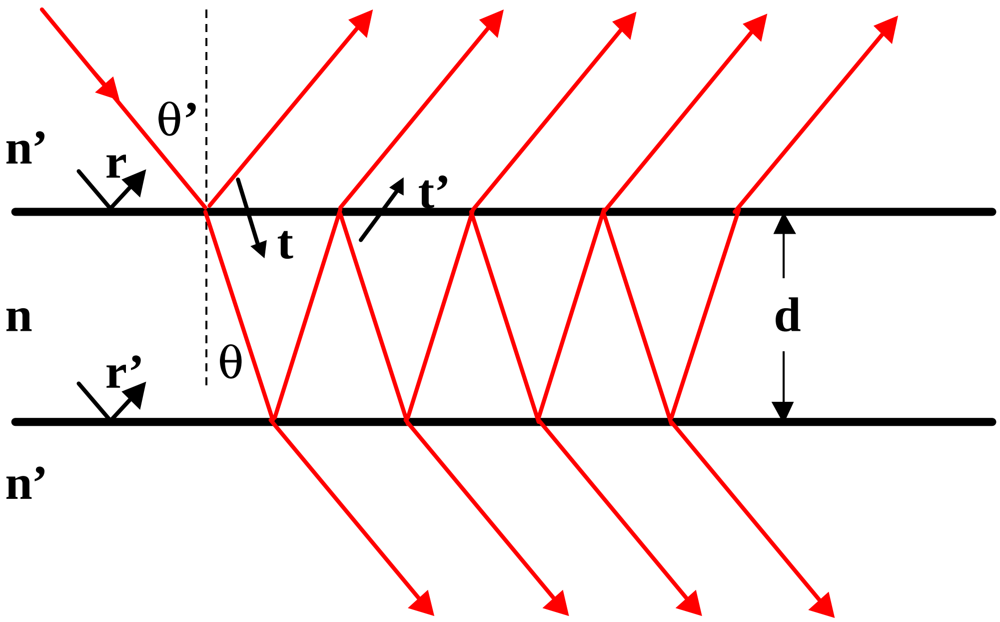
7. Multiple Beam Interference

7.1 Airy's Formula

We will first derive Airy's formula for the case of no absorption.

7.1.1 Basic reflectance and transmittance



Reflected light

$$E_r = E_i\,(r + t\,t'\,r'\,e^{i\delta} + \cdots + t\,t'\,r'^{(2p-3)}\,e^{i\,(p-1)\,\delta})$$

Transmitted light

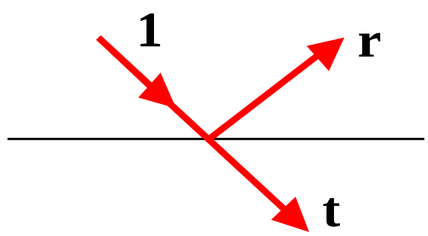
$$E_t = E_i\,t\,t'\,(1 + r'^2\,e^{i\delta} + r'^4\,e^{i\,2\,\delta} + \cdots + r'^{2\,(p-1)}\,e^{i\,(p-1)\,\delta})$$

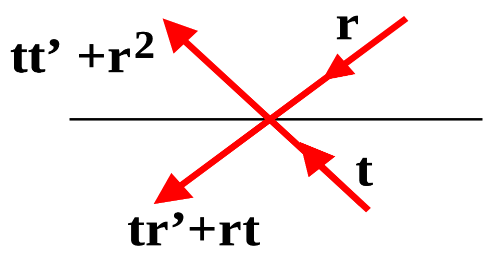
Phase due to OPD

$$\delta = \frac{2\,\pi}{\lambda_o}\,2\,n\,d\,\mathrm{Cos}[\theta]$$

7.1.2 Stokes Relations

If there are no losses a wave's propagation must be reversible.





Reversibility

Equations derived from Stokes Relations

$$t\,t' + r^2 = 1$$

$$t\,r' + r\,t = 0$$

$$t\,t' = 1 - r^2$$

$$r = -r'$$

Therefore

$$R + T = 1; \quad t\,t' = T; \quad R = r^2 = r'^2$$

7.1.3 Reflected Light

$$E_r = E_i\,(r + t\,t'\,r'\,e^{i\delta}\,(1 + r'^2\,e^{i\delta} + \cdots + r'^{2(p-2)}\,e^{i(p-2)\delta}))$$

$$E_r = E_i\left(r + t\,t'\,r'\,e^{i\delta}\sum_{n=0}^{p-2} r'^{2n}\,e^{i n\delta}\right)$$

$$E_r = E_i\left(r + t\,t'\,r'\,e^{i\delta}\left(\frac{1 - r'^{2(p-1)}\,e^{i(p-1)\delta}}{1 - r'^2\,e^{i\delta}}\right)\right)$$

Substitute r = -r' and let p→∞.

$$E_r = E_i\left(r - \frac{t\,t'\,r\,e^{i\delta}}{1 - r^2\,e^{i\delta}}\right)$$

$$E_r = E_i\,r\left(\frac{1 - r^2\,e^{i\delta} - t\,t'\,e^{i\delta}}{1 - r^2\,e^{i\delta}}\right)$$

$$E_r = E_i\,r\left(\frac{1 - (r^2 + t\,t')\,e^{i\delta}}{1 - r^2\,e^{i\delta}}\right)$$

$$E_r = E_i\left(\frac{\sqrt{R}\,(1 - e^{i\delta})}{1 - R\,e^{i\delta}}\right)$$

$$I_r = E_r\,E_r^* = I_i\left(\frac{R\,(2 - 2\,\mathrm{Cos}[\delta])}{1 + R^2 - 2\,R\,\mathrm{Cos}[\delta]}\right)$$

But $1 - \mathrm{Cos}[\delta] = 2\,\mathrm{Sin}[\delta/2]^2$

$$\frac{I_r}{I_i} = \frac{4\,R\,\mathrm{Sin}[\delta/2]^2}{(1-R)^2 + 4\,R\,\mathrm{Sin}[\delta/2]^2}$$

7.1.4 Transmitted Light

$$E_t = E_i\, t\, t' \,(1 + r'^2\, e^{i\delta} + r'^4\, e^{i\,2\,\delta} + \cdots + r'^{2\,(p-1)}\, e^{i\,(p-1)\,\delta})$$

$$E_t = E_i\, t\, t' \left(\frac{1 - r'^{2\,p}\, e^{i\,p\,\delta}}{1 - r'^2\, e^{i\delta}}\right)$$

Let p→∞.

$$E_t = E_i\, \frac{t\, t'}{1 - r'^2\, e^{i\delta}}$$

$$E_t = E_i\, \frac{T}{1 - R\, e^{i\delta}}$$

$$\frac{I_t}{I_i} = \frac{T^2}{1 + R^2 - 2\,R\,\mathrm{Cos}[\delta]}$$

$$\frac{I_t}{I_i} = \frac{T^2}{(1-R)^2 + 4\,R\,\mathrm{Sin}[\delta/2]^2}$$

7.1.5 Comments on Airy's Formula

$\frac{I_r}{I_i}$ and $\frac{I_t}{I_i}$ are known as Airy's formula. Note that if there are no losses so $T + R = 1$, $\frac{I_r}{I_i} + \frac{I_t}{I_i} = 1$.

Let F, the coefficient of finesse, be given by

$$F = \frac{4\,R}{(1-R)^2}$$

Then

$$\frac{I_r}{I_i} = \frac{F\,\mathrm{Sin}[\delta/2]^2}{1 + F\,\mathrm{Sin}[\delta/2]^2}$$

$$\frac{I_t}{I_i} = \frac{1}{1 + F\,\mathrm{Sin}[\delta/2]^2}$$

This shows again that the reflected and transmitted light are complementary.

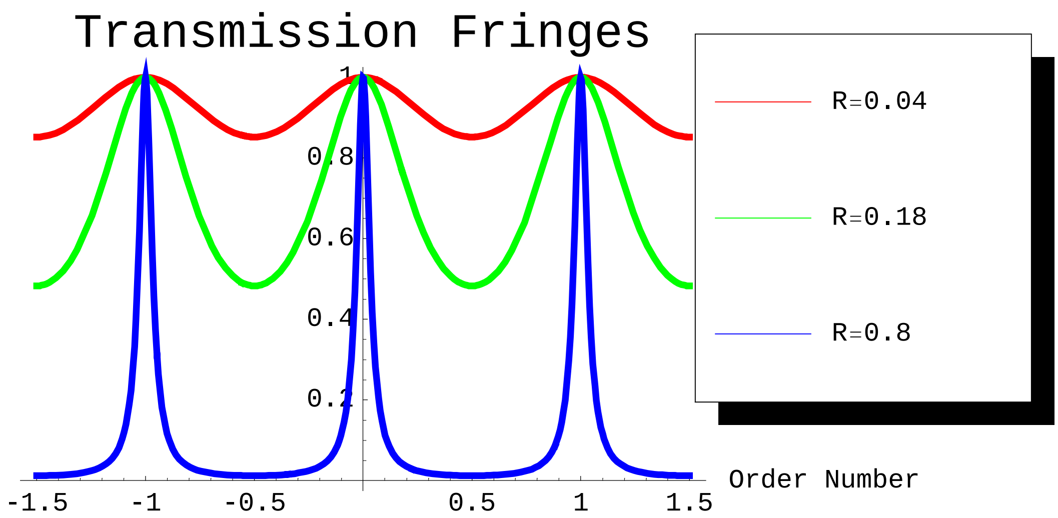
$\frac{I_r}{I_i}$ does go to zero for all values of R, but it goes to 1 only in the limit that $R \to 1$.

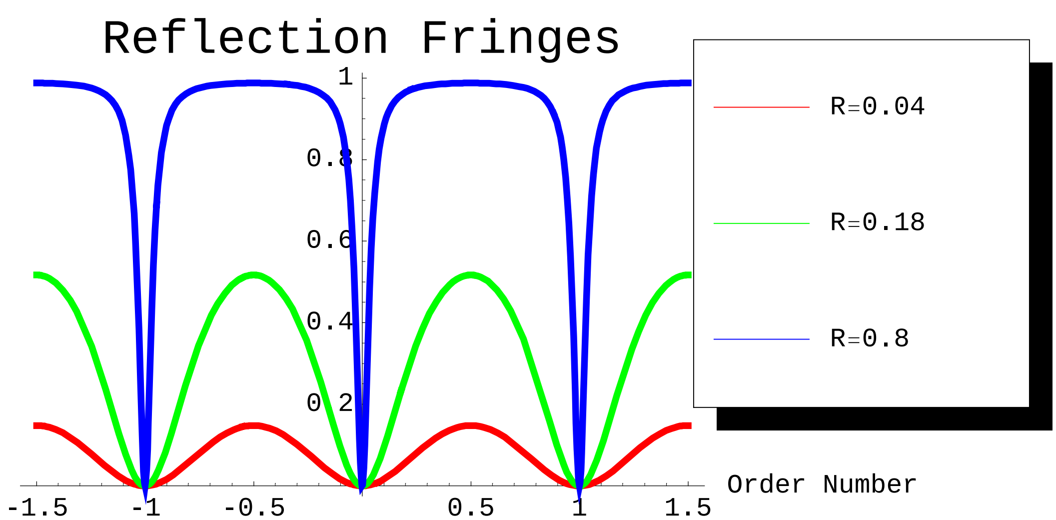
$\frac{I_t}{I_i}$ goes to 1 for all values of R, but it goes to zero only in the limit that $R \to 1$.

As R→1 the transmitted light becomes narrow bright fringes on a dark background.

As R→1 the reflected light becomes narrow dark fringes on a bright background.

7.1.6 Plots of Transmission and Reflection





For $\frac{I_t}{I_i}$ a maximum $\text{Sin}[\delta/2]^2=0$.

$$\frac{\delta}{2} = \frac{2\pi}{\lambda_o} n\, d\, \text{Cos}[\theta] = m\pi$$

or for a maximum in the transmitted light

$$2\, n\, d\, \mathrm{Cos}[\theta] = m\, \lambda$$

Going through the same procedure for reflected light we find that for a minimum in the reflected light

$$2\, n\, d\, \mathrm{Cos}[\theta] = m\, \lambda$$

This is the same result found previously for two beam interference.

For zero reflection all beams after the first reflection are subtracting from the first reflection.

7.1.7 Low Reflectivity Approximations

If R is small, F is also small, and the equations for the reflected and transmitted light can be approximated as

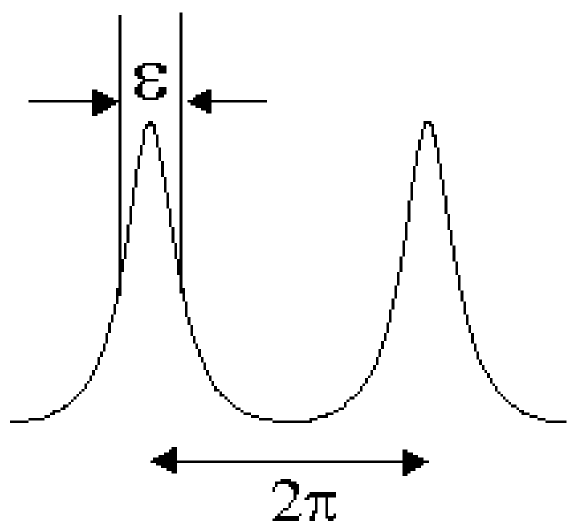
$$\frac{I_r}{I_i} \approx F\, \mathrm{Sin}[\delta/2]^2 = \frac{F}{2}\,(1 - \mathrm{Cos}[\delta])$$

$$\frac{I_t}{I_i} \approx 1 - F\, \mathrm{Sin}[\delta/2]^2 = 1 - \frac{F}{2}\,(1 - \mathrm{Cos}[\delta])$$

These equations are characteristic of two-beam interference.

7.1.8 Fringe Sharpness

Sharpness of fringes conveniently measured by their half intensity width which for transmitted light is the width between points on either side of maximum where intensity has fallen to half its maximum value.



$$\text{finesse} = \frac{\text{separation of adjacent fringes}}{\text{width of half max}}$$

For intensity at half max

$$\delta = 2\, m\, \pi \pm \frac{\epsilon}{2}$$

Thus

$$\frac{1}{2} = \frac{1}{1 + F\,\mathrm{Sin}\left[\frac{\epsilon}{4}\right]^2}$$

If F is very large ϵ is small so

$$\mathrm{Sin}\left[\frac{\epsilon}{4}\right]^2 \approx \left(\frac{\epsilon}{4}\right)^2$$

$$\mathrm{finesse} = \frac{2\pi}{\epsilon} = \frac{2\pi}{4/\sqrt{F}}$$

$$\mathrm{finesse} = \frac{\pi\sqrt{F}}{2} = \frac{\pi\sqrt{R}}{(1-R)}$$

7.2 Absorbing Coatings

If the two surfaces of the plate are identical, but there are losses, the value of $\frac{I_t}{I_i}$ can be determined as follows.

$$\delta = \frac{2\pi}{\lambda_o} 2\,n\,d\,\mathrm{Cos}[\theta] + 2\,\phi$$

ϕ is the phase change upon reflection for each surface.

We can still write

$$\frac{I_t}{I_i} = \frac{T^2}{(1-R)^2 + 4\,R\,\mathrm{Sin}[\delta/2]^2}$$

or

$$\frac{I_t}{I_i} = \frac{T^2}{(1-R)^2}\,\frac{1}{1 + F\,\mathrm{Sin}[\delta/2]^2}$$

However, now we have losses of an amount A so we must write

$$R + T + A = 1 \quad \mathrm{or} \quad T = (1-R) - A$$

It follows that

$$\frac{I_t}{I_i} = \left(1 - \frac{A}{1-R}\right)^2 \frac{1}{1 + F\,\mathrm{Sin}[\delta/2]^2}$$

$$T_{\mathrm{max}} = \left(1 - \frac{A}{1-R}\right)^2 = \left(\frac{1}{1+\frac{A}{T}}\right)^2$$

The effect of absorption is to reduce transmitted intensity and shift fringes. For the maximum transmitted intensity the important quantity is $\frac{A}{T}$. Even though A may be very small, if T is also small (R large), $\frac{A}{T}$ may become large and the maximum transmitted intensity may be very small. As an example let R = 99.7% and A = 0.2%, so T is approximately 0.1%. T_{max} is now 11%. However, let R = 99.7% and A = 0.29%. Now T is 0.01% and T_{max}

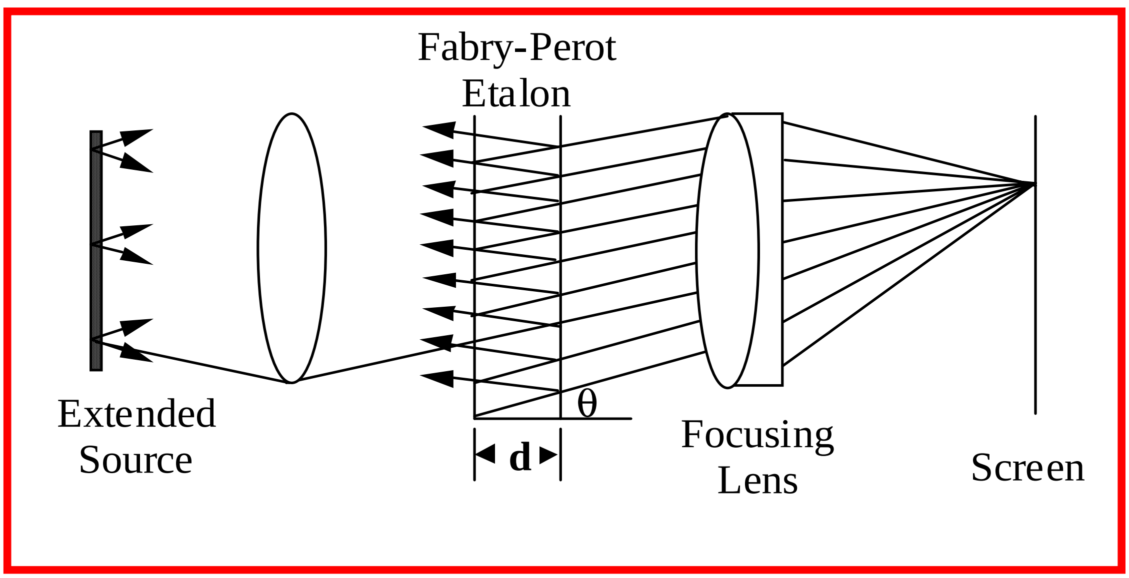
becomes 0.11%. What is happening physically is that while for each reflection there is very little loss, the reflectivity is so high that there are many effective reflections and the total loss becomes large.

While the phase shift due to ϕ is normally not a problem at normal incidence there may be a problem at non-normal incidence because ϕ is a function of polarization. At normal incidence ϕ is an equivalent to an increase of $\phi\, \lambda_o/2\pi$ in optical thickness of the plate.

The reflection case is more complicated because first reflection experiences no absorption. As a result the interference pattern does not go to zero.

7.3 Fabry-Perot

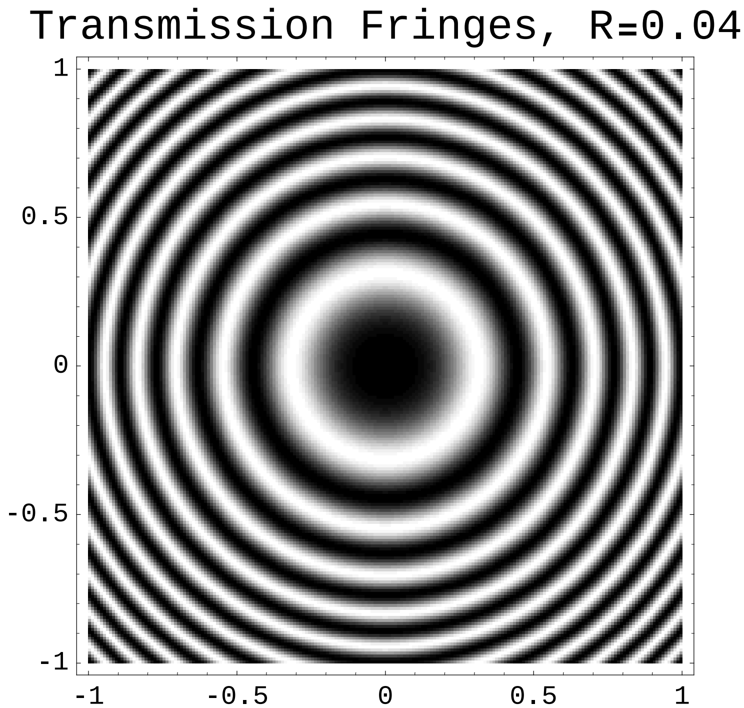
The multiple beam interference fringes from two highly reflecting surfaces illuminated near normal incidence are used in the classical Fabry-Perot interferometer. A Fabry-Perot interferometer is useful for spectroscopy.

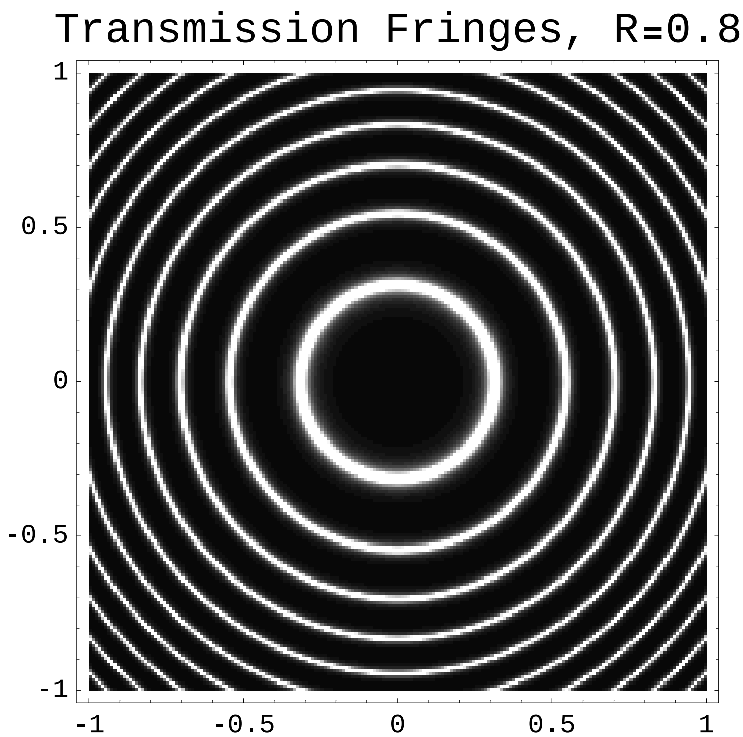


Narrow bright circular fringes are obtained. For a bright fringe of order m

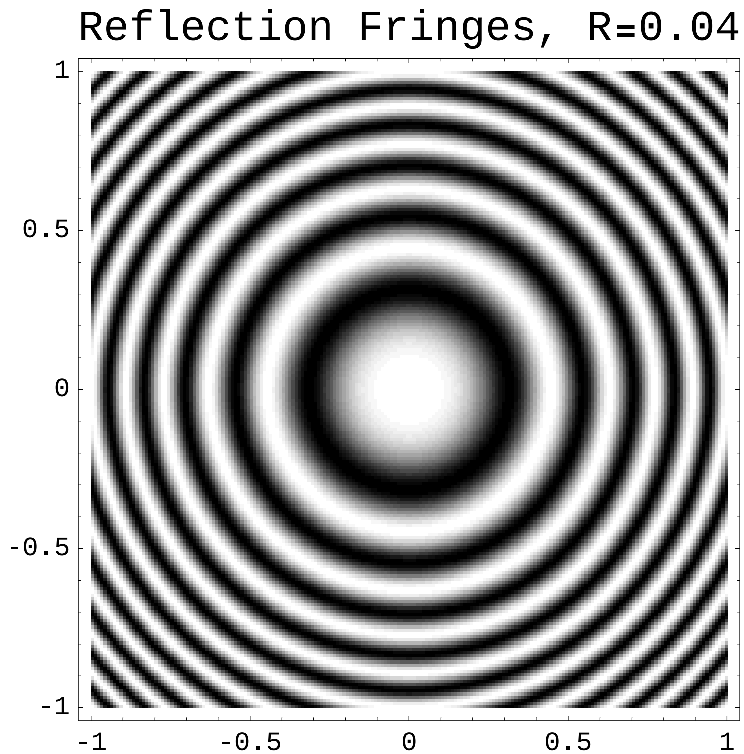
$$m = \frac{\delta}{2\pi} = \frac{2\, n\, d\, \mathrm{Cos}[\theta]}{\lambda_o} + \frac{\phi}{\pi}$$

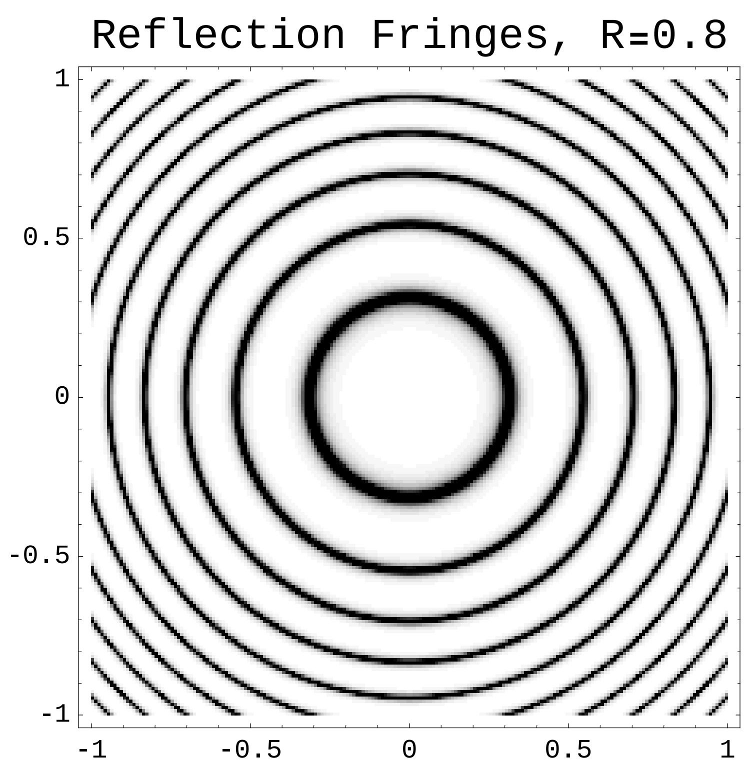
Transmission Fringes





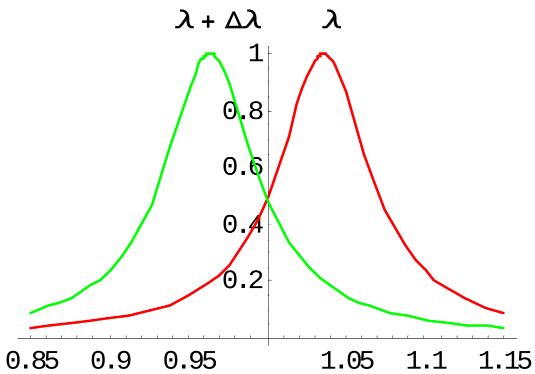
Reflection Fringes

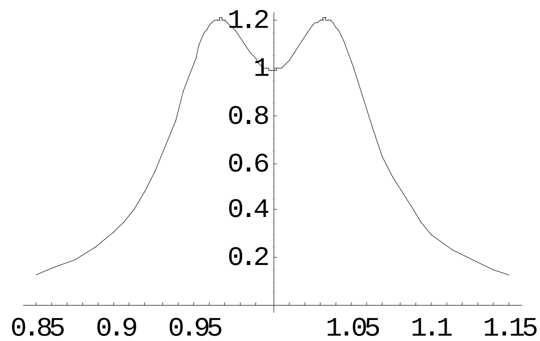




7.3.1 Resolving power

If more than one wavelength is present we see a superposition of the transmission pattern for each wavelength. Let there be two wavelengths present, λ_1 and $\lambda_2 = \lambda_1 + \Delta\lambda$. Our criterion for resolution is that the lines are just resolvable if the half maximum intensity of the peak of order m for one wavelength coincides with the half maximum intensity of the peak of order m for the second wavelength.





The left side of the figure shows the individual intensity contours of two Fabry-Perot fringes that are just resolved. The right side shows the two intensity contours added to give the observed effect.

It should be noted that some books such as Born & Wolf have a different criterion. They choose a separation such that the sum of the two intensities equals 0.811 that of the maximum. This agrees with the Rayleigh criterion that if we had a sinc^2 function the intensity maximum of one line would coincide with the minimum of the second line. Using this criterion we would get a resolution equal to 0.97 the resolution we get using our criterion.

The phase difference between the two interfering beams is

$$\delta = \frac{4\pi}{\lambda} n\, d \operatorname{Cos}[\theta] + 2\phi$$

where 2ϕ is much smaller than $\frac{4\pi}{\lambda} n\, d$ Cos[θ].

$$\Delta\delta_{1/2} = \frac{2\pi}{\text{finesse}}$$

$$\Delta\delta = -\frac{1}{\lambda}\,\delta\,\Delta\lambda$$

$$\left|\frac{\lambda}{\Delta\lambda}\right| = \frac{\delta}{\Delta\delta} = \frac{2\pi m}{\frac{2\pi}{\text{finesse}}} = m\,(\text{finesse})$$

$$\text{resolving power} = \frac{\lambda}{\Delta\lambda} = m\,(\text{finesse}) = \frac{m\,\pi\sqrt{F}}{2} = \frac{m\,\pi\sqrt{R}}{(1-R)}$$

near normal incidence

$$m \approx \frac{2\,n\,d}{\lambda}$$

$$\text{resolving power} \approx \frac{2\,n\,d}{\lambda}\,\text{finesse}$$

Thus the resolution is proportional to the mirror separation.

As an example let the finesse be 30 (R $\approx$ 0.9), nd = 4 mm, λ = 500 nm, then the resolving power is $\approx 5 \times 10^5$ and $\Delta\lambda$ is 0.001 nm.

The question is why not keep increasing the resolution by increasing the separation of the Fabry-Perot plates indefinitely? The problem is that we would have an overlapping of orders. The wavelength difference at which overlapping takes place is called the free spectral range.

7.3.2 Free Spectral Range

Overlapping takes place when order m of wavelength $\lambda_2 = \lambda_1 + \Delta\lambda$ falls on top of order m+1 of wavelength λ_1.

$$(m+1)\,\lambda_1 = m\,\lambda_2 = m\,(\lambda_1 + \Delta\lambda)$$

Thus

$$\Delta\lambda = \frac{\lambda}{m}$$

$$\Delta\lambda_{\mathrm{FSR}} = \frac{\lambda}{m} = \frac{\lambda^2}{2\,n\,d\,\mathrm{Cos}[\theta]}$$

Near normal incidence

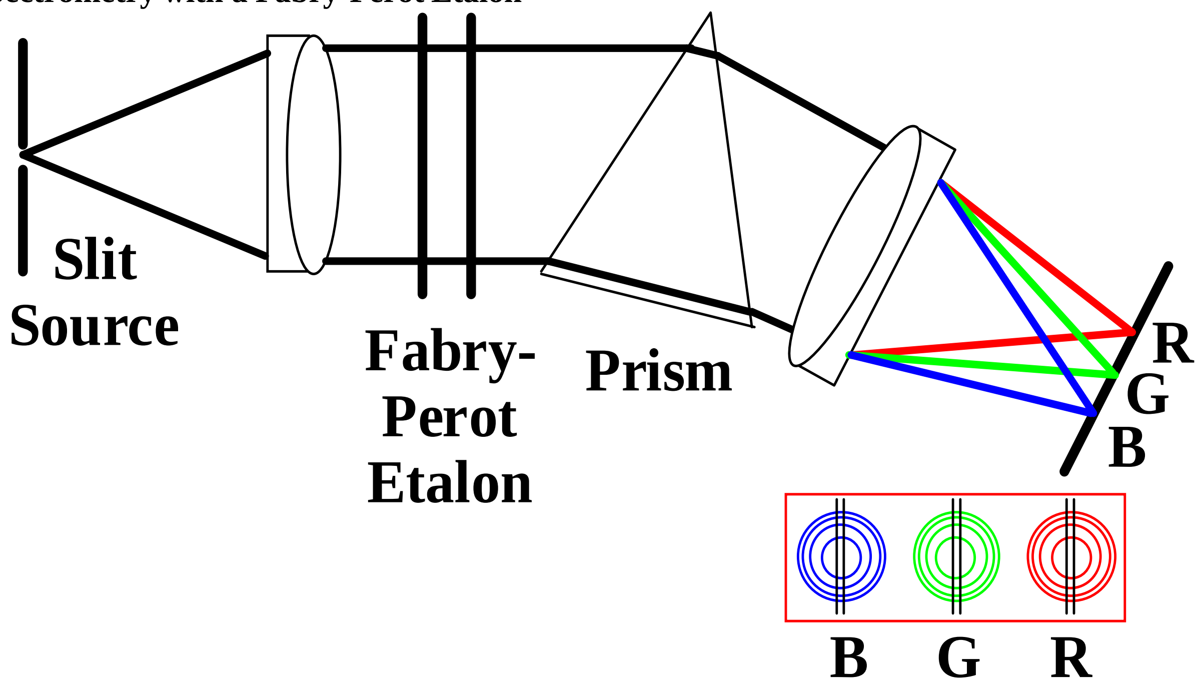
$$\Delta\lambda_{\mathrm{FSR}} = \frac{\lambda^2}{2\,n\,d}; \qquad \Delta\nu_{\mathrm{FSR}} = \frac{c}{2\,n\,d}$$

So increasing the resolving power by increasing the cavity thickness gives a reduction in the FSR.

$$\Delta\lambda_{\mathrm{FSR}} = \frac{\lambda}{m}; \qquad \Delta\lambda_{\mathrm{res}} = \frac{\lambda}{m\,(\mathrm{finesse})}$$

$$\frac{\Delta\lambda_{\mathrm{FSR}}}{\Delta\lambda_{\mathrm{res}}} = \mathrm{finesse} = \frac{\pi\sqrt{F}}{2} = \frac{\pi\sqrt{R}}{1-R}$$

7.3.3 Spectrometry with a Fabry-Perot Etalon



The prism does gross separation to eliminate, or at least reduce, the FSR problem.
The Fabry-Perot gives high resolution.

The following figure shows some actual interference fringes obtained using a Fabry-Perot etalon with a prism spectrometer (Ref: Born & Wolf).

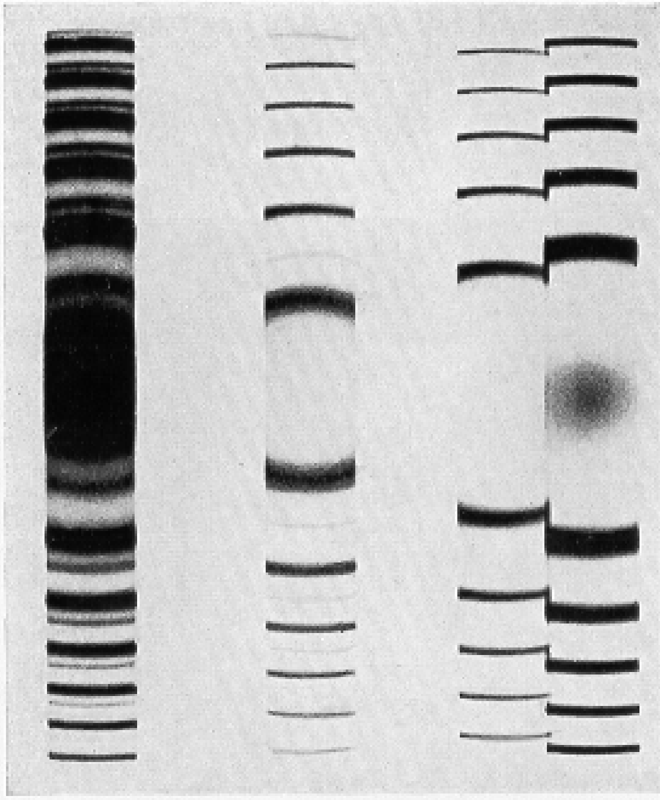
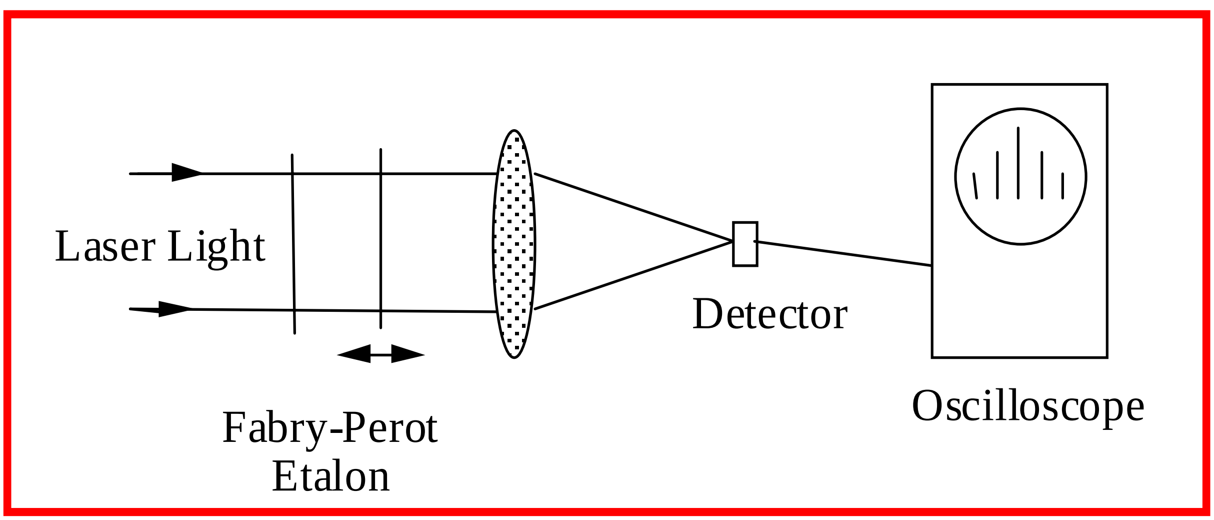


Fig. 7.64. FABRY–PEROT fringes from lines in the emission spectrum of helium (photographic negative).
[After K. W. MEISSNER, *J. Opt. Soc. Amer.*, **31** (1941), 416.]

7.3.4 Scanning Fabry-Perot

The scanning Fabry-Perot is useful when only a few discrete wavelengths are present as is often the case with a laser.



The scanning can be achieved by mounting one of the Fabry-Perot mirrors on a PZT. Since for a given fringe

$$2\,d = m\,\lambda; \qquad 2\,\Delta\,d = m\,\Delta\lambda = \frac{2\,d}{\lambda}\,\Delta\lambda$$

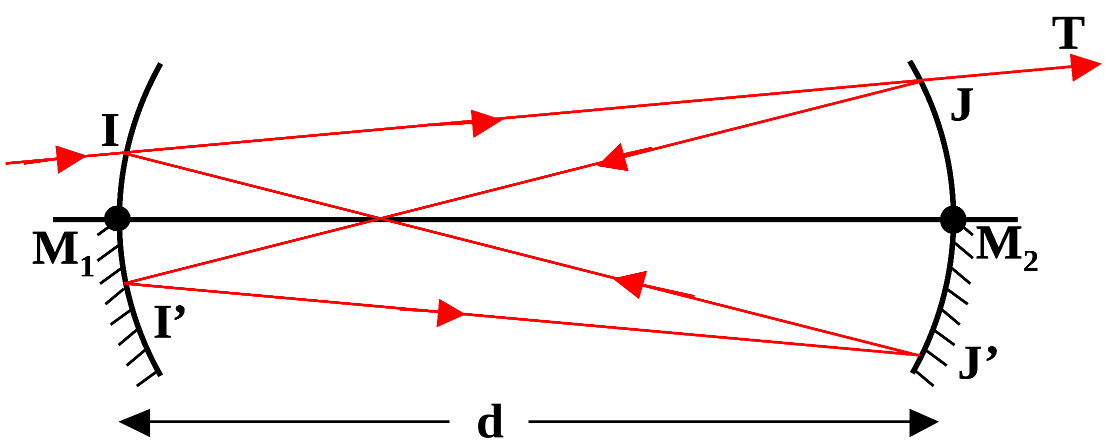
Thus

$$\frac{\Delta d}{d} = \frac{\Delta\lambda}{\lambda}$$

As d is varied different wavelengths will be transmitted through the Fabry-Perot and the oscilloscope display will show the wavelengths present.

7.3.5 Spherical Fabry-Perot

The figure below shows one form of a spherical Fabry-Perot. In the drawing the lower half of each spherical mirror is totally reflecting and the upper half is semi-transparent. The center of curvature of each mirror is located on the opposite mirror.



In the paraxial region the path difference between the initial ray I J and ray I J I' J' I J is equal to 4 d, where d is the distance between M_1 and M_2.

Instead of obtaining a series of parallel emerging rays originating from a single incident ray, as is the case with a plane parallel plate Fabry-Perot, we have a series of overlapping rays travelling along JT. The phase difference between consecutive rays is given by

$$\phi = \frac{2\,\pi}{\lambda}\,(4\,d)$$

which is independent of the inclination of the rays and their azimuth within the limits of the Gaussian approximation.

The intensity expression is the same as for a regular parallel plate Fabry-Perot, except we have a flat tint all over the field. If rays are inclined to the axis third-order aberrations produce variations of the path difference and we find the flat tint surrounded by circular fringes. We can reduce aberrations by placing two identical circular diaphragms centered on M_1 and M_2.

This interferometer is well suited to the large path differences corresponding to a high spectral resolution.

7.4 FECO (Fringes of Equal Chromatic Order)

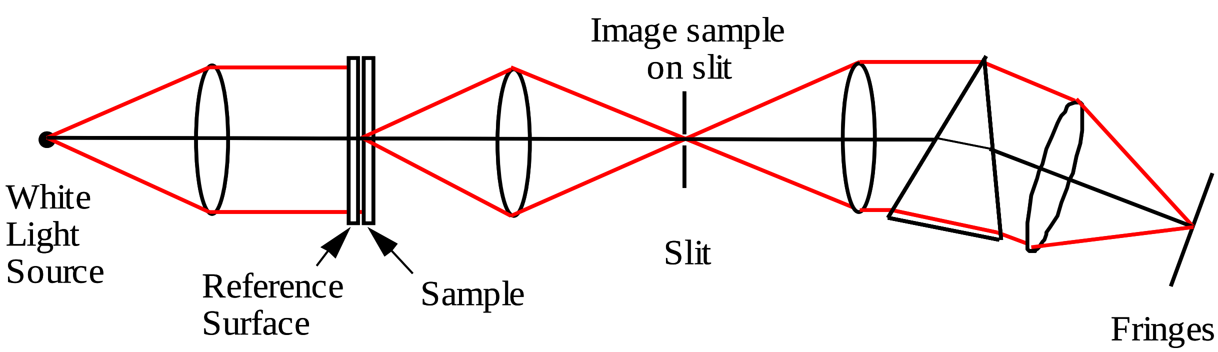
Previously we were concerned with multiple-beam fringes produced by monochromatic radiation. In some cases it is better to use a white light source. In this section we will combine a multiple beam interferometer with a spectrometer to measure thickness variations.

For transmission

$$I_t = \frac{I_{\max}}{1 + F\,\mathrm{Sin}[\delta/2]^2} \quad \text{where} \quad \delta = \frac{2\pi}{\lambda_o} 2\, n\, d\, \mathrm{Cos}[\theta] + 2\phi$$

ϕ is the phase change on reflection at each surface.

A schematic diagram of a FECO interferometer is shown below. Both the sample and the reference surface must have high reflectivity so high finesse multiple beam interference fringes are obtained. The sample is imaged onto the entrance slit of a spectrometer



If n = 1 and $\theta = 0°$ for a bright fringe of order m

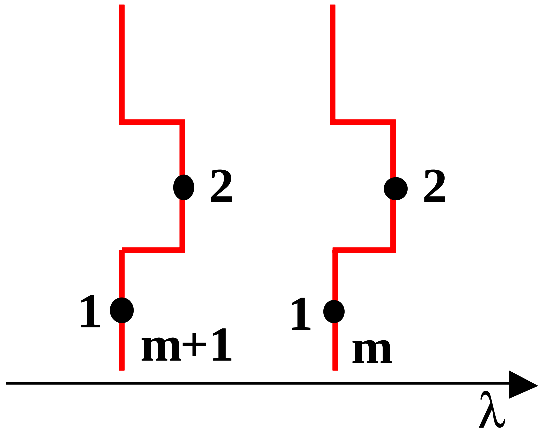
$$\frac{\phi}{\pi} + 2\frac{d}{\lambda} = m$$

It should be noted that for a given fringe $\frac{d}{\lambda}$= constant and

$$\lambda_m = \frac{2\,d}{m - \frac{\phi}{\pi}}$$

Solving for the height difference across a sample is complicated since $\phi = \phi[\lambda]$. However, with many coatings ϕ can be considered to be independent of λ over the small spectral region used for the analysis. (For more details see Born & Wolf or Jean Bennett, JOSA 54, p. 612 (1964).

The following drawing shows two fringes in the FECO output. The goal is to find the surface height difference between points 1 and 2.



$d = \left(m - \frac{\phi}{\pi}\right) \frac{\lambda}{2}$ and $d_2 - d_1 = \left(m - \frac{\phi}{\pi}\right) \left(\frac{\lambda_{2,m} - \lambda_{1,m}}{2}\right)$

For point 1 and fringe orders m and m + 1

$$\left(m - \frac{\phi}{\pi}\right) \lambda_{1,m} = \left(m + 1 - \frac{\phi}{\pi}\right) \lambda_{1,m+1}$$

Thus,

$$\left(m - \frac{\phi}{\pi}\right) = \frac{\lambda_{1,m+1}}{\lambda_{1,m} - \lambda_{1,m+1}}$$

and

$$d_2 - d_1 = \frac{\lambda_{1,m+1}}{\lambda_{1,m} - \lambda_{1,m+1}} \left(\frac{\lambda_{2,m} - \lambda_{1,m}}{2}\right)$$

The following figure shows some actual FECO interference fringes (Ref: Born & Wolf).

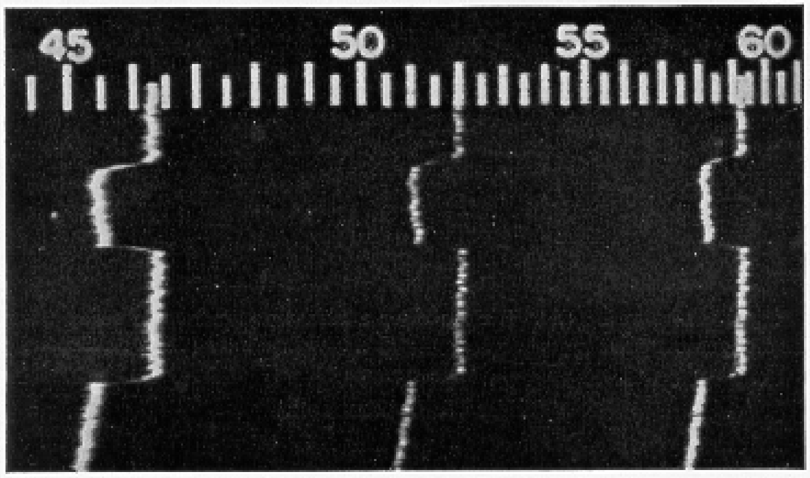


Fig. 7.79. Fringes of equal chromatic order given by a section of a diamond crystal surface. The scale is of wavelength in hundreds of Ångstroms.
(After S. Tolansky and W. L. Wilcock, *Proc. Roy. Soc.*, A, **191** (1947), 192.)

Since $d_2 - d_1$ is proportional to $\lambda_{2,m} - \lambda_{1,m}$, the profile of the cross-section of an unknown surface is obtained by plotting a single fringe on a scale proportional to the wavelength.

The spectroscopic slit is in effect selecting a narrow section of the interference system and each fringe is a profile of the variation of d in that section since there is exact point-to-point correspondence between the selected region and its image on the slit.
Small changes in d are determined by measuring small changes in λ. There are no ambiguities as to whether a region is a hill or a valley. There are no ambiguities at a discontinuity as we would have with monochromatic light where it is difficult to determine which order belongs to each fringe. Surface height variations in the Angstrom range can be determined.

Two disadvantages are
1) we are getting data only along a line and
2) the sample being measured must have a high reflectivity.