11.5 Optical Transfer Function

Reference: Goodman, *Introduction to Fourier Optics*

Basic Definitions

Let $\text{Abs}[h[\xi, \eta]]^2$ be the point spread function, PSF, and let $I_g[\xi, \eta]$ be the intensity of the geometrical image, then

$$I_i[u, v] = \kappa \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Abs}[h[u-\xi, v-\eta]]^2 \, I_g[\xi, \eta] \, d\xi \, d\eta$$

We will now look at the normalized spatial frequency of I_g and I_i

$$G_g[f_x, f_y] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_g[u, v] \, e^{-i 2\pi (f_x u + f_y v)} \, du \, dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_g[u, v] \, du \, dv}$$

$$G_i[f_x, f_y] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_i[u, v] \, e^{-i 2\pi (f_x u + f_y v)} \, du \, dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_i[u, v] \, du \, dv}$$

The normalized transfer function of the system is given by

$$H[f_x, f_y] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Abs}[h[u, v]]^2 \, e^{-i 2\pi (f_x u + f_y v)} \, du \, dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Abs}[h[u, v]]^2 \, du \, dv}$$

which is the normalized Fourier transform of the PSF.

From the convolution theorem

$$G_i[f_x, f_y] = H[f_x, f_y] \, G_g[f_x, f_y]$$

$H[f_x, f_y]$ is called the optical transfer function, OTF. The modulus of the OTF is called the modulation transfer function, MTF. From the above we see the OTF is the normalized Fourier transform of the PSF.

Relating OTF to pupil function.

The OTF is given by the Fourier transform of the PSF. The PSF is the square of the absolute value of the Fourier transform of the pupil function.

From the autocorrelation theorem we have

$$\text{FT}\left[\text{Abs}[g[x, y]]^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G[\xi, \eta] \, G^*[\xi - f_x, \eta - f_y] \, d\xi \, d\eta$$

Therefore if $P[x', y']$ is the pupil function

$$H[f_x, f_y] = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P[x', y'] \, P^*[x' - \lambda z_i f_x, y' - \lambda z_i f_y] \, dx' \, dy'}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Abs}[P[x', y']]^2 \, dx' \, dy'}$$

We can do a simple change of variables

$$x = x' - \frac{\lambda z_i f_x}{2} \text{ and } y = y' - \frac{\lambda z_i f_y}{2}$$

$$H[f_x, f_y] = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left[x + \frac{\lambda z_i f_x}{2}, y + \frac{\lambda z_i f_y}{2}\right] P^*\left[x - \frac{\lambda z_i f_x}{2}, y - \frac{\lambda z_i f_y}{2}\right] dx\, dy\right) \Big/ \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Abs}[P[x, y]]^2\, d\xi\, dy\right)$$

That is, the OTF is given by the autocorrelation of the pupil function.

General properties of OTF

- **1) H[0,0]=1**

  Follows directly from equation for OTF

- **2) $H[-f_x, -f_y] = H^*[f_x, f_y]$**

  Fourier transform of real function is Hermitian.

- **3) $\text{Abs}[H[f_x, f_y]] \le \text{Abs}[H[0, 0]]$**

  Follows directly from fact OTF is given by the autocorrelation of the pupil function.

What does OTF mean?

object

$$I[x] = I_o\,(1 + m\,\text{Sin}[\nu\, x])$$

Perfect Image

Assume unit magnification

$$I[x] = I_o\left(1 + \frac{m}{2\,i}\left(e^{i\,\nu\,x} - e^{-i\,\nu\,x}\right)\right)$$

$$\text{OTF}[\nu] = \text{Abs}[\text{OTF}[\nu]]\, e^{i\,\theta[\nu]}, \text{ Hermitian}$$

Actual image

$$I[x] = I_o\left(1 + \frac{m}{2\,i}\,\text{Abs}[\text{OTF}[\nu]]\left(e^{i\,(\nu\,x + \theta[\nu])} - e^{-(i\,\nu\,x + \theta[\nu])}\right)\right)$$

$$I[x] = I_o\,(1 + m\,\text{MTF}[\nu]\,\text{Sin}[\nu\, x + \theta[\nu]])$$

The modulus of the OTF (MTF) changes the contrast of the image and the phase of the OTF shifts the pattern. Since $\theta[\nu]$ depends on spatial frequency, the shift depends upon spatial frequency.