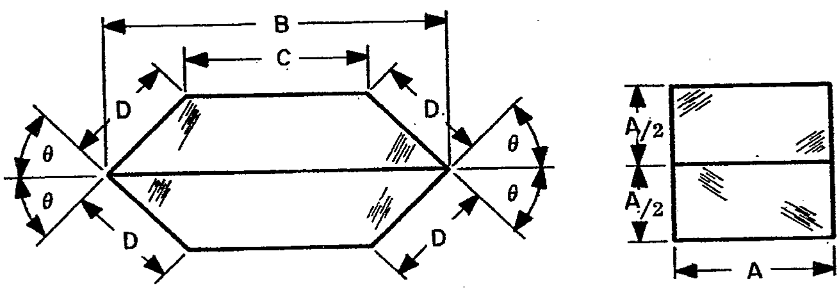


Figure 13.50-Dove prism tunnel diagram.

13.10.11 Double Dove Prism. This twin prism consists of two Harting-Dove prisms. Their reflecting surfaces are silvered and then the two halves are cemented together. This method cuts the length of the single Harting-Dove prism in half. This prism performs the duties of a single Harting-Dove prism, and it too must be placed in parallel light only.





$A = 1.00 \quad n = 1.5170 \quad C = B - A = 1.1136 \quad \theta = 45° \quad t = \dfrac{nA}{2 \sin\theta \sqrt{n^2 - \sin^2\theta} - \sin\theta} = nAC = 1.6893$

$t/n = 1.1135$

$$B = \frac{A}{2}\left[\frac{\sqrt{n^2 - \sin^2\theta} + \sin\theta}{\sqrt{n^2 - \sin^2\theta} - \sin\theta} + 1\right] = 2.1136A = 2.1136 \qquad D = \frac{A}{2\cos\theta} = 0.7071A = 0.7071$$

Figure 13.51-Double Dove prism.

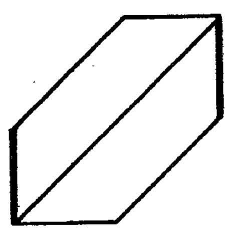
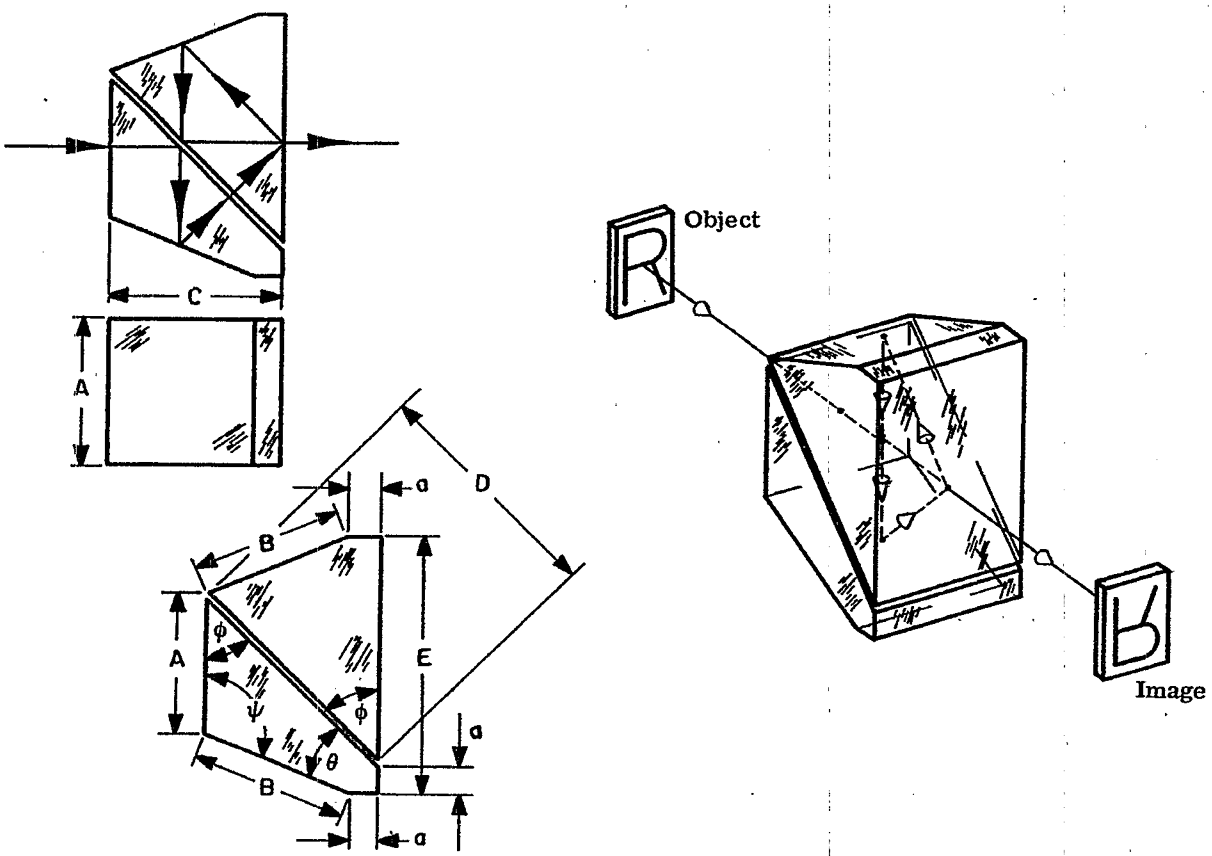


Figure 13.52-Double Dove prism tunnel diagram.

13.10.12 Pechan Prism. The prism performs the same duties as the Harting-Dove prism but it has one great advantage over the latter inasmuch as it may be placed in convergent or divergent light. This will permit the reduction in length or height of the instrument. It will invert (as shown) or revert the image, depending on its orientation. It may displace the line of sight if not properly centered but it will not deviate it. The surfaces marked B are silvered and covered with a protective coating. The unsilvered reflecting surfaces of the prism are separated by a distance of about 0.002 inch.



A = 1.00 n = 1.5170 θ = 22° 30' ϕ = 45° ω = 67° 30' ψ = 112° 30' a = 0.2071A = 0.2071
B = 1.0824A = 1.0824 C = 1.2071A = 1.2071 D = 1.7071A = 1.7071 E = 1.8284A = 1.8284
t = 4.6213A = 4.6213 t/n = 3.0464

Figure 13.53-Pechan prism.

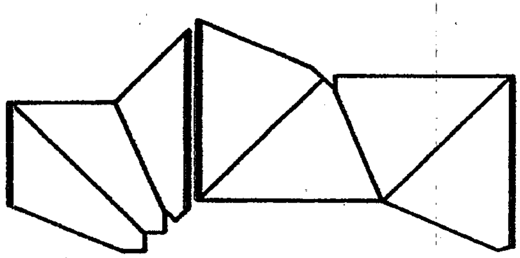
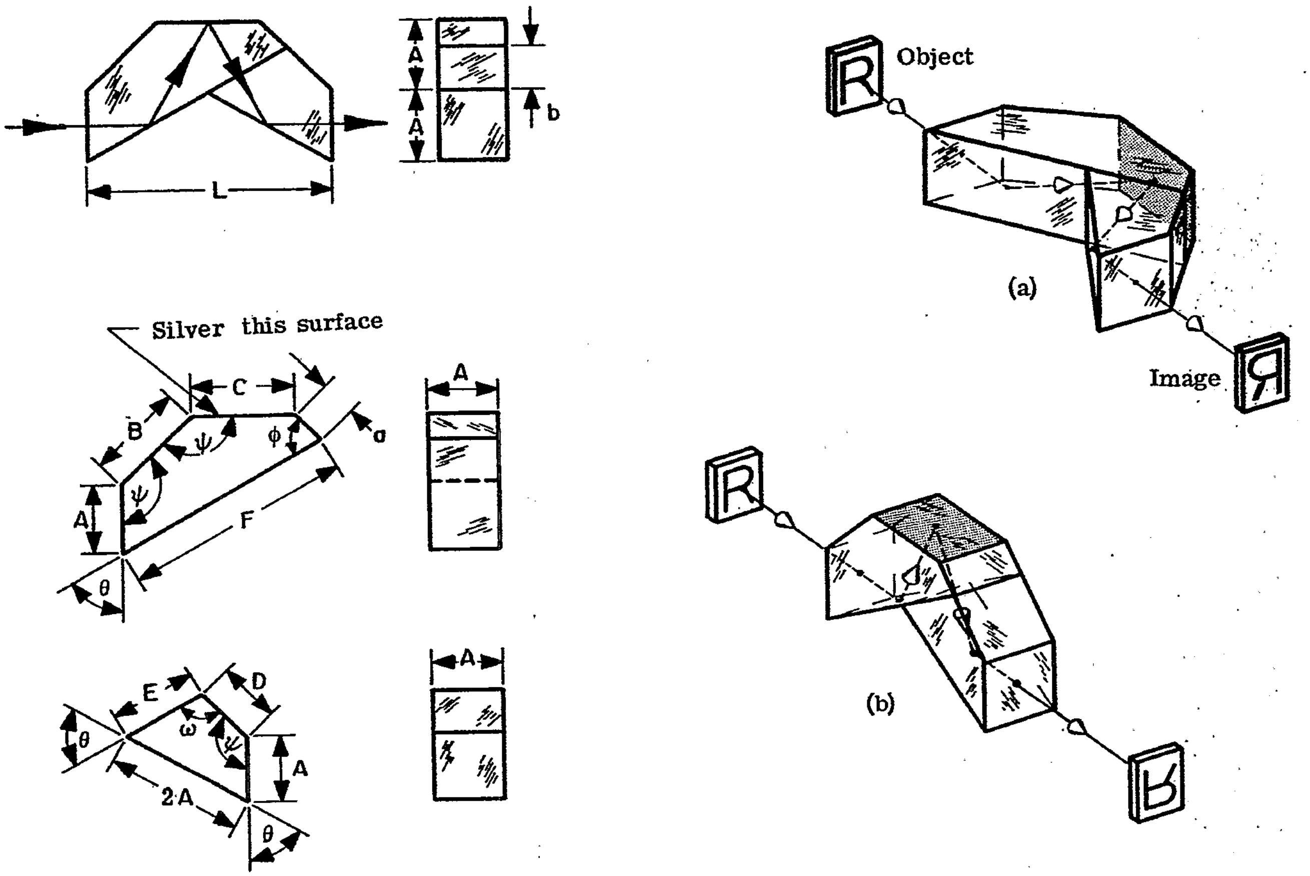


Figure 13.54-Pechan prism tunnel diagram.

13.10.13 Reversion Prism. This prism, which is a modification of the Abbe prism type A, consists of two elements which are cemented together. Like the Pechan prism, it may be placed in the path of parallel, converging, or diverging beams of light. Since three reflections are involved, it may be used to revert (a) or invert (b) the image, depending on its orientation. If not properly centered vertically, it will displace the line of sight by twice the centering error but will not deviate the sight line.



$A = 1.00$ $n = 1.5170$ $a = 0.5176A$ $b = 0.6340A$ $B = 1.4142A$ $C = 1.4641A$
$\theta = 60°$ $\phi = 75°$ $\psi = 135°$ $\omega = 105°$ $D = 0.8966A$ $E = 1.2679A$
$F = 3.2679A$ $L = 3.4641A$ $t/n = 3.4253A$ $t = 5.1962A$

Figure 13.55-Reversion prism.

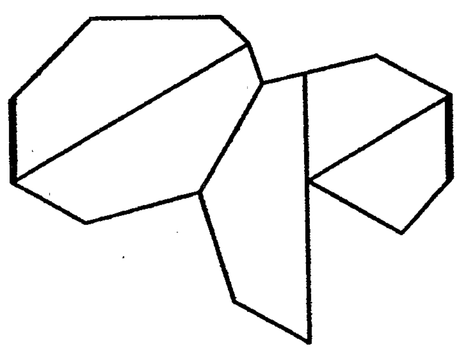
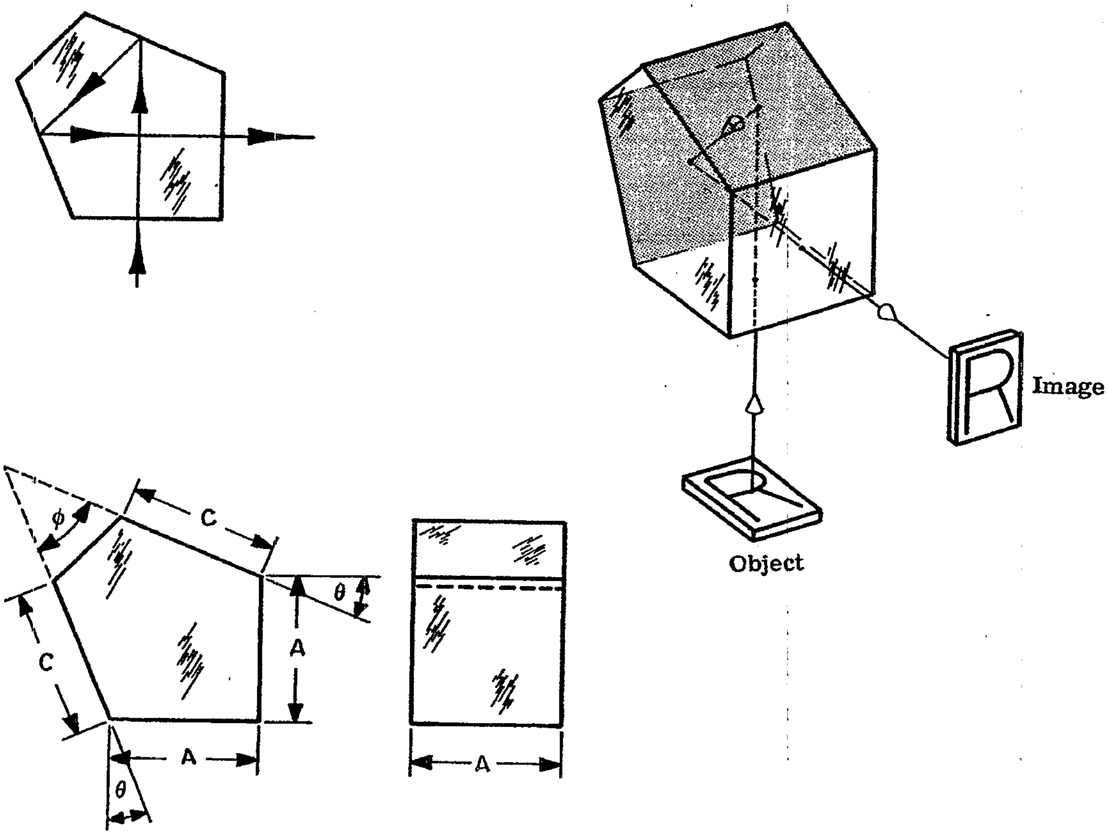


Figure 13.56-Reversion prism tunnel diagram.

13.10.14 Penta Prism. This prism will neither revert nor invert the image but will merely deviate the line of sight through an angle of 90°. The surfaces marked C in Figure 13.57 must be silvered and covered with a protective coating.



A = 1.00 n = 1.5170 θ = 22° 30' ϕ = 45° B = 0.4142 A = 0.4142
C = 1.0824 A = 1.0824 t = 3.4142 A = 3.4142 t/n = 2.2506

Figure 13.57-Penta prism.

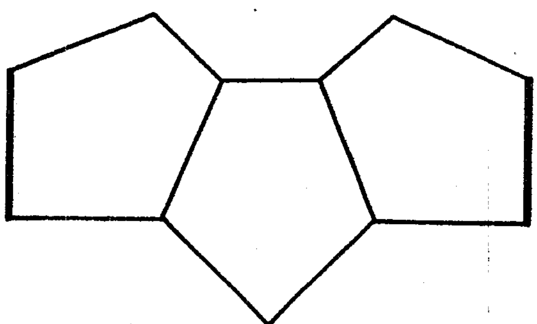
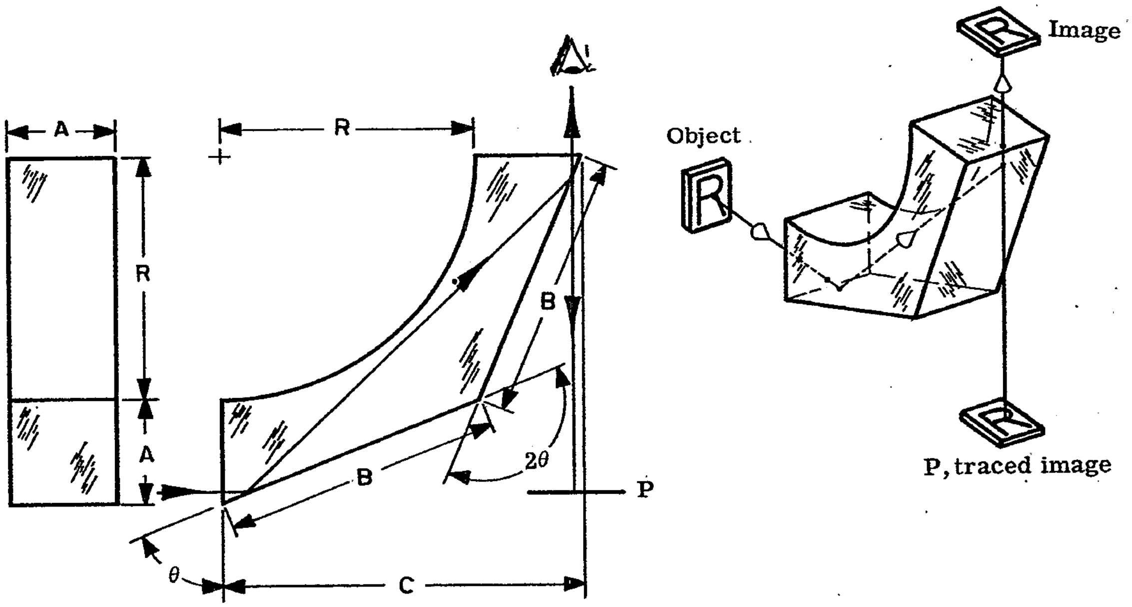


Figure 13.58-Penta prism tunnel diagram.

13.10.15 Wollaston Prism. Between the years of 1766 and 1828, the English scientist W. H. Wollaston designed a prism which has been named after him. It is made in one piece of glass and will neither invert nor revert the image, but it will deviate a beam of light through an angle of 90°. It is not used in military instruments due to its unfavorable shape. However, it is still used in an instrument known as "Camera Lucida," or "Camera Clara," the theory of which is explained here. If the observer's eye is placed right above the upper corner of the prism as shown in Figure 13.59, and a sheet of paper P is placed on the table about 10 inches from the eye, the observer will be able, with the aid of a pen, to trace the image of the object on the paper.



A = 1.00 n = 1.5170 θ = 67° 30' B = 2.6131A = 2.6131 C = 3.4142A = 3.4142
R = 2.4142A = 2.4142 t = 2R = 4.8284A = 4.8284 t/n = 3.1829

Figure 13.59-The Wollaston prism.

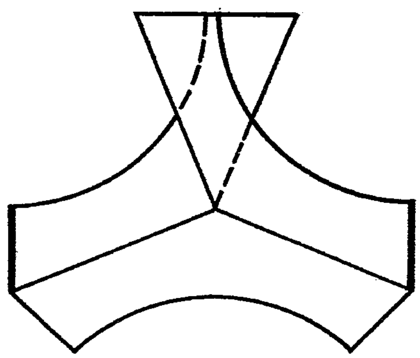
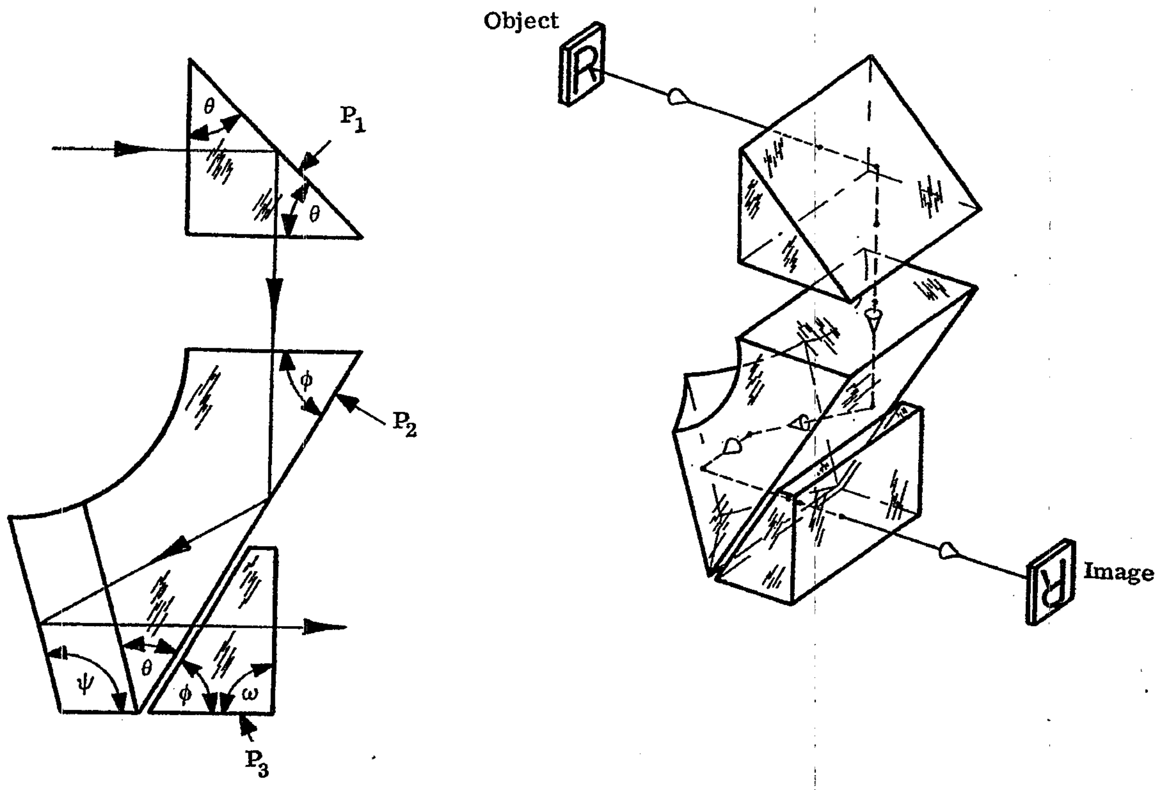


Figure 13.60-The Wollaston prism tunnel diagram.

13.10.16 Carl Zeiss Prism System. This combination consists of three single prisms (see Figure 13.61). As a rule the objective is placed between P_1 and P_2 ; however, it may also be placed in front of the objective prism P_1 . This system will invert and revert the image but will not deviate the line of sight. The line of sight will be displaced an amount depending on the distance between the prisms P_1 and P_2 .



$\theta = 45°$ $\phi = 60°$ $\omega = 90°$ $\psi = 105°$

Figure 13.61-A Carl Zeiss prism system.

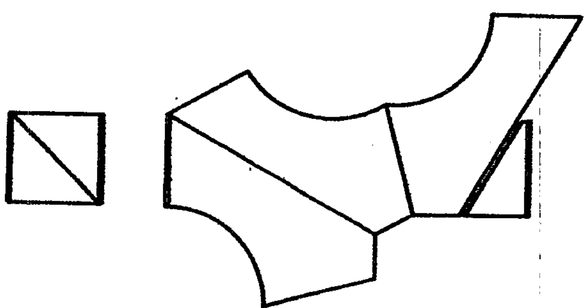
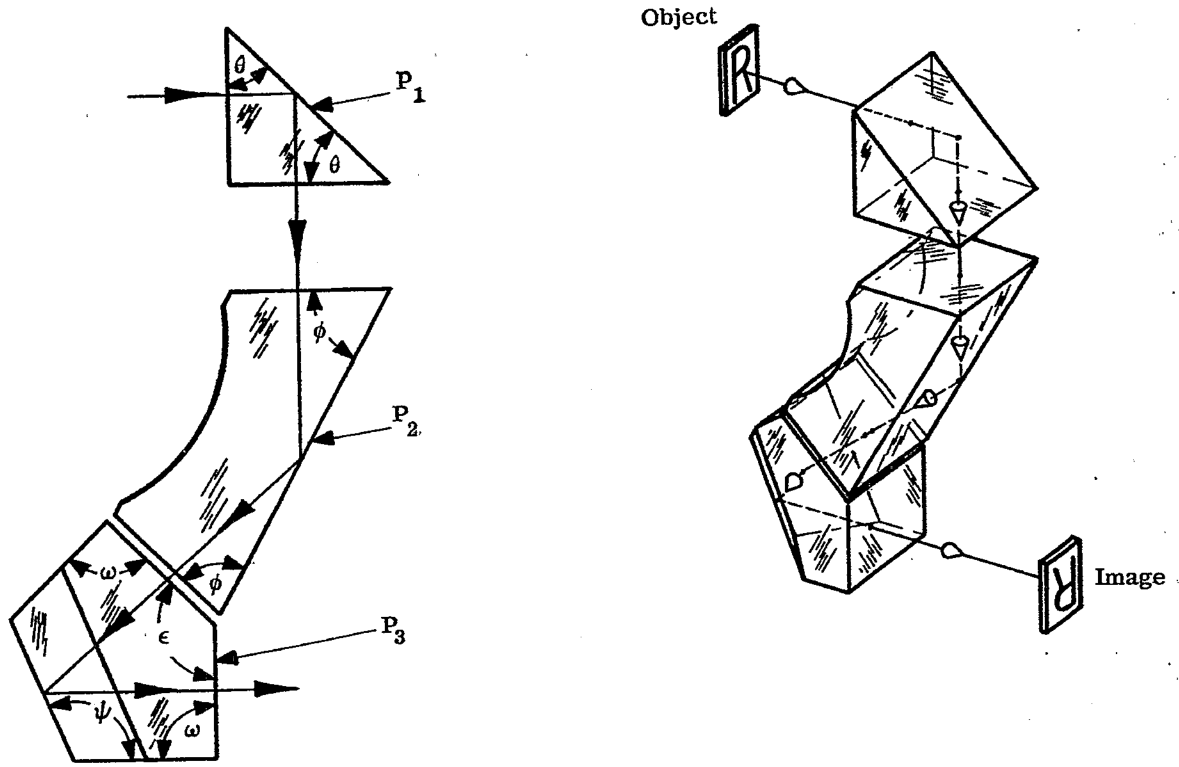


Figure 13.62-Carl Zeiss prism system tunnel diagram.

13.10.17 C. P. Goerz Prism System. This prism system consists of three single prisms as illustrated in Figure 13.63. The light is received by prism P_1, also known as the objective prism. The objective, usually placed between P_1 and P_2, may also be placed in front of P_1. This system will invert and revert the image. The line of sight will not be deviated from its original direction but will be displaced by an amount depending on the distance between the prisms P_1 and P_2.



$\theta = 45°$ $\phi = 67° 30'$ $\omega = 90°$ $\psi = 112° 30'$ $\epsilon = 135°$

Figure 13.63-A Goerz prism system.

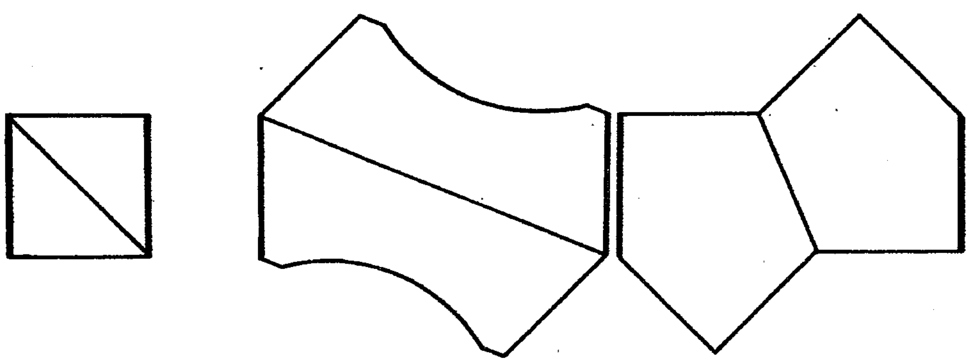


Figure 13.64-A Goerz prism system tunnel diagram.

13.10.18 Carl Zeiss Ocular Prism. This prism system, used in coincidence type range-finders, is made up of four single prisms, which are cemented together (see Figure 13.65). Light from the right will enter the system through the rhomboid prism P_1 and, after two internal reflections in this prism, and then three more in P_2 (the last reflection takes place on the silvered portion), the ray will emerge from the prism P_4 and then enter the eye of the observer. The image will be erect but reverted. Light from the left will enter the system through the prism P_3 and, after two internal reflections in this prism it will emerge from the system through P_4 and then it will also enter the eye of the observer. The image will appear inverted and reverted. In Figure 13.65 the refracting angles of the prism P_2, P_3, and P_4 are 22° 30'', and the light is deviated through an angle of 45°. This value may easily be varied by changing the refracting angles of the prisms.

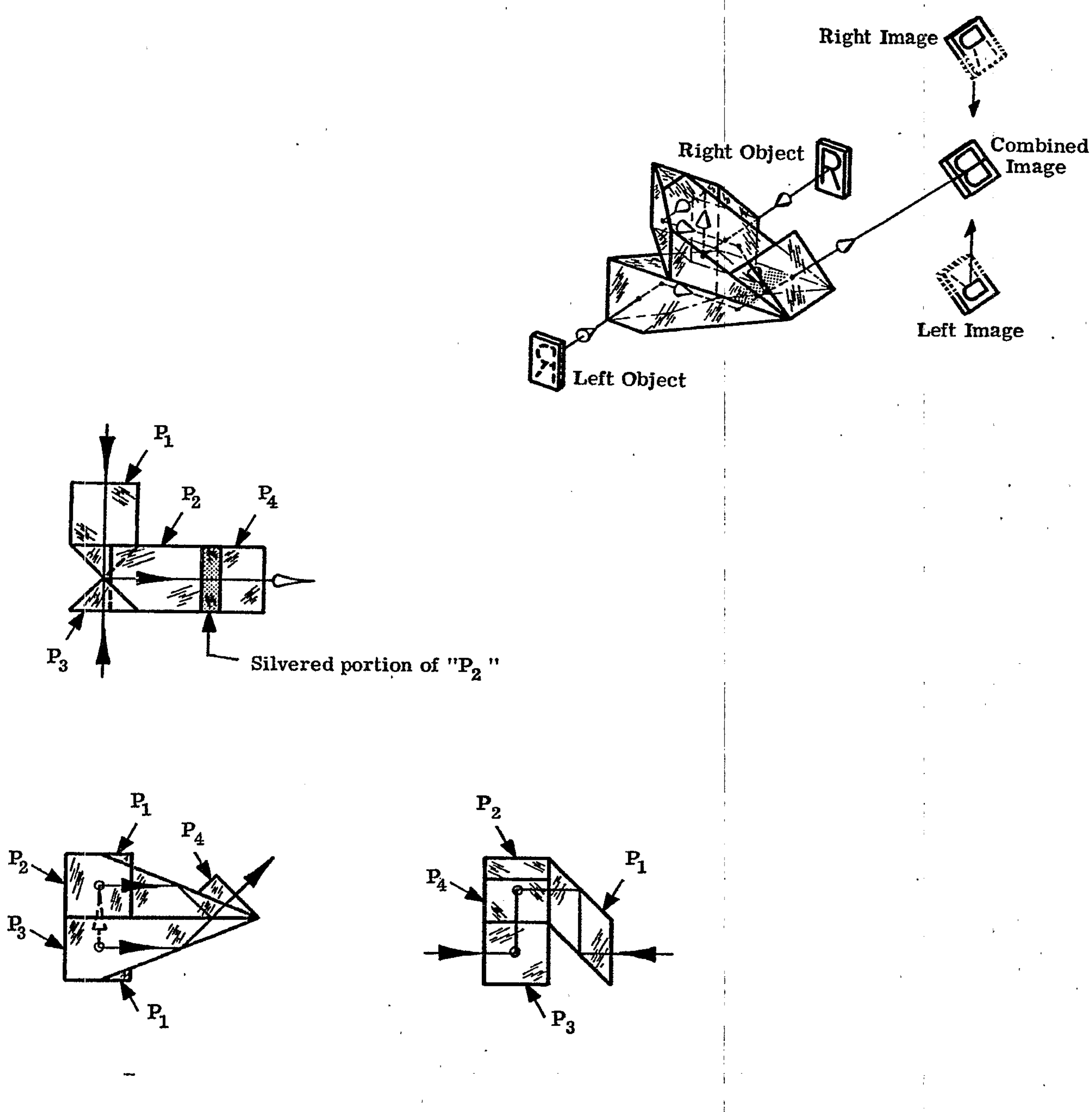


Figure 13.65-An ocular prism by Zeiss.

13.10.19 Barr and Stroud Ocular Prism. This ocular prism system, consisting of four single prisms and a cover, all cemented together, was used during the second world war by Research Enterprise Limited. It has one advantage over the Zeiss prism inasmuch as no silvered surface is required in producing the dividing line between the two images. On the other hand, the production division claims that the cost of manufacturing this prism is about five times that of the Zeiss prism, due to the great difficulties encountered in producing a well defined dividing line. The prisms P_1, P_2, and P_3 are made of a borosilicate crown glass ($n = 1.509$) and the prism P_4 of an extra dense flint glass ($n = 1.654$). The paths through the prism system of the various rays are illustrated in diagrams (a) and (b) of Figure 13.66. The rays of light, after passing through the right objective will enter the prism system through the prism P_1. After a reflection on the hypotenuse of this prism the rays will enter prism P_2, and, after three internal reflections in this prism, they will pass undeviated through the prism P_3 and the cover C and will then proceed towards the eyepiece. The image seen through this part of the prism system will appear inverted and reverted. The rays of light passing through the left objective will enter the prism system through the prism P_4, and will be reflected twice before they reach the dividing line between this prism and prism P_3. Due to the fact that the refractive index of P_4 is much greater than that of prism P_3, the rays will be reflected in an upward direction and emerge from the prism system parallel to the other rays. The image seen through this portion of the prism system will be erect but reverted.

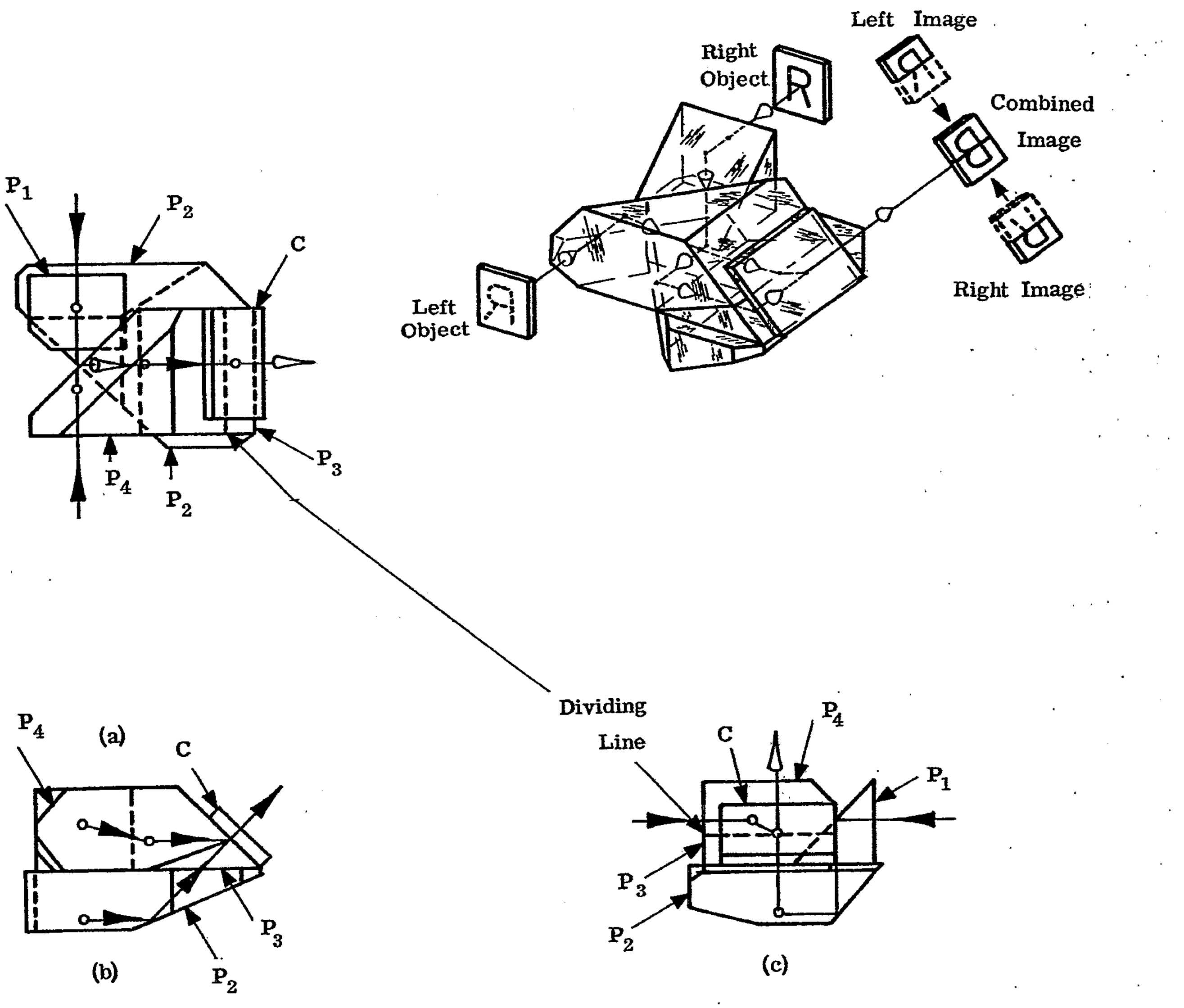
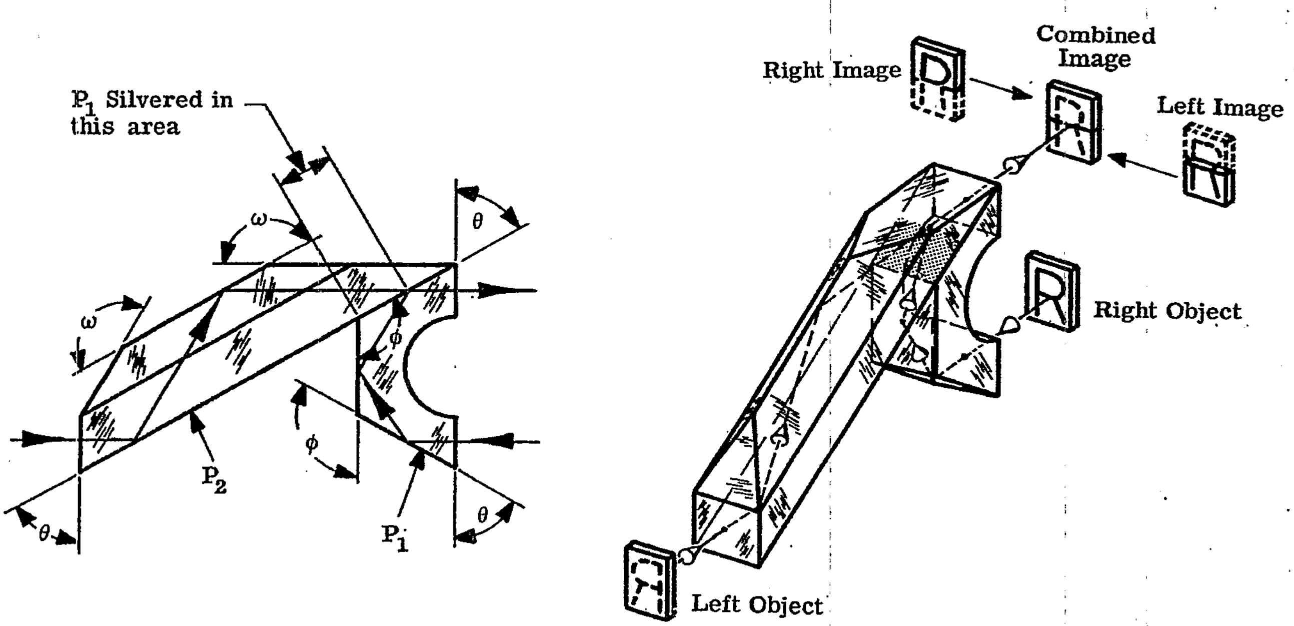


Figure 13.66-A Barr and Stroud ocular prism.

13.10.20 Carl Zeiss Coincidence Prism System. This prism system, illustrated in Figure 13.67, consists of two single prisms, P_1 and P_2. The lower half of the upper reflecting surface of P_1 is silvered and then the two prisms are cemented together. Light from the right will enter first P_1 at the lower entrance surface and, after three internal reflections it will emerge from the system at the upper exit surface. The image will appear reverted. Light from the left will enter through the prism P_2 and, after three reflections in the prism, it will enter prism P_1 through the unsilvered portion of the reflecting surface. The light will pass through P_1 undeviated before emerging from the prism system. The image will also appear reverted. This system is used in long base range finders. It is placed between the left objective and its image plane. The images formed by the left and right objective are formed in a plane normal to the line of sight through the dividing line of the silvered and the unsilvered portions of the reflecting surface. A lens erecting system will then transmit these images into the front focal plane of the ocular.



$\theta = 60°$ $\phi = 120°$ $\omega = 150°$

Figure 13.67-A Zeiss coincidence prism system.

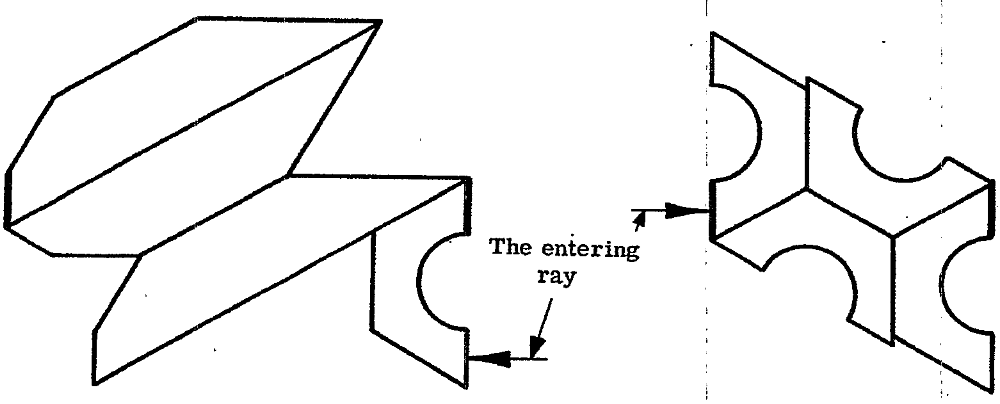
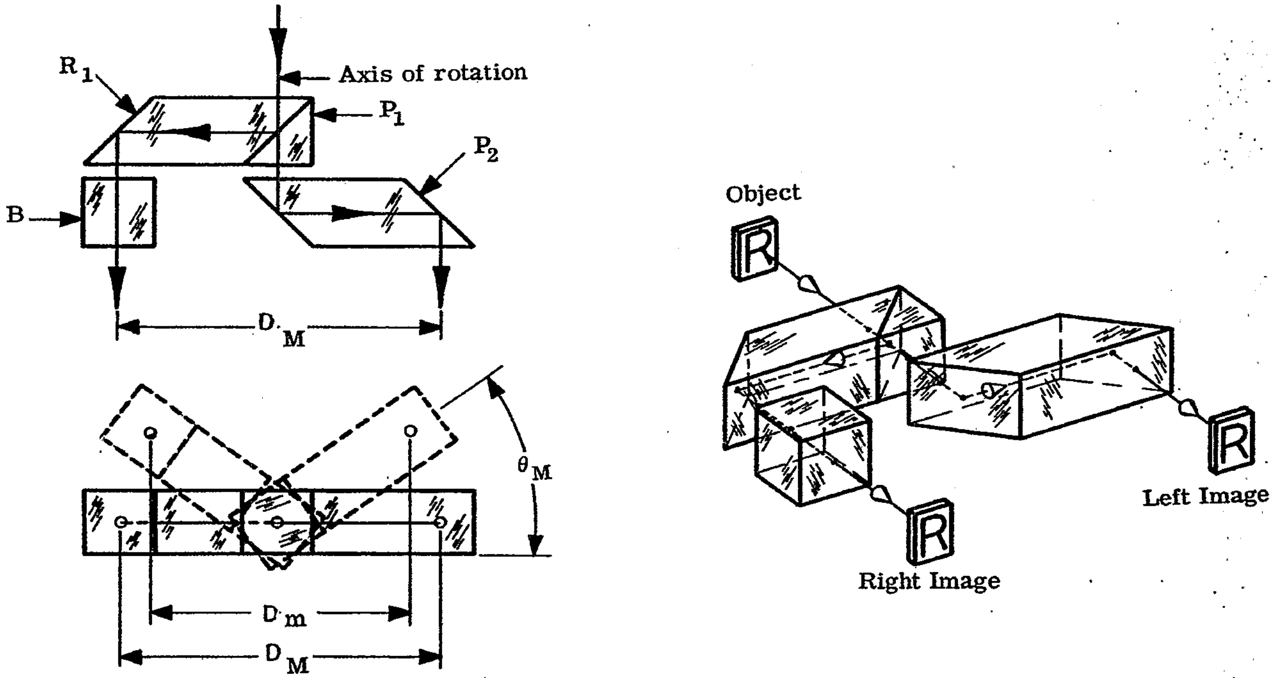


Figure 13.68-Zeiss coincidence prism tunnel diagram.

13.10.21 Carl Zeiss Binocular-Ocular Prism System. This system, illustrated in Figure 13.69, is used in binocular telescopes (or microscopes) when both eyes are to view the image presented by the objective. This system is made up of four single prisms, namely, the right angle prism P_1 cemented to the rhomboid prism R_1 ; the cemented surface will split the beam of light. The light passing through R_1 and P_1 will, before entering the eye, pass through the prism P_2. The other ray will pass through the block B which has been added to the system to equalize the length of the light-paths in glass. The interpupillary distance is designated by the letter D. Its value varies between the limits of D_m = 58 mm = 2.283 inches and D_M = 72 mm = 2.835 inches.



$$\cos \theta_M = \frac{D_m}{D_M} = \frac{2.283}{2.835} = 0.805291 \qquad \theta_M = 36° \, 22' 3''$$

Figure 13.69-A Carl Zeiss binocular-ocular prism system.

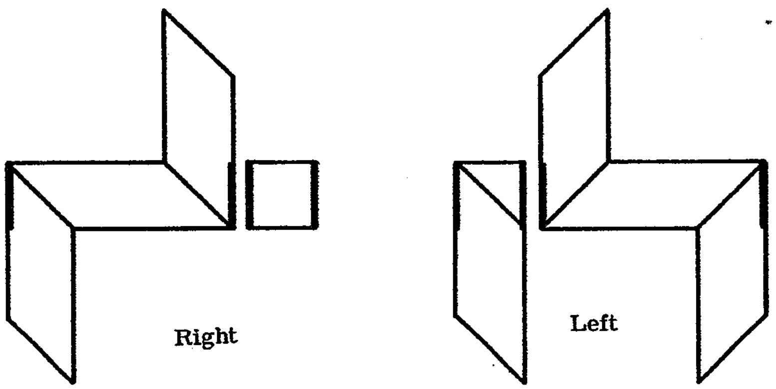
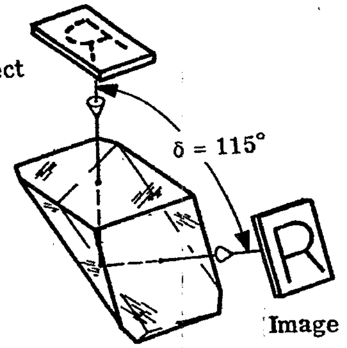
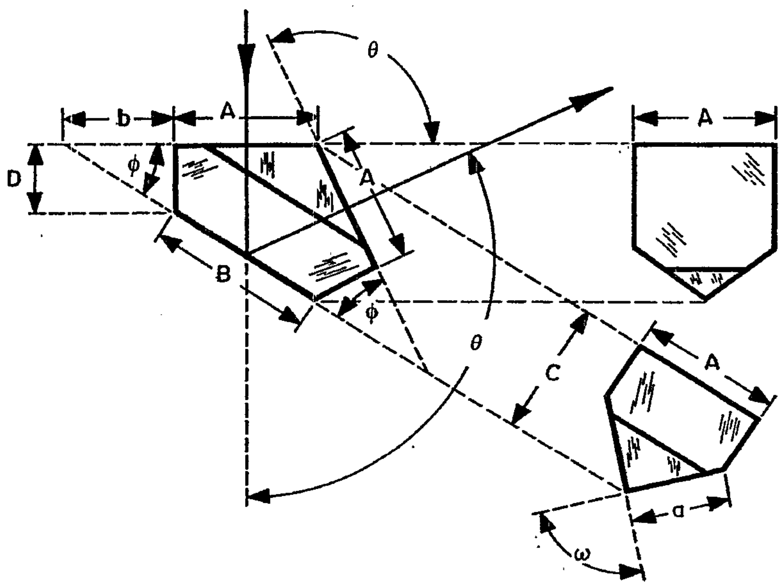


Figure 13.70-A Zeiss binocular-ocular prism tunnel diagram.

13.10.22 Frankford Arsenal Prism No. 1. This prism will revert the image and, at the same time, it will deviate the line of sight through an angle $\delta = 115°$.





$A = 1.00$ $n = 1.5170$ $\theta = 115°$ $\phi = 32° 30'$ $\omega = 90°$
$a = 0.7071A = 0.7071$ $b = 0.7320A = 0.7320$ $B = 1.1857A = 1.1857$ $C = 0.9306A = 0.9306$
$D = 0.4613A = 0.4613$ $t = 1.5697A = 1.5697$ $t/n = 1.0347$

Figure 13.71-Frankford Arsenal prism No. 1.

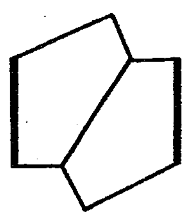
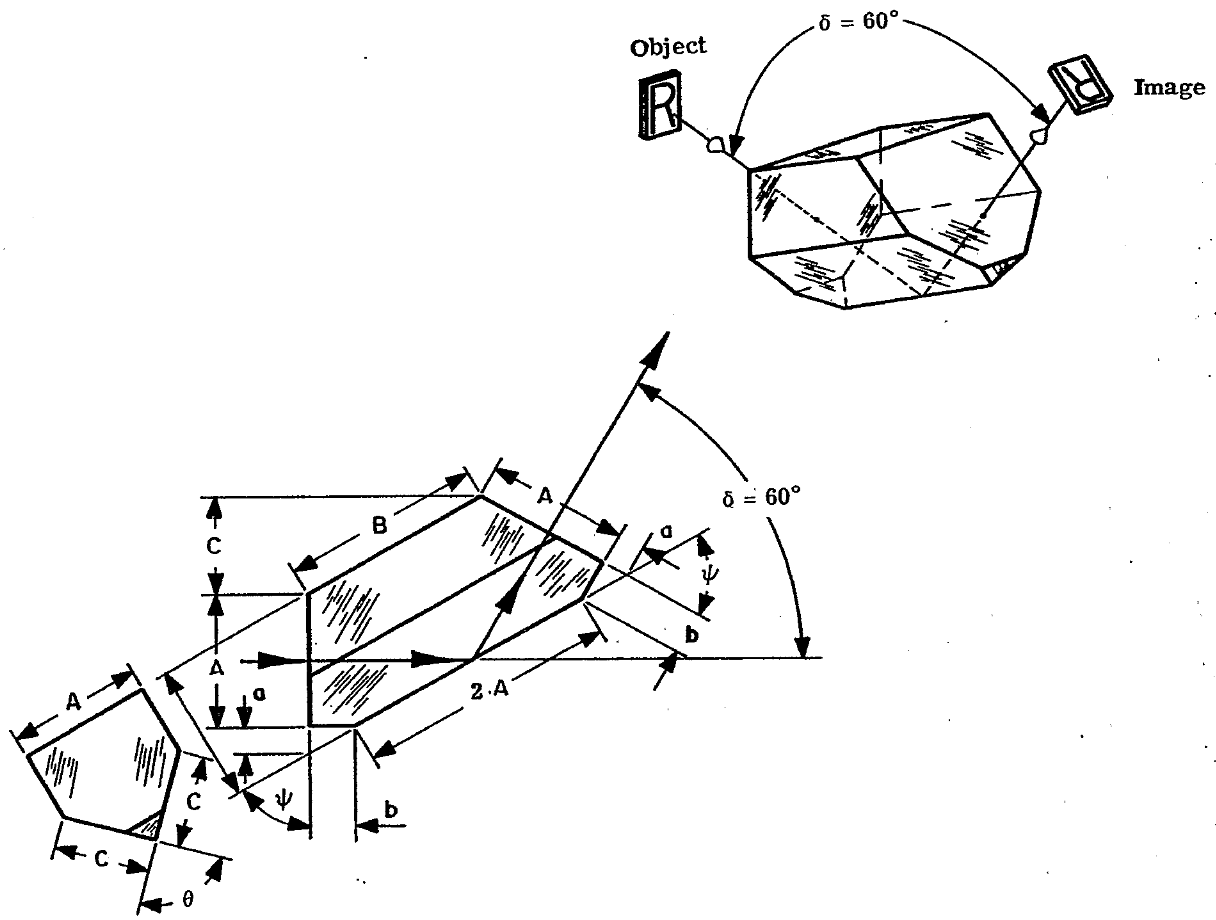


Figure 13.72-Frankford Arsenal prism No. 1 tunnel diagram.

13.10.23 Frankford Arsenal Prism No. 2. This prism is made in one piece. It will invert and revert the image and, at the same time, it will deviate the line of sight through an angle of $\delta = 60°$.



$A = 1.00$ $n = 1.5170$ $\theta = 90°$ $\psi = 60°$ $a = 0.1547A = 0.1547$ $b = 0.2680A = 0.2680$
$B = 1.4641A = 1.4641$ $C = 0.7321A = 0.7321$ $t = 2.2680A = 2.2680$ $t/n = 1.4951$

Figure 13.73-Frankford Arsenal prism No. 2.

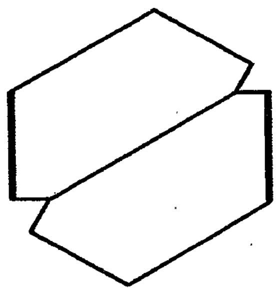
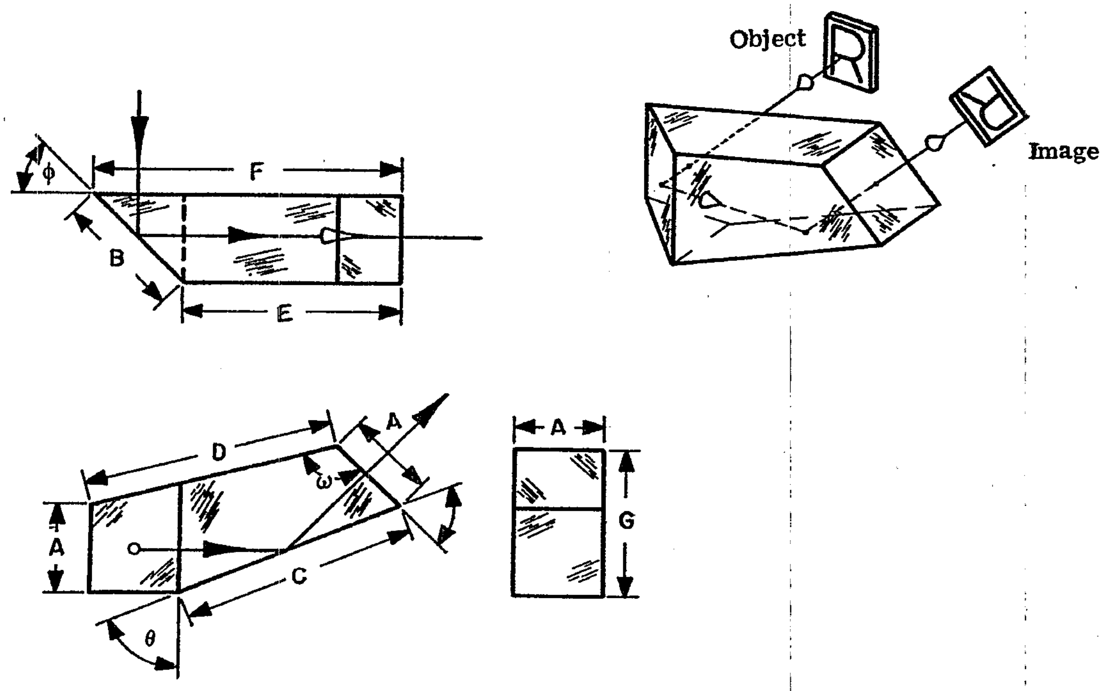


Figure 13.74-Frankford Arsenal prism No. 2 tunnel diagram.

13.10.24 Frankford Arsenal Prism No. 3. This prism is made in one piece. It will deviate the line of sight through an angle of 90° in the horizontal plane and, at the same time, through an angle of 45° in an upward direction. The observer, standing at right angles to the line of sight, will see an inverted and reverted image.



A = 1.00 n = 1.5170 θ = 67° 30' ϕ = 45° ω = 120° 21' 40" B = 1.4142A = 1.4142
C = 2.6131A = 2.6131 D = 2.7979A = 2.7979 E = 2.4142A = 2.4142 F = 3.4142A = 3.4142
G = 1.7071A = 1.7071 t = 3.4142A = 3.4142 t/n = 2.2506

Figure 13.75-Frankford Arsenal prism No. 3.

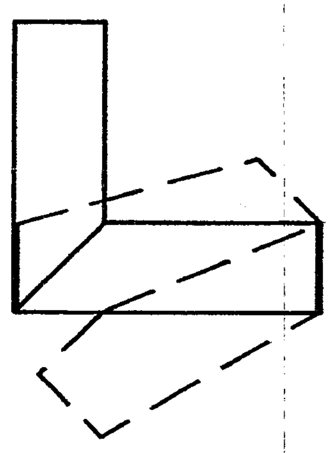
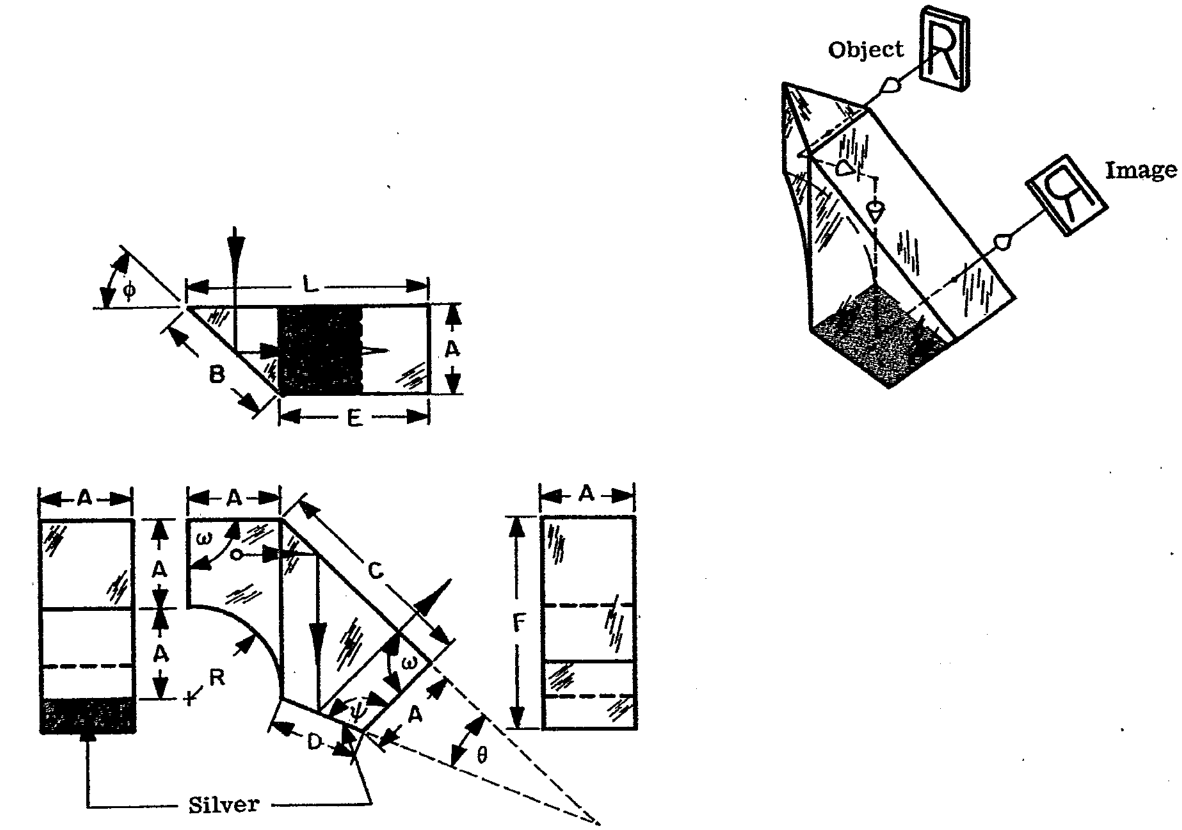


Figure 13.76-Frankford Arsenal prism No. 3 tunnel diagram.

13.10.25 Frankford Arsenal Prism No. 4. This prism is made of one piece of glass. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 45° in the vertical plane. The observer, standing at right angles to the line of sight, will see the image reverted.



A = 1.00 n = 1.5170 θ = 22° 30' ϕ = 45° ω = 90° ψ = 112° 30' B = 1.4142A = 1.4142
C = 2.4142A = 2.4142 D = 1.0824A = 1.0824 E = 1.7071A = 1.7071 F = 2.4142A = 2.4142
L = 2.7071A = 2.7071 R = A = 1.00 t = 4.4142A = 4.4142 t/n = 2.9098

Figure 13.77-Frankford Arsenal prism No. 4.

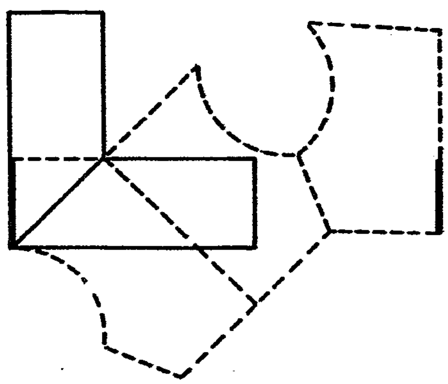
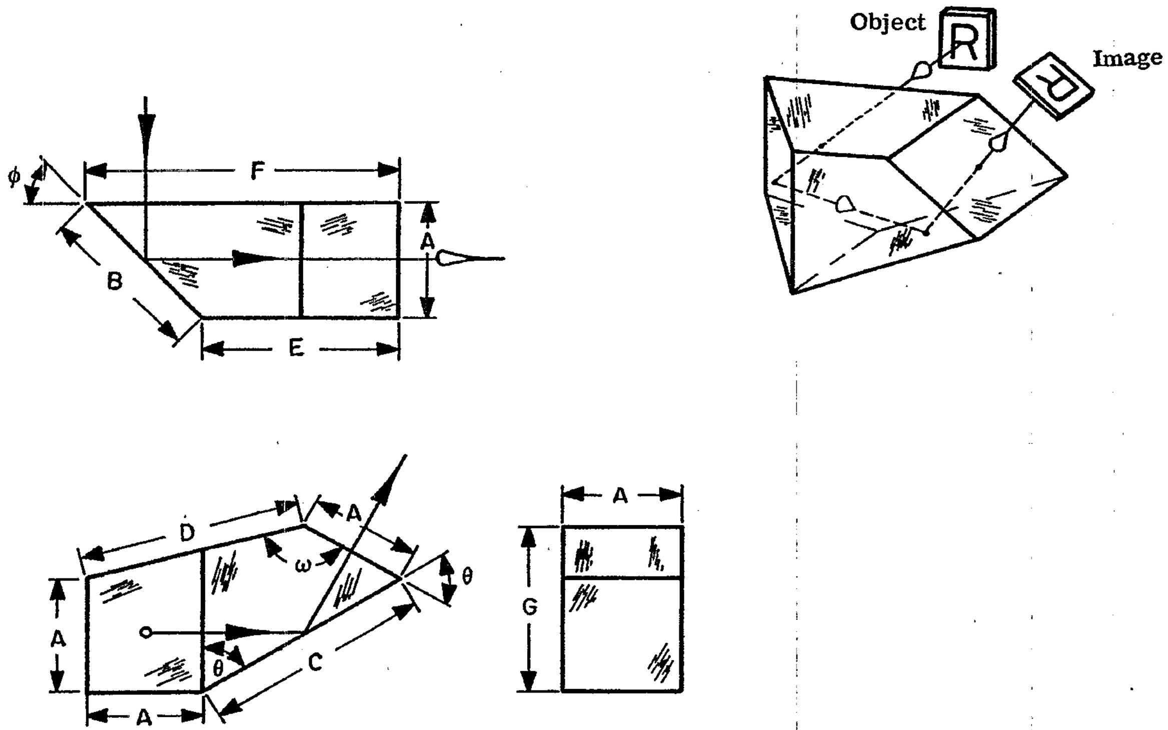


Figure 13.78-Frankford Arsenal prism No. 4 tunnel diagram.

13.10.26 Frankford Arsenal Prism No. 5. This prism is made in one piece. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 60° in the vertical plane. The observer, standing at right angles to the line of sight, will see the image inverted and reverted.



A = 1.00 n = 1.5170 θ = 60° φ = 45° ω = 135° B = 1.4142A = 1.4142
C = 2.000A = 2.000 D = 1.9318A = 1.9318 E = 1.7321A = 1.7321 F = 2.7321A = 2.7321
G = 1.500A = 1.500 t = 2.7437A = 2.7431 t/n = 1.8086

Figure 13.79-Frankford Arsenal prism No. 5.

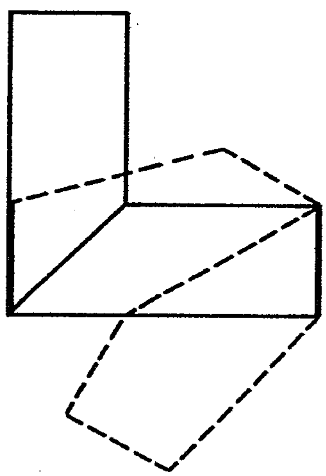
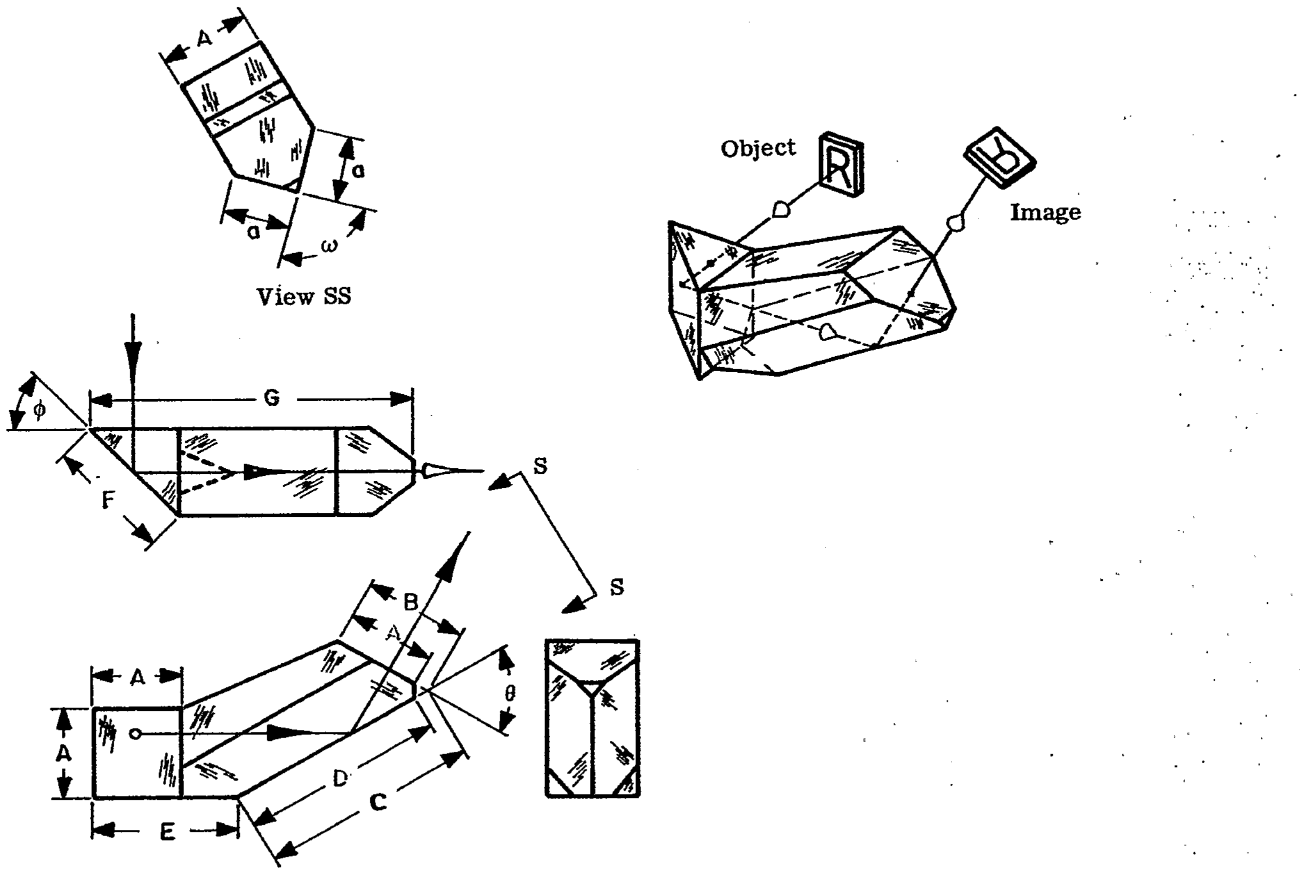


Figure 13.80-Frankford Arsenal prism No. 5 tunnel diagram.

13.10.27 Frankford Arsenal Prism No. 6. This prism is made in one piece. It will deviate the line of sight through an angle of 90° in the horizontal plane and through an angle of 60° in the vertical plane. The prism will invert the image.



$A = 1.00$ $n = 1.5170$ $\theta = 60°$ $\phi = 45°$ $\omega = 90°$ $a = 0.7071A = 0.7071$ $t/n = 2.4180$
$B = 1.2071A = 1.2071$ $C = 2.4142A = 2.4142$ $D = 2.2071A = 2.2071$ $E = 1.5774A = 1.5774$
$F = 1.4142A = 1.4142$ $G = 3.4888A = 3.4888$ $H = 1.8107A = 1.8107$ $t = 3.6681A = 3.6681$

Figure 13.81-Frankford Arsenal prism No. 6.

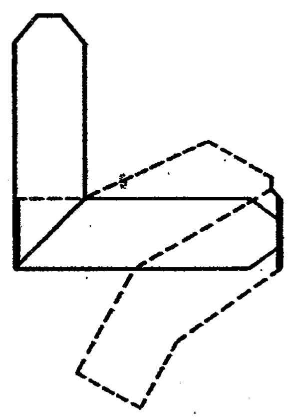
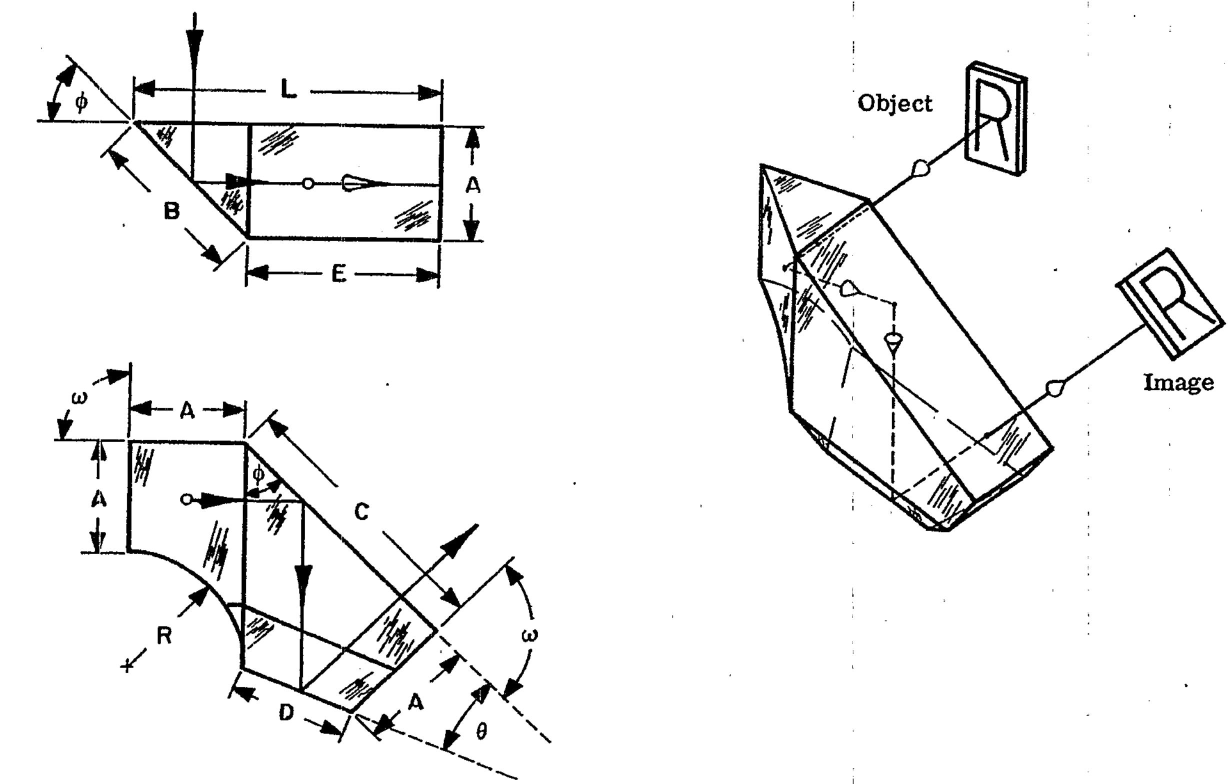


Figure 13.82-Frankford Arsenal prism No. 6 tunnel diagram.

13.10.28 Frankford Arsenal Prism No. 7. This prism is made in one piece. The line of sight is deviated through an angle of 90° in the horizontal plane and, simultaneously, through an angle of 45° in the vertical plane. The observer, standing at right angles to the line of sight, will see a normal image of the target since the prism neither inverts nor reverts the image.



$A = 1.00$ $n = 1.5170$ $\theta = 22°\,30'$ $\omega = 90°$ $\phi = 45°$ $B = 1.4142A = 1.4142$ $R = A = 1.00$
$C = 2.4142A = 2.4142$ $D = 1.0824A = 1.0824$ $E = 1.7071A = 1.7071$ $L = 2.7071A = 2.7071$
$t = 4.4142A = 4.4142$ $t/n = 2.9098$

Figure 13.83-Frankford Arsenal prism No. 7.

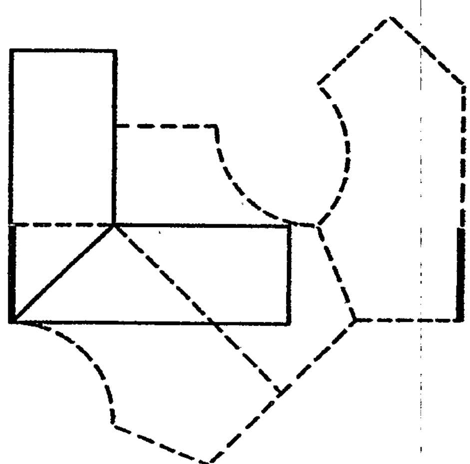


Figure 13.84-Frankford Arsenal prism No. 7 tunnel diagram.