Diffraction

D - 1

In class we derived the expression

$$U(P_o) = \frac{1}{4\pi}\int_{\Sigma}\frac{e^{i k r_{01}}}{r_{01}}\left(\frac{\partial U}{\partial n} - i k U \operatorname{Cos}[\vec{n}, \vec{r}_{01}]\right) dS$$

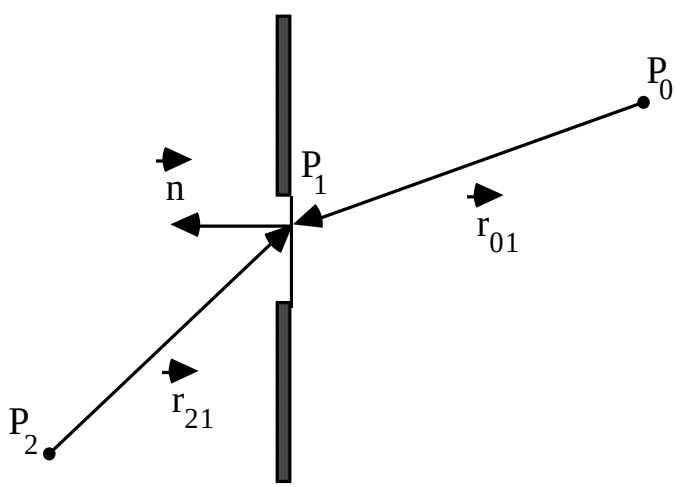
We said that if the aperture is illuminated by a single spherical wave,

$$U(P_1) = \frac{A e^{i k r_{21}}}{r_{21}}$$

arising from *a* source at P_2, *a* distance r_{21} from P_1 and r_{21} is many wavelengths long, the above expression can be written as

$$U(P_o) = \frac{A}{i\lambda}\int_{\Sigma}\frac{e^{i k (r_{21}+r_{01})}}{r_{21} r_{01}}\left(\frac{\operatorname{Cos}[\vec{n}, \vec{r}_{01}] - \operatorname{Cos}[\vec{n}, \vec{r}_{21}]}{2}\right) dS$$

Show that the above statement is correct.



Solution

For a point source at P_2

$$U(P_1) = \frac{A e^{i k r_{21}}}{r_{21}}$$

The amplitude at P_o is then given by

$$U(P_o) = \frac{1}{4\pi}\int_{\Sigma} \frac{e^{i k r_{01}}}{r_{01}} \left(\frac{\partial U}{\partial n} - i k U \operatorname{Cos}[\vec{n}, \vec{r}_{01}]\right) dS$$

$$\frac{\partial U(P_1)}{\partial n} = \operatorname{Cos}[\vec{n}, \vec{r}_{21}] \frac{\partial U}{\partial r_{21}} = \frac{A e^{i k r_{21}}}{r_{21}} \left(i k - \frac{1}{r_{21}}\right) \operatorname{Cos}[\vec{n}, \vec{r}_{21}]$$

Since $r_{21} >> \lambda$

$$\frac{\partial U(P_1)}{\partial n} \approx i k \frac{A e^{i k r_{21}}}{r_{21}} \operatorname{Cos}[\vec{n}, \vec{r}_{21}]$$

Therefore,

$$U(P_o) = \frac{A}{i\lambda}\int_{\Sigma} \frac{e^{i k (r_{21}+r_{01})}}{r_{21} r_{01}} \left(\frac{\operatorname{Cos}[\vec{n}, \vec{r}_{01}] - \operatorname{Cos}[\vec{n}, \vec{r}_{21}]}{2}\right) dS$$

D - 2

Consider the use of the Green's function

$$G_+(P_1) = \frac{e^{i k r_{01}}}{r_{01}} + \frac{e^{i k r_{01}}}{\tilde{r}_{01}}$$

in the Rayleigh-Sommerfeld theory.

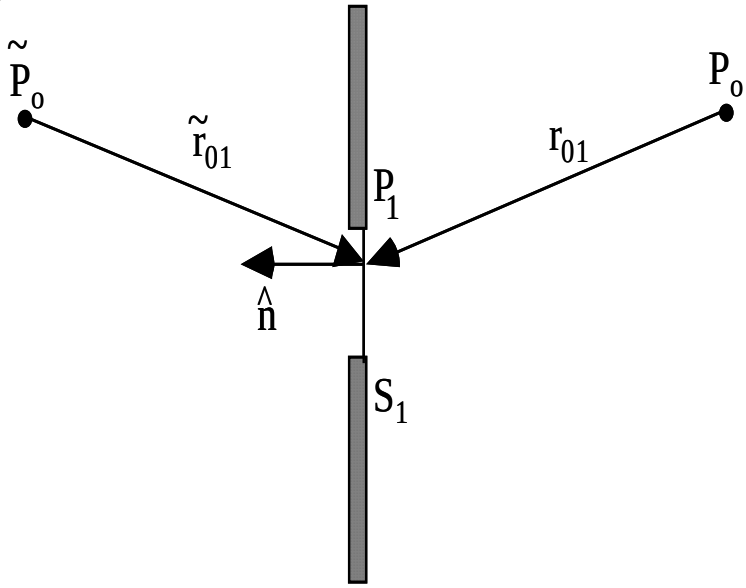
a) Show that the normal derivative of G_+ vanishes across the plane of the aperture.

b) Using this Green's function, find the expression for $U(P_0)$ in terms of an arbitrary disturbance across the aperture. What boundary conditions must be applied to obtain this result?

c) Using the result of (b), find an expression for $U(P_0)$ when the aperture illumination consists of a spherical wave diverging about the point P_2.

Solution

- **a)**



$$\frac{\partial G_+(P_1)}{\partial n} = \hat{n}\cdot\frac{\partial G_+(P_1)}{\partial r} = \mathrm{Cos}[\hat{n}, \vec{r}_{01}]\,\frac{\partial}{\partial r}\left(\frac{e^{i k r_{01}}}{r_{01}}\right) + \mathrm{Cos}[\hat{n}, \vec{r}_{01}]\,\frac{\partial}{\partial r}\left(\frac{e^{i k r_{01}}}{\tilde{r}_{01}}\right) =$$

$$\mathrm{Cos}[\hat{n}, \vec{r}_{01}]\left(i k - \frac{1}{r_{01}}\right)\frac{e^{i k r_{01}}}{r_{01}} + \mathrm{Cos}[\hat{n}, \vec{r}_{01}]\left(i k - \frac{1}{\tilde{r}_{01}}\right)\frac{e^{i k r_{01}}}{\tilde{r}_{01}}$$

Let $\tilde{P}_0$ be chosen to be the mirror image of P_o with respect to the aperture plane. Then $r_{01} = \tilde{r}_{01}$ and $\mathrm{Cos}[\hat{n}, \vec{r}_{01}] = -\mathrm{Cos}[\hat{n}, \vec{r}_{01}]$. Therefore,

$$\frac{\partial G_+(P_1)}{\partial n} = 0 \text{ for all points on } S_1.$$

- **b)**

$$U(P_o) = \frac{1}{4\pi}\int_{S_1}\left(\frac{\partial U}{\partial n}G_+ - U\frac{\partial G_+}{\partial n}\right)dS = \frac{1}{4\pi}\int_{S_1}\frac{\partial U}{\partial n}G_+\,dS = \frac{1}{2\pi}\int_{\Sigma}\frac{\partial U(P_1)}{\partial n}\frac{e^{i k r_{01}}}{r_{01}}\,dS$$

Boundary Conditions

1) Across Σ, $\frac{\partial U}{\partial n}$ is the same as it would have been in the absence of a screen.

2) Over the portion of S_1 in the geometrical shadow of the screen $\frac{\partial U}{\partial n}$ =0.

Nothing needs to be concerned concerning U.

- **c)**

$$U(P_1) = A\,\frac{e^{i k r_{21}}}{r_{21}}$$

$$\frac{\partial U(P_1)}{\partial n} = A\,\mathrm{Cos}[\hat{n}, \hat{r}_{21}]\left(i\,k - \frac{1}{r_{21}}\right)\frac{e^{i\,k r_{21}}}{r_{21}}$$

If r_{21} >> k

$$\frac{\partial U(P_1)}{\partial n} = A\,\mathrm{Cos}[\hat{n}, \hat{r}_{21}]\,(i\,k)\,\frac{e^{i\,k r_{21}}}{r_{21}}$$

$$U(P_o) = -\frac{A}{i\,\lambda}\int_{\Sigma}\frac{e^{i\,k\,(r_{21}+r_{01})}}{r_{21}\,r_{01}}\,\mathrm{Cos}[\hat{n}, \hat{r}_{21}]\,dS$$

D - 3

a) What is the major difference in the diffraction equation for the Rayleigh-Sommerfeld formulation and the equation for the Kirchhoff formulation?
b) How do we normally satisfy the Fraunhofer diffraction approximations in a small laboratory?

Solution

- **a)**

The obliquity factor is different.

Kirchhoff

$$\frac{\mathrm{Cos}[\hat{n}, \hat{r}_{01}] - \mathrm{Cos}[\hat{n}, \hat{r}_{21}]}{2}$$

Rayleigh-Sommerfeld

$$\mathrm{Cos}[\hat{n}, \hat{r}_{01}]$$

- **b)**

We use a lens to image infinity at the focal plane of the edge.

D - 4

a) What is the fundamental assumption used in deriving the diffraction equation for the Kirchhoff formulation that is not present in the derivation for the Rayleigh-Sommerfeld formulation?
b) We have seen that as a wave propagates there is a change in the relative phases of the components of the angular spectrum. Give a physical explanation of this change in the relative phases.

Solution

- **a)**

The Kirchhoff boundary conditions state that
1) Across the open portion of the aperture the field distribution U and its derivative $\frac{\partial U}{\partial n}$ are exactly the same as they would be in the absence of the screen.
2) In the geometrical shadow of the screen the field distribution U and its derivative $\frac{\partial U}{\partial n}$ are identically zero.

In the Rayleigh-Sommerfield formulation the need to impose the boundary conditions on <u>both</u> U and its derivative $\frac{\partial U}{\partial n}$ is removed.

- **b)**

The different plane waves travel a different distance hence there is a Cos[θ] factor that appears in the phases.

D - 5

Assuming unit-amplitude normally incident plane-wave illumination, find the angular spectrum of
a) a circular aperture of diameter d.
b) a circular opaque disk of diameter d.

Solution

- **a)**

Fourier transform of circular aperture

$$\frac{J_1[\pi d f]}{\pi d f}; \quad f = \text{spatial frequency} = \sqrt{f_x^2 + f_y^2}$$

- **b)**

Use Babinet's Principle

$$\delta[f_x, f_y] - \frac{J_1[\pi d f]}{\pi d f}$$

D - 6

Use the plane wave spectrum approach to look at the propagation of two plane waves between two planes separated by an unknown distance. One plane wave makes an angle of 5 degrees with respect to the z axis, and the second plane wave is along the z-axis. The phase difference between the two given components of the angular spectrum changes by 45 degrees as the beam propagates between the two planes? The wavelength is 500 nm. What is the distance between the two planes?

Solution

$$\text{phaseDifference} = \frac{2\pi}{\lambda}(1 - \text{Cos}[\theta])\, z$$

$$z = \frac{\text{phaseDifference}}{2\pi(1 - \text{Cos}[\theta])}\ \lambda\ /.\ \{\text{phaseDifference} \to 45\ \text{Degree},\ \lambda \to 0.5\ \mu\text{m},\ \theta \to 5\ \text{Degree}\}$$

16.4245 μm

D - 7

A circular aperture 208 microns in diameter is illuminated with a quasi-monochromatic plane wave having an irradiance of 5 watts/cm^2 and a wavelength of 516 nm. What is the minimum and maximum on-axis irradiance of the Fresnel diffraction pattern?

Solution

The on-axis irradiance varies between zero and 4 times the unobstructed irradiance. Thus, the minimum on-axis irradiance is 0 and the maximum is 20 watts/cm^2.

D - 8

A monochromatic point source S emits light of wavelength 600 nm. It falls onto an aperture A 15 cm away and then onto a screen 20 cm beyond A.

a) What is the value of the radius of the first Fresnel half-period zone at A?

b) The aperture A is a circle of radius 1 cm. How many Fresnel half-period zones does it contain?

c) A Fresnel zone plate is made with vertical polarizers covering the odd zones and horizontal polarizers covering the even zones. How does its behavior compare with a conventional zone plate with even zones opaque and odd zones transparent?

Solution

- **a)**

If y_1 is the radius of the first Fresnel zone

$$\frac{y_1^2}{2\,r_1} + \frac{y_1^2}{2\,r_2} = \frac{\lambda}{2}$$

```
Solve[y1^2/(2 r1) + y1^2/(2 r2) == λ/2, y1] /. {r1 → 15 cm, r2 → 20 cm, λ → 0.6 10^-4 cm}
```

$$\left\{\left\{y_1 \to -\frac{0.0226779\sqrt{\text{cm}}}{\sqrt{\frac{1}{\text{cm}}}}\right\}, \left\{y_1 \to \frac{0.0226779\sqrt{\text{cm}}}{\sqrt{\frac{1}{\text{cm}}}}\right\}\right\}$$

```
Im[cm] ^= 0; Im[μm] ^= 0; Positive[cm] ^= True;
```

```
FullSimplify[0.022677 Sqrt[cm]/Sqrt[1/cm]] /. cm -> 10^4 μm
```

226.77 μm

- **b)**

$$\frac{y^2}{2\,r_1} + \frac{y^2}{2\,r_2} = m\,\frac{\lambda}{2}$$

```
Solve[y^2/(2 r1) + y^2/(2 r2) == m λ/2, m] /. {r1 → 15 cm, r2 → 20 cm, λ → 0.6 10^-4 cm, y → 1 cm}
```

{{m → 1944.44}}

- **c)**

The odd and even zones will operate independently since the polarization of the light transmitted through them is orthogonal.
Independent of the polarization of the incident light the amount of light in the orders is the same as for a conventional zone plate.

D - 9

A zone plate is made by interfering two spherical waves of wavelength 633 nm. Let the distance from the zone plate to one point source be 10 cm and the distance from the zone plate to the second point source be 15 cm.
a) Where do the two first orders come to focus if the zone plate is illuminated with a plane wave of wavelength 633 nm? Where do the two third orders come to focus?
b) Repeat part a for plane wave illumination of wavelength 500 nm.

Solution

- **a)**

Since the problem is a little unclear as to whether the zone plate is made using two diverging (or converging) spherical waves or if it is made using one diverging wave and one converging wave I will work the problem both ways.

First orders

If the zone plate is made using two diverging waves or two converging waves then

```
fPositive = (1/(10 cm) - 1/(15 cm))^-1; fNegative = -fPositive;
```

```
distance = {fPositive, fNegative}
```

```
{30 cm, -30 cm}
```

If the zone plate is made using one diverging wave and one converging spherical wave then

```
fPositive2 = (1/(10 cm) + 1/(15 cm))^-1; fNegative2 = -fPositive2;
```

```
distance2 = {fPositive2, fNegative2}
```

```
{6 cm, -6 cm}
```

Third orders

If the zone plate is made using two diverging waves or two converging waves then

```
distance / 3
```

```
{10 cm, -10 cm}
```

If the zone plate is made using one diverging wave and one converging spherical wave then

```
distance2 / 3
```

```
{2 cm, -2 cm}
```

- **b)**

First orders

If the zone plate is made using two diverging waves or two converging waves then

```
633/500 distance // N
```

{37.98 cm, -37.98 cm}

or if one diverging wave and one converging wave

```
633/500 distance2 // N
```

{7.596 cm, -7.596 cm}

Third orders

If the zone plate is made using two diverging waves or two converging waves then

```
633/500 distance/3 // N
```

{12.66 cm, -12.66 cm}

or if one diverging wave and one converging wave

```
633/500 distance2/3 // N
```

{2.532 cm, -2.532 cm}

D - 10

A zone plate is made by interfering a plane wave and a spherical wave of wavelength 633 nm. Let the distance from the zone plate to the point source be 20 cm.

a) Where do the two first orders come to focus if the zone plate is illuminated with a plane wave of wavelength 500 nm? Where do the two second orders come to focus?

b) Where do the two first orders come to focus if the zone plate is illuminated with a 25 cm radius of curvature spherical wavefront of wavelength 633 nm.

Solution

- **a)**

$$\text{focalLength} = \pm 20\ \text{cm}\ \frac{633}{\lambda}$$

The two orders come to focus at

```
20 cm 633/500 {+1, -1} // N
```

{25.32 cm, -25.32 cm}

- **b)**

$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$; where p is the distance to the reconstructing source and q is the distance to the image.

```
ans = Solve[1/f == 1/p + 1/q, q]
```

$\{\{q \to -\frac{f\,p}{f - p}\}\}$

```
ans /. {f -> {20 cm, -20 cm}, p -> 25 cm} // N
```

{{q → {100. cm, -11.1111 cm}}}

The two images are -11.11 cm to the left and 100 cm to the right.

D - 11

I bought a Fresnel zone plate from Professor Gaskill for $10. The salesman (Gaskill) claimed the zone plate has a focal length of 10 cm for a wavelength of 500 nm.

a) Let me illuminate the zone plate with a spherical wave coming from a source 5 cm to the left of the plate. The spherical wave has a wavelength of 500 nm. Where do the two first orders come to focus?

b) For some reason the zone plate produces a second order. What could be wrong with the zone plate? (Gaskill claims that if I bought the $20 version I would not have this problem.)

Solution

- **a)**

$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$; $z_2 = \left(\frac{1}{f} - \frac{1}{z_1}\right)^{-1}$

$f = \pm 10\text{ cm}$

$z_2 = \left(\frac{1}{10\text{ cm}} - \frac{1}{5\text{ cm}}\right)^{-1} = -10\text{ cm}$ (image to left)

$z_2 = \left(\frac{1}{-10\text{ cm}} - \frac{1}{5\text{ cm}}\right)^{-1} = -3.33\text{ cm}$ (image to left)

- **b)**

The binary zone plate does not have a 50% duty cycle.

D - 12

A zone plate transmits 100% of the incident light, and every other zone is covered with a thin film of thickness d and refractive index 1.5. The zone plate has a focal length of 10 cm for a wavelength of 550 nm.

a) What is the smallest d such that the zone plate has no zero order for a wavelength of 550 nm?

b) What is the smallest d such that the zone plate has no zero order for wavelengths of 550 nm and 450 nm.

c) The zone plate is illuminated with a spherical wave coming from a source 5 cm to the left of the plate. Where do the two first orders come to focus for a wavelength of

i) 550 nm?

ii) 450 nm?

Solution

a)

Optical thickness of adjacent zones must differ by one-half wavelength.

$$(n-1)\,d = \frac{\lambda}{2}$$

$$\mathbf{d = \frac{\lambda}{2\,(n-1)}\ /.\ \{\lambda \to 550\,\mathrm{nm},\ n \to 1.5\}}$$

550. nm

b)

We want the smallest d such that the optical thickness between adjacent zones is one-half wavelength for both wavelengths.

$$(n-1)\,d = (2\,m+1)\,\frac{\lambda_1}{2} = (2\,m-1)\,\frac{\lambda_2}{2}$$

$$\mathbf{Solve\left[(2\,m+1)\,\frac{\lambda_1}{2} == (2\,m-1)\,\frac{\lambda_2}{2},\ m\right]}$$

$$\left\{\left\{m \to -\frac{\lambda_1+\lambda_2}{2\,(\lambda_1-\lambda_2)}\right\}\right\}$$

$$\mathbf{m = -\frac{\lambda_1+\lambda_2}{2\,(\lambda_1-\lambda_2)}\ /.\ \{\lambda_1 \to 450\,\mathrm{nm},\ \lambda_2 \to 550\,\mathrm{nm}\}}$$

5

$$\mathbf{d = \frac{(2\,m+1)\,\lambda_1}{2\,(n-1)}\ /.\ \{\lambda_1 \to 0.450\,\mu\mathrm{m},\ n \to 1.5\}}$$

4.95 μm

- **c)**

```
Clear[p, q, f];
```

i) f = ± 10 cm

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

```
Solve[1/p + 1/q == 1/f, q] /. {p → 5. cm, f → 10 cm}
```

{{q → -10. cm}}

```
Solve[1/p + 1/q == 1/f, q] /. {p → 5. cm, f → -10 cm}
```

{{q → -3.33333 cm}}

ii)

```
fplus = 10. cm (550 nm)/(450 nm)
```

12.2222 cm

```
Solve[1/p + 1/q == 1/f, q] /. {p → 5. cm, f → fplus}
```

{{q → -8.46154 cm}}

```
Solve[1/p + 1/q == 1/f, q] /. {p → 5. cm, f → -fplus}
```

{{q → -3.54839 cm}}

D - 13

a) Give a physical explanation for the 2nd order being missing for a binary Fresnel zone plate.
b) A source of 500 nm wavelength is 1 meter away from an observation screen. How many Fresnel zones are there for a 5 mm diameter circular region on the observation screen?

Solution

- **a)**

For the second order the OPD between consecutive openings is 2 λ. For the binary Fresnel zone plate we are assuming a 50% duty cycle, so there is 1 λ OPD across an opening. We can divide this opening into two regions where for each point in one half there is a corresponding point in the second half having an OPD difference of $\lambda/2$ or a phase difference of 180°. Thus, the light from the two halves of the opening cancel each other out.

■ **b)**

$$\text{OPD} = \frac{y^2}{2\,r}; \quad \text{numberFresnelZones} = \frac{\text{OPD}}{\frac{\lambda}{2}};$$

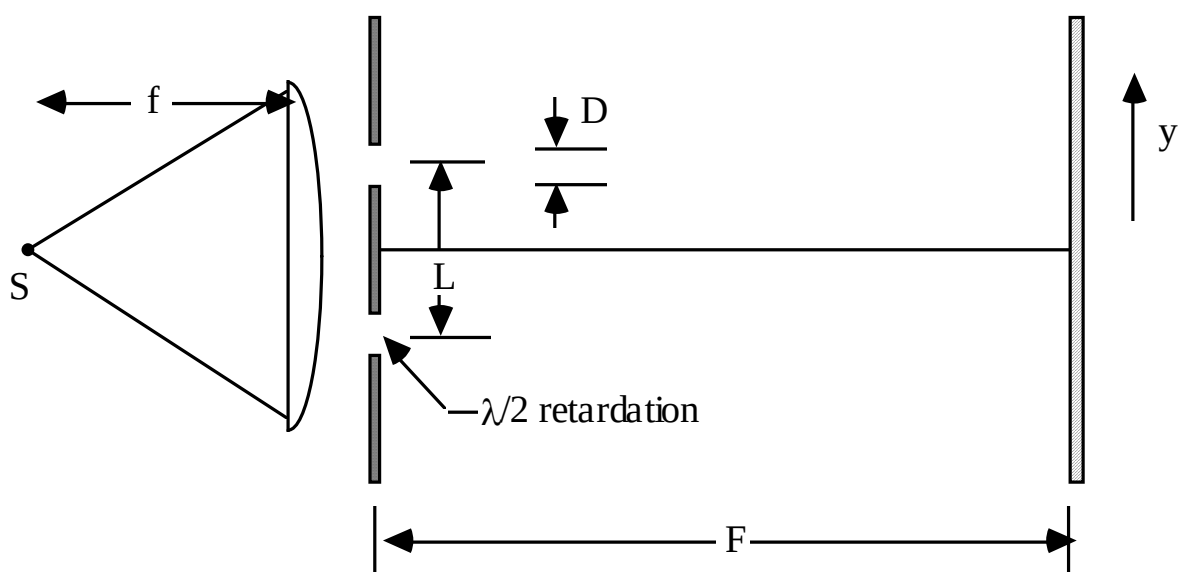
$$\text{numberFresnelZones} = \frac{y^2}{\lambda\, r} = \frac{(2.5\times 10^{-3}\ \text{m})^2}{(0.5\times 10^{-6}\ \text{m})\ 1\ \text{m}} = 12.5$$

D - 14

Find the Fresnel diffraction pattern of a wavefront which impinges upon two identical slits of width D and separation L. The source S is at infinity and the observation plane is at a distance F=1,000,000 wavelengths. D=1000 wavelengths and L=3000 wavelengths. The lower slit has a retardation of 1/2 wavelength. Plot either the absolute value of the amplitude or the irradiance for the coordinate y in the range

$$-5000\lambda \le y \le 5000\lambda.$$

Indicate clearly on your plot where the geometrical shadow is located. Do not spend a lot of time finding detail. Pick out a few minima and maxima and then make an educated guess as to what the remainder of the curve looks like.



Solution

```
amplitude[u1_, u2_] := 1/√2 (FresnelC[u2] -
    FresnelC[u1] + i (FresnelS[u2] - FresnelS[u1]))
```

$$u = y\left(\frac{2}{\lambda F}\right)^{1/2} = y\left(\frac{2}{\lambda\,(1,000,000\,\lambda)}\right)^{1/2} = \frac{y\sqrt{2}}{1000\,\lambda}$$

Top Slit

$$a = \frac{(y - 2000\,\lambda)\,\sqrt{2}}{1000\,\lambda},\quad b = \frac{(y - 1000\,\lambda)\,\sqrt{2}}{1000\,\lambda}$$

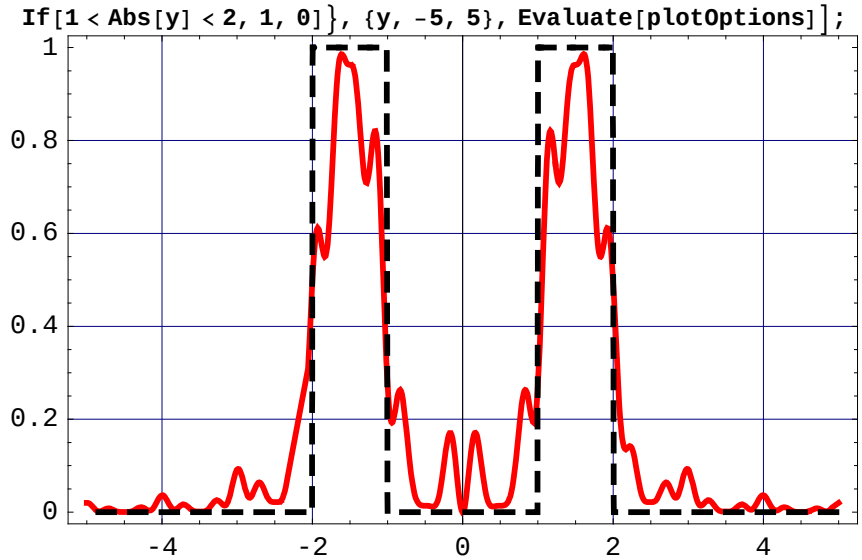
Bottom Slit

$$a = \frac{(y + 1000\,\lambda)\,\sqrt{2}}{1000\,\lambda},\quad b = \frac{(y + 2000\,\lambda)\,\sqrt{2}}{1000\,\lambda}$$

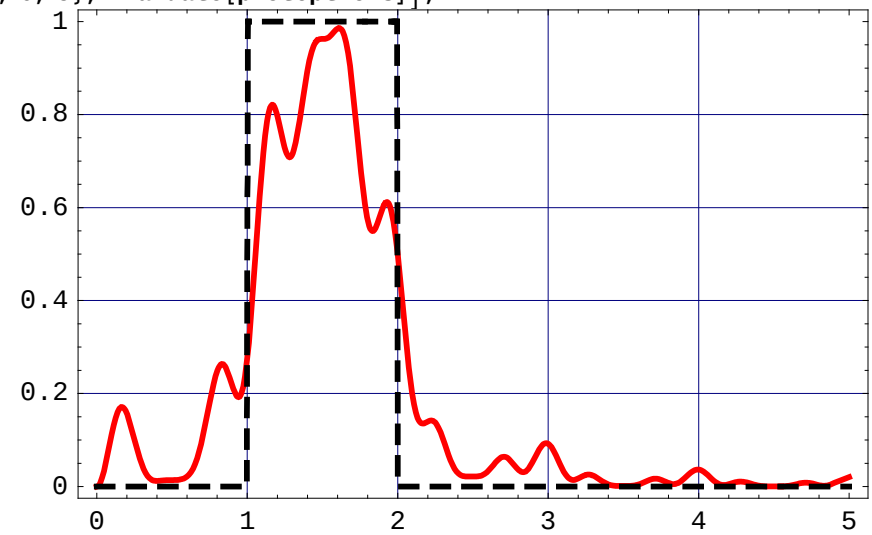
On the graph the geometrical shadow goes from -2 to -1 and 1 to 2.

```
plotOptions = {PlotRange → All, AxesOrigin → {0, 0}, Frame → True,
   GridLines → Automatic, PlotStyle → {{RGBColor[1, 0, 0], Thickness[.0075]},
     {Thickness[.0075], Dashing[{.0200}]}}, Background → White};
```

```
Plot[{(Abs[amplitude[√2 (y - 2), √2 (y - 1)] - amplitude[√2 (y + 1), √2 (y + 2)]])^2,
      If[1 < Abs[y] < 2, 1, 0]}, {y, -5, 5}, Evaluate[plotOptions]];
```



```
Plot[{(Abs[amplitude[√2 (y - 2), √2 (y - 1)] - amplitude[√2 (y + 1), √2 (y + 2)]])^2,
      If[1 < y < 2, 1, 0]},
  {y, 0, 5}, Evaluate[plotOptions]];
```



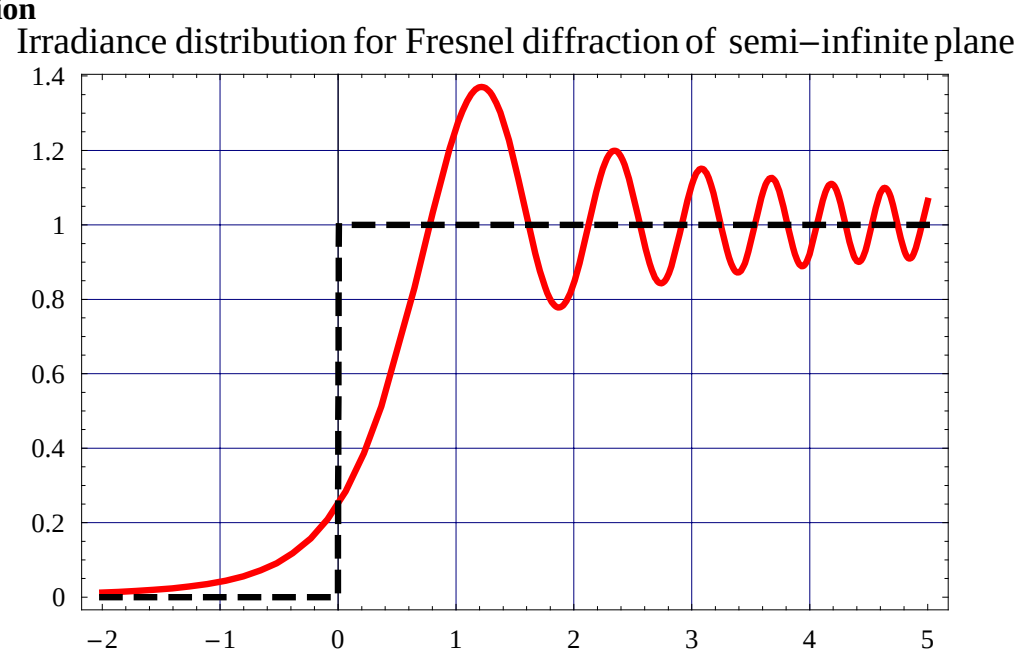
D - 15

Roughly sketch the irradiance distribution for the Fresnel diffraction pattern of a knife edge. Be sure to point out the edge of the geometrical shadow. Give a physical description for

a) the origin of the light in the geometrical shadow and

b) the structure of the irradiance distribution outside the shadow.

Solution



- **a)**

The light comes from the edge of the knife edge.

- **b)**

This is the interference of the light coming from the edge of the knife edge and the straight through transmitted light.

D - 16

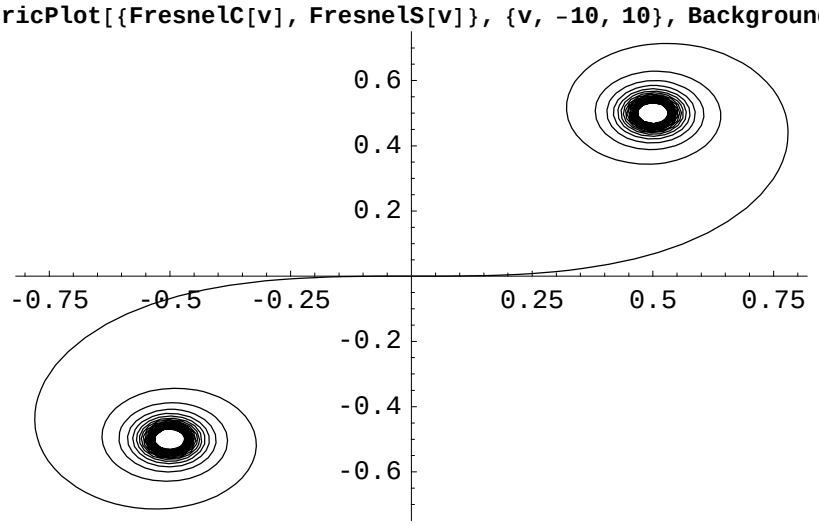
a) How would you construct a Cornu spiral? Roughly sketch a Cornu spiral.
b) Briefly explain how to use the Cornu spiral to determine the Fresnel diffraction pattern of
   i) a rectangular aperture.
   ii) a circular aperture.

Solution

- **a)**

The Cornu spiral is a plot of the sine and cosine Fresnel integrals. The spiral can be thought of as being constructed by placing end to end infinitesimal vectors $e^{i\pi v^2/2}$ dv as v increases from -∞ to ∞.

```
ParametricPlot[{FresnelC[v], FresnelS[v]}, {v, -10, 10}, Background -> White];
```



- **b)**

The Cornu spiral is used for rectangular apertures, it is not used for circular apertures.

Basically just select the portion of the vectors from v_1 to v_2 making up the open portion of the aperture and the point in the diffraction pattern being calculated. As we look at the different points in the diffraction pattern we will use different portions of the vectors, but $v_2 - v_1$ will be a constant. The resultant vector gives the magnitude and phase of the corresponding point in the diffraction pattern.

D - 17

Describe the on-axis irradiance for the Fresnel diffraction pattern of a
a) circular obstacle.
b) circular aperture.
c) a knife edge where the knife transmits 100 percent, but introduces a 180 degree phase change for the light transmitted through the knife.
d) a knife edge where the knife transmits 70 percent, but introduces a 180 degree phase change for the light transmitted through the knife.

Solution

- **a)**

As long as we are observing at a distance from the circular obstacle large enough so the Fresnel approximations are valid the on-axis irradiance is equal to the unobstructed irradiance.

- **b)**

As long as we are observing at a distance from the circular aperture large enough so the Fresnel approximations are valid the on-axis irradiance varies between zero and four times the unobstructed irradiance.

- **c)**

As shown below the on-axis irradiance is zero.

The amplitude is given by

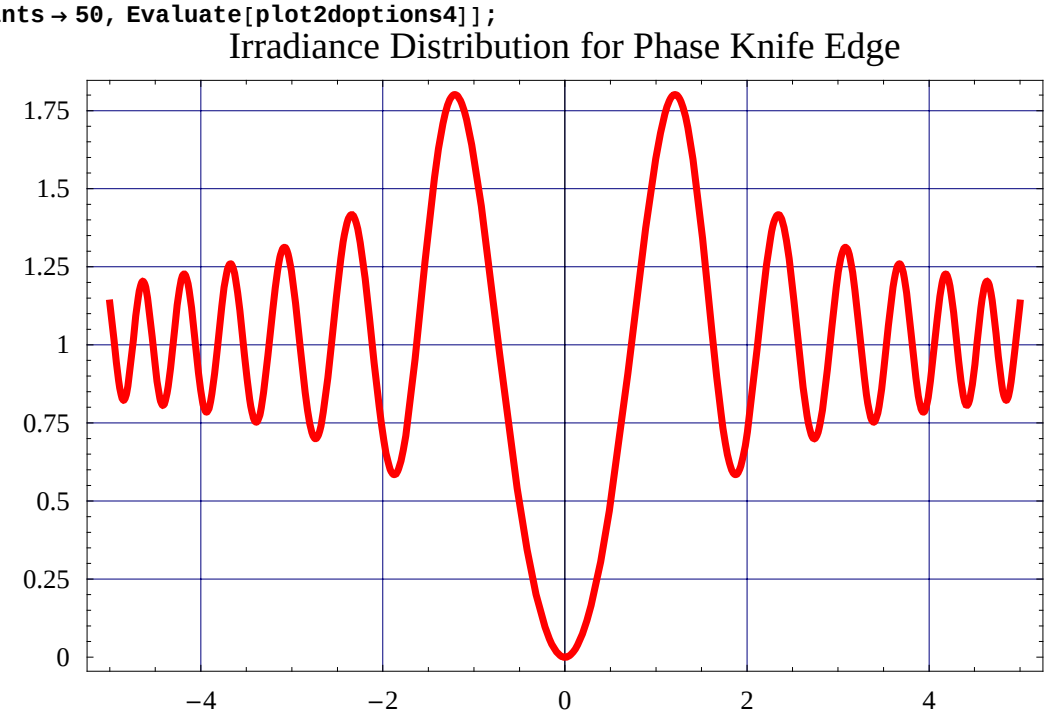
```
amplitude[u1_, u2_] := 1/√2 (FresnelC[u2] - FresnelC[u1] + ⅈ (FresnelS[u2] - FresnelS[u1]))
```

The on-axis irradiance is given by

```
(Abs[amplitude[0, -∞] - amplitude[∞, 0]])^2
0
```

The following gives a plot of the irradiance.

```
Plot[(Abs[amplitude[η, -∞] - amplitude[∞, η]])^2, {η, -5, 5}, PlotRange → All,
  PlotLabel -> StyleForm["Irradiance Distribution for Phase Knife Edge", FontSize → 14],
  PlotPoints → 50, Evaluate[plot2doptions4]];
```



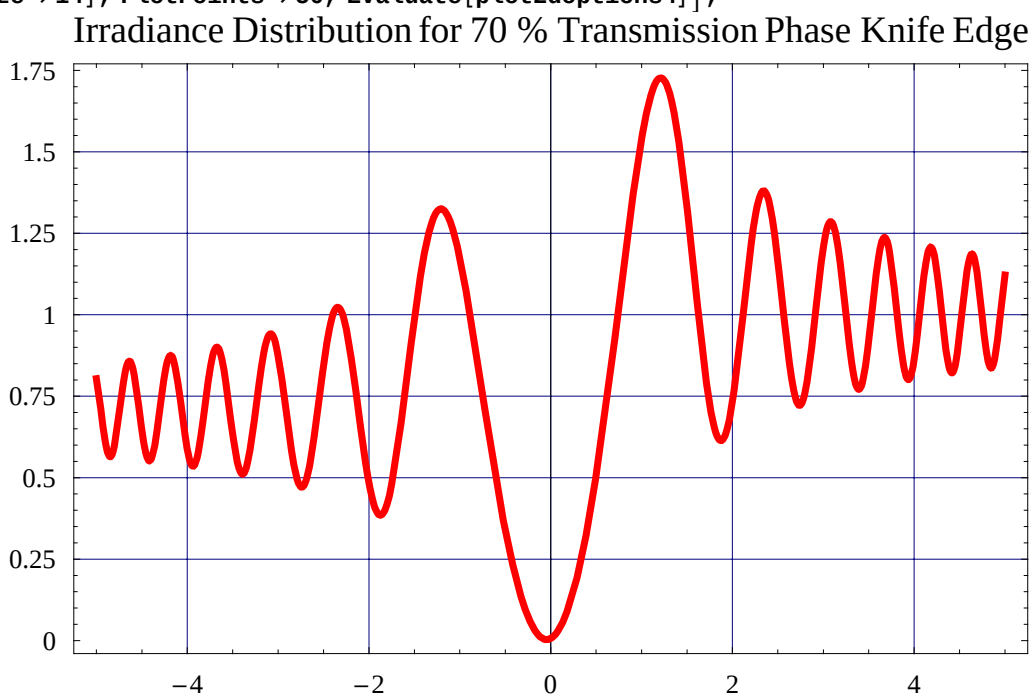
- **d)**

The on-axis irradiance will be small, but not zero. The value is found below to be 0.007 times the unobstructed value.

```
(Abs[amplitude[0, -∞] - √0.7 amplitude[∞, 0]])^2
0.00666999
```

The following gives a plot of the irradiance.

```
Plot[(Abs[amplitude[η, -∞] - √0.7 amplitude[∞, η]])^2,
  {η, -5, 5}, PlotRange → All, PlotLabel ->
   StyleForm["Irradiance Distribution for 70 % Transmission Phase Knife Edge",
    FontSize → 14], PlotPoints → 50, Evaluate[plot2doptions4]];
```

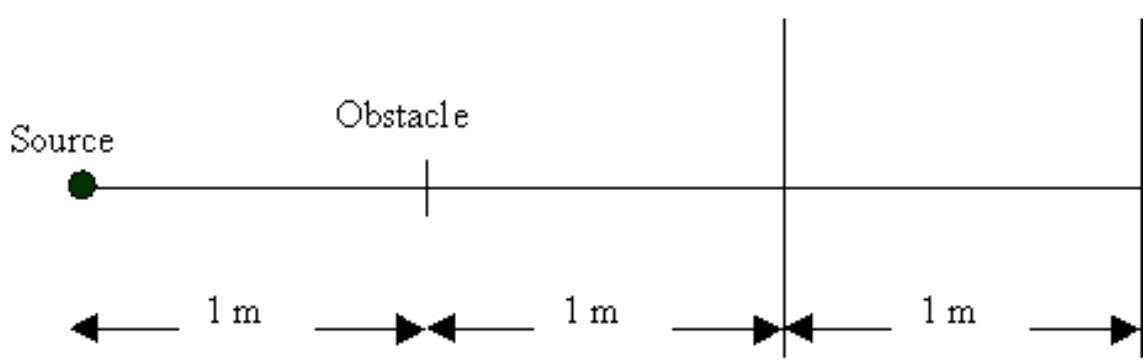


D - 18

A 1 mm diameter circular obstacle is illuminated with a spherical wave diverging from a point source 1 meter to the left of the circular obstacle. In the plane of the circular obstacle the irradiance of the illuminating beam is 1 watt/cm^2. What is the on-axis irradiance

a) 1 meter to the right of the circular obstacle?

b) 2 meters to the right of the circular obstacle?



Solution

- **a)**

$$\frac{1\,\text{watt}/\text{cm}^2}{2^2} \;//\; N$$

$$\frac{0.25\,\text{watt}}{\text{cm}^2}$$

- **b)**

$$\frac{1\,\text{watt}/\text{cm}^2}{3^2} \;//\; N$$

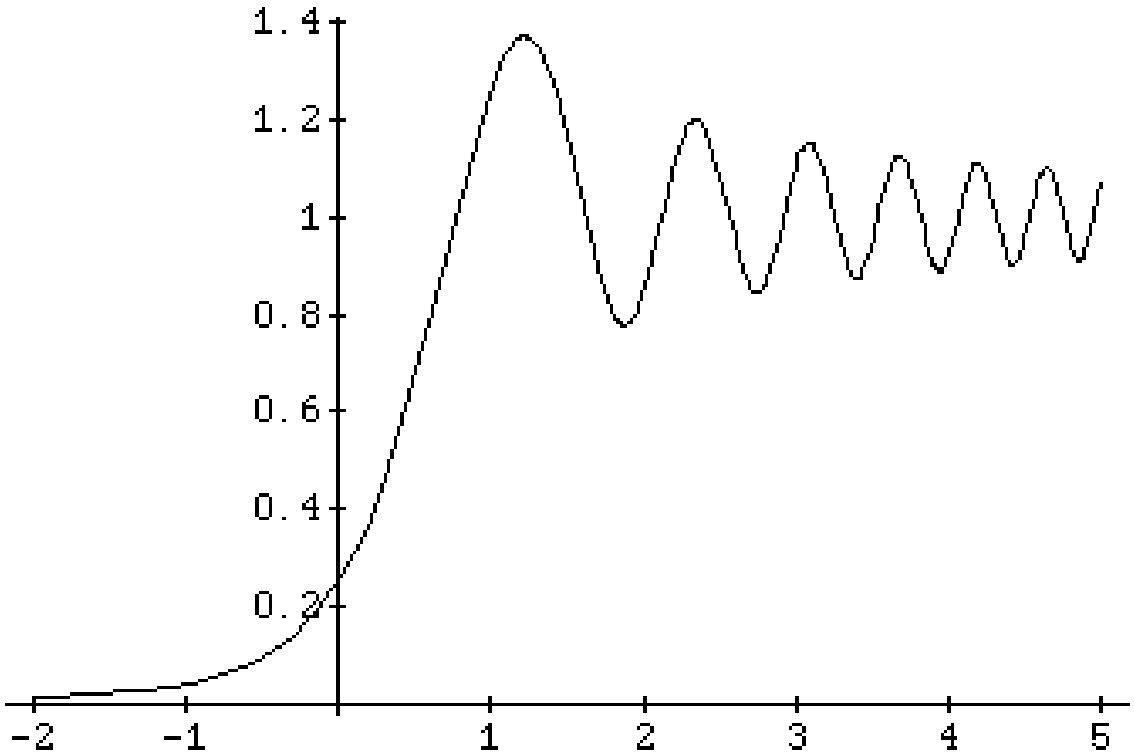
$$\frac{0.111111\,\text{watt}}{\text{cm}^2}$$

D - 19

a) The figure below shows the Fresnel diffraction pattern of what object?

b) Give a physical explanation for the fringes becoming closer together as we move toward the right side of the pattern.

c) Give a physical explanation for there being no fringes on the left side of the pattern.



Solution

- **a)**

  Fresnel diffraction of a straight edge.

- **b)**

  We are seeing the interference of the straight through beam and the light coming from the edge of the aperture. As we move toward the right side of the pattern the angle between the two interfering beam increases.

- **c)**

  There is no straight through beam. The only beam we have is the light coming from the edge of the aperture.

D - 20

In a plane z = 0 the amplitude transmittance of a screen is

$$t[x, y] = \frac{1}{2}\left(1 + m\,\mathrm{Cos}\left[\frac{2\pi}{\lambda}\,0.1\,x\right]\right)$$

Assume normally incident plane wave illumination and neglect the finite aperture extent.
What is the amplitude of the Fresnel diffraction pattern of the screen at a distance of z = 100λ?

Solution

The period of the grating is 10 λ. Thus the diffraction angle of the ± first orders is $\frac{\lambda}{\text{period}}$ = 0.1 radians. The phase of the transmitted light will go as $\frac{2\pi}{\lambda}$ z Cos[θ] ≈ $\frac{2\pi}{\lambda}$ z $\left(1 - \frac{\theta^2}{2}\right)$. Thus the amplitude can be written as

$$\frac{1}{2}\, e^{i\,\frac{2\pi}{\lambda}\,100\,\lambda}\left(1 + m\, e^{-i\,\frac{2\pi}{\lambda}\,100\,\lambda\left(-\frac{.1^2}{2}\right)}\,\mathrm{Cos}\left[\frac{2\pi}{\lambda}\,0.1\,x\right]\right) =$$

$$\boxed{\frac{1}{2}\left(1 - m\,\mathrm{Cos}\left[\frac{2\pi}{\lambda}\,0.1\,x\right]\right)}$$

Thus, the image is a phase reversal of the object.

$$\frac{2\pi}{\lambda}\,100\,\lambda\left(\frac{.1^2}{2}\right)$$

3.14159

$$e^{-i\pi}$$

-1

D - 21

Assuming normally incident plane-wave illumination, and neglecting finite aperture extent:

a) Find the Fresnel diffraction pattern of a screen with the following transmittance function:

$$t(x,y) = 1/2(1 + m \cos 2\pi\, f_o\, x)$$

b) Given that m<<1, at what distances z from the aperture is the field distribution across a parallel plane (1) purely amplitude modulated over space? (2) approximately phase modulated over space?

Solution

- **a)**

Thus the diffraction angle of the ± first orders is $\pm\lambda\, f_o$. The phase of the transmitted light will go as $\frac{2\pi}{\lambda}$ z Cos[θ] ≈ $\frac{2\pi}{\lambda}$ z $\left(1-\frac{\theta^2}{2}\right)$. Thus the amplitude can be written as

$$\text{amplitude} = \frac{1}{2}\, e^{i k z}\left(1 + m\, e^{-i \pi z \lambda f_o^2} \cos 2\pi f_o x\right)$$

- **b)**

$$\text{amplitude} = \frac{1}{2}\, e^{i k z}\, e^{i m e^{-i \pi z \lambda f_o^2} \cos 2\pi f_o x}$$

This can be written as

$$\text{amplitude} = \frac{1}{2}\, e^{i k z}\, e^{m\,(i\,\text{Cos}[\pi z \lambda f_o^2] + \text{Sin}[\pi z \lambda f_o^2])\cos 2\pi f_o x}$$

This can be further reduced to

$$\text{amplitude} = \frac{1}{2}\, e^{i k z}\, e^{i m\,\text{Cos}[\pi z \lambda f_o^2] \cos 2\pi f_o x}\, e^{m\,\text{Sin}[\pi z \lambda f_o^2] \cos 2\pi f_o x}$$

Amplitude modulated

$$\text{Cos}[\pi z \lambda f_o^2] = 0$$

$$\pi z \lambda f_o^2 = \left(m + \frac{1}{2}\right) \pi$$

$$z = \frac{(m + \frac{1}{2})}{\lambda f_o^2}$$

Phase modulated

$$\text{Sin}[\pi z \lambda f_o^2] = 0$$

$$\pi z \lambda f_o^2 = m \pi$$

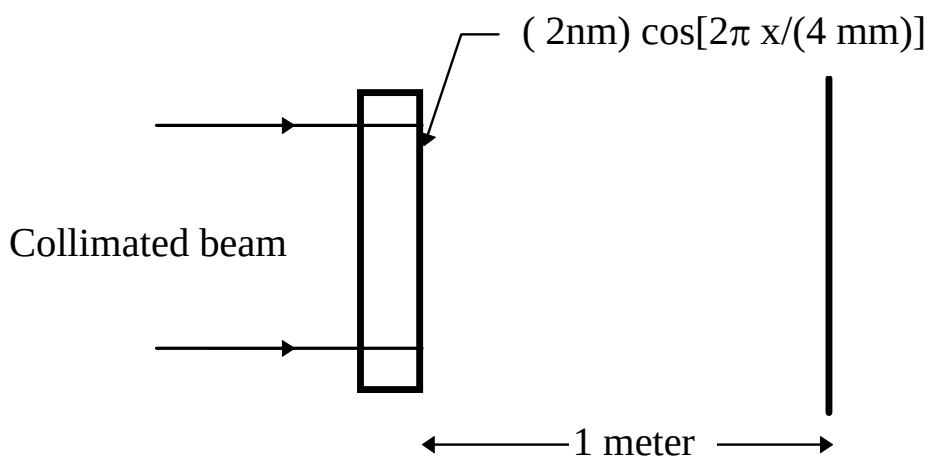
$$z = \frac{m}{\lambda f_o^2}$$

D - 22

A uniform collimated beam of wavelength 500 nm is transmitted through a glass window of refractive index 1.5 as shown below. One surface of the window is flat, while the second surface has a height variation given by (2 nm) cos[2π x/(4 mm)]. Give a simple expression for the phase variation across the wavefront

a) leaving the glass window.

b) 1 meter to the right of the glass window.



Solution

- **a)**

$$\phi = \frac{2\pi}{\lambda} (n - 1)\, d = \frac{2\pi}{500\,\text{nm}} (0.5)\,(2\,\text{nm})\,\text{Cos}\left[2\pi \frac{x}{4\,\text{mm}}\right]$$
$$= (4 \times 10^{-3})\, \pi\, \text{Cos}\left[2\pi \frac{x}{4\,\text{mm}}\right] = 0.0125664\, \text{Cos}\left[\pi \frac{x}{2\,\text{mm}}\right]$$

Note that the phase modulation is very small so a 1 meter propagation will probably change the phase a very small amount.

b)

Let $m = 4\times 10^{-3}\,\pi$ and $f_o = \frac{1}{4\,\text{mm}}$ then the wavefront can be written as

$$e^{i\,m\,\text{Cos}[2\,\pi\,f_o\,x]} \approx 1 + i\,m\,\text{Cos}[2\,\pi\,f_o\,x]$$

As the beams propagate the phase of the various plane wave spectrum components will go as

$$\frac{2\,\pi}{\lambda}\,(z\,\text{Cos}[\theta]) \approx \frac{2\,\pi}{\lambda}\,z\left(1 - \frac{\theta^2}{2}\right)$$

Since

$$\theta = \lambda\,f_o$$

the phase will go as

$$\frac{2\,\pi}{\lambda}\,z - \frac{2\,\pi}{\lambda}\,z\,\frac{(\lambda\,f_o)^2}{2}$$

$$\frac{2\,\pi\,z}{\lambda} - \pi\,z\,\lambda\,f_o^2$$

It is interesting to calculate $\pi\,z\,\lambda\,f_o^2$ for z = 1 m

$$\pi\,z\,\lambda\,f_o^2\ /.\ \left\{f_o \to \frac{1}{4\,\text{mm}},\ \lambda \to 0.5\times 10^{-3}\,\text{mm},\ z \to 10^3\,\text{mm}\right\}$$

0.0981748

After propagating a distance z the amplitude is given by

$$U[x,\,y,\,z] = e^{i\,k\,z}\left(1 + i\,m\,e^{-i\,\pi\,\lambda\,z\,f_o^2}\,\text{Cos}[2\,\pi\,f_o\,x]\right)$$

Thus

$$U[x,\,y,\,z] \approx e^{i\,k\,z}\,e^{i\,m\,e^{-i\,\pi\,\lambda\,z\,f_o^2}\,\text{Cos}[2\,\pi\,f_o\,x]}$$

This can be written as

$$U[x,\,y,\,z] \approx e^{i\,k\,z}\,e^{m\,\text{Cos}[2\,\pi\,x\,f_o]\,(i\,\text{Cos}[\pi\,z\,\lambda\,f_o^2] + \text{Sin}[\pi\,z\,\lambda\,f_o^2])}$$

This can be further reduced to

$$U[x,\,y,\,z] \approx e^{i\,k\,z}\,e^{m\,\text{Sin}[\pi\,z\,\lambda\,f_o^2]\,\text{Cos}[2\,\pi\,x\,f_o]}\,e^{i\,m\,\text{Cos}[\pi\,z\,\lambda\,f_o^2]\,\text{Cos}[2\,\pi\,x\,f_o]}$$

The phase variation is

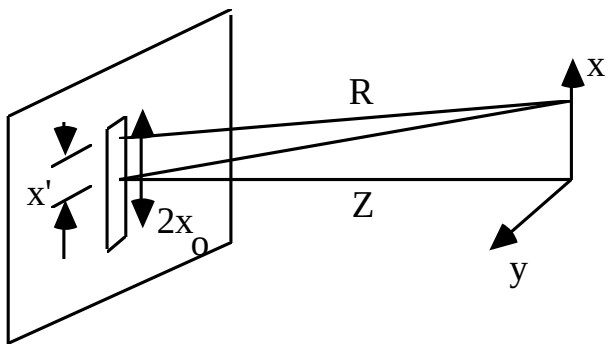
```
phaseVariation =
  m Cos[π z λ f_o^2] Cos[2 π x f_o] /. {m -> 4×10^-3 π, f_o -> 1/(4 mm), λ -> 0.5×10^-3 mm, z -> 10^3 mm}
```

$$0.0125059\,\text{Cos}\left[\frac{\pi\,x}{2\,\text{mm}}\right]$$

The propagation has changed the phase very little.

D - 23

An experimenter is studying Fraunhofer diffraction by a slit using the setup shown in the figure below. The slit height 2 x_o is 1 mm; the wavelength is 500 nm. How large does z have to be if the maximum value of the neglected x'^2 term in the phase k R is to be 0.1 radians.



Solution

We demand that the term $\frac{1}{2}\,\frac{k\,x'^2}{z}$ be less than 0.1

```
Solve[1/2 (2 π x'^2)/(λ z) == 0.1, z] /. {x' -> 0.5×10^-3 m, λ -> 0.5×10^-6 m}
```

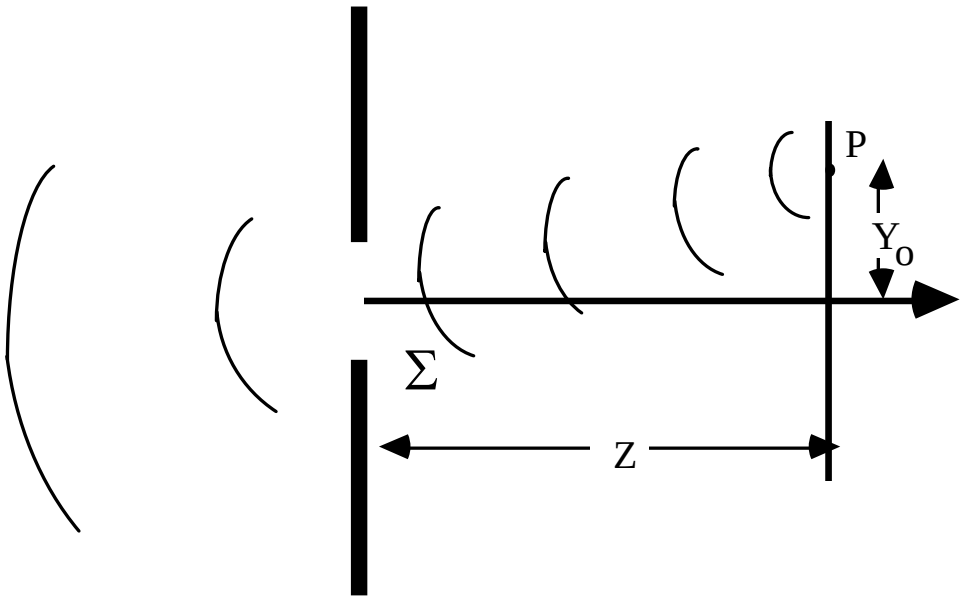
{{z → 15.708 m}}

D - 24

An aperture Σ in an opaque screen is illuminated by a spherical wave converging toward a point P located in a parallel plane a distance Z behind the aperture, as shown in the figure below.

a) Find a quadratic approximation to the illuminating wavefront in the plane of the aperture, assuming that (1) P lies on the z axis, and (2) P lies at the coordinates (0, Y_o).

b) Assuming Fresnel diffraction from the plane of the aperture to the observation plane containing P, show that in both of the above cases the observed irradiance distribution is the Fraunhofer diffraction pattern of the aperture, centered on the point P.



Solution

- **a)**

Illuminating wave is

$$\frac{a\, e^{-i k r_{01}}}{r_{01}}$$

$$r_{01} = \sqrt{\left(z^2 + (x_o - x_1)^2 + (y_o - y_1)^2\right)} \approx z + \frac{(x_o - x_1)^2}{2 z} + \frac{(y_o - y_1)^2}{2 z}$$

For $x_o = y_o = 0$

$$u[x_1, y_1] = \frac{a\, e^{-i k z}}{z}\, e^{-\frac{i k}{2 z}\left(x_1^2 + y_1^2\right)}$$

Let $x_o = 0$, $y_o = Y_o$

$$u[x_1, y_1] = \frac{a\, e^{-i k z}}{z}\, e^{-\frac{i k}{2 z}\left(x_1^2 + (Y_o - y_1)^2\right)} = \frac{a\, e^{-i k z}}{z}\, e^{-\frac{i k}{2 z}\left(x_1^2 + y_1^2 - 2 y_1 Y_o + Y_o^2\right)}$$

- **b)**

$$u[x_o, y_o] = \frac{e^{i k z}}{i \lambda z}\, e^{\frac{i k}{2 z}\left(x_o^2 + y_o^2\right)} \int_{-\infty}^{\infty} u[x_1, y_1]\, e^{\frac{i k}{2 z}\left(x_1^2 + y_1^2\right)}\, e^{-\frac{i k}{z}\left(x_o x_1 + y_1 y_o\right)}\, dx_1\, dy_1$$

If we illuminate the aperture with the spherical wave given in part a we get

$$u[x_o, y_o] = \frac{a\, e^{\frac{i k}{2 z}\left(x_o^2 + y_o^2\right)}}{i \lambda z^2} \int_{\Sigma} e^{-\frac{i k}{2 z} Y_o^2}\, e^{-\frac{i k}{z}\left(x_o x_1 + (y_o - Y_o) y_1\right)}\, dx_1\, dy_1$$

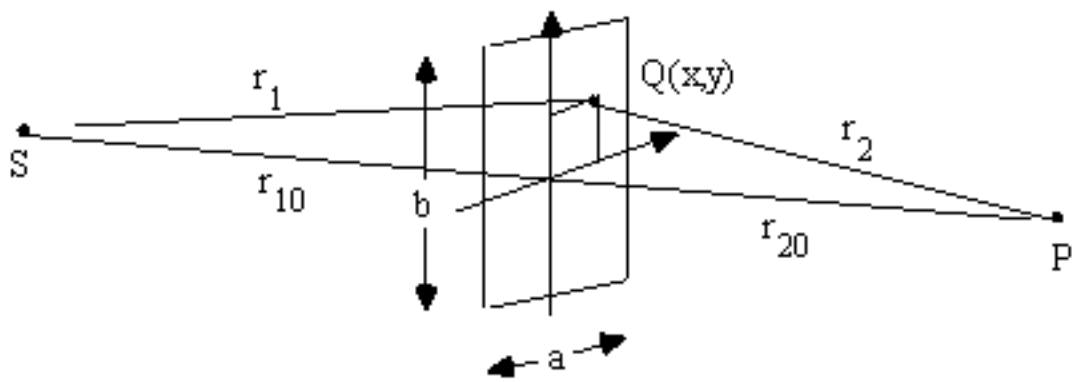
If T represents the Fourier transform of the aperture

$$I[x_o, y_o] = \left(\frac{a}{\lambda z^2}\right)^2 \text{Abs}\left[T\left[\frac{x_o}{\lambda z}, \frac{(y_o - Y_o)}{\lambda z}\right]\right]^2$$

which is a scaled by $\frac{1}{z^2}$ version of the Fraunhofer diffraction pattern of the aperture centered at $(x_o, y_o) = (0, Y_o)$.

D - 25

Consider the rectangular aperture shown below. Let a, b, the wavelength, and the distance from the source to the observing screen be a constant. Given a value of r_{10}, there is in general one other value for which the diffraction pattern is essentially the same except for size. What is the relation between these two distances, and what is the relation between the sizes of the pattern?



Solution

We can select a coordinate system where the source point and observation point is on axis. Letting $z_1 = r_{10}$ and $z_2 = r_{20}$ the disturbance at the observation point is

$$u[p] = \left(\frac{a}{i \lambda z_1 z_2}\right) e^{i k (z_1 + z_2)} \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} e^{i \frac{\pi}{\lambda} (\xi^2 + \eta^2) \left(\frac{1}{z_1} + \frac{1}{z_2}\right)} d\xi \, d\eta$$

This is symmetrical with respect to z_1 and z_2. Thus z_1 can be the distance from the source to the aperture and z_2 can be the distance from the aperture to the observation point, or z_2 can be the distance from the source to the aperture and z_1 can be the distance from the aperture to the observation point.

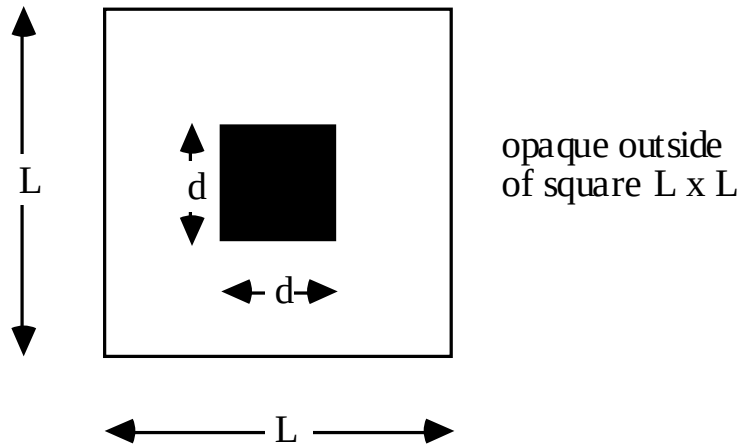
By moving the aperture sideways to sweep out the diffraction pattern we see that the size of the diffraction pattern goes as

```
(sourceApertureDistance + apertureObservationPointDistance) /
 sourceApertureDistance
```

However, the numerator above is constant, so the size of the pattern is inversely proportional to the distance from the source to the aperture.

D - 26

Find an expression for the intensity distribution in the Fraunhofer diffraction pattern of the aperture shown below. Assume unit-amplitude, normally incident plane-wave illumination.



Solution

Use Babinet's Principle.

$$i[\theta, \phi] = \left(\frac{L^2}{\lambda z}\frac{\text{Sin}[\beta_L]}{\beta_L}\frac{\text{Sin}[\alpha_L]}{\alpha_L} - \frac{d^2}{\lambda z}\frac{\text{Sin}[\beta_d]}{\beta_d}\frac{\text{Sin}[\alpha_d]}{\alpha_d}\right)^2$$
$$= \left(\frac{1}{\lambda z}\right)^2 \left(L^4\,\text{Sinc}[\beta_L, \alpha_L]^2 + d^4\,\text{Sinc}[\beta_d, \alpha_d]^2 - 2\,L^2 d^2\,\text{Sinc}[\beta_L, \alpha_L]\,\text{Sinc}[\beta_d, \alpha_d]\right)$$

$$\beta_L = \frac{\pi}{\lambda} L\,\text{Sin}[\phi]; \qquad \alpha_L = \frac{\pi}{\lambda} L\,\text{Sin}[\theta];$$
$$\beta_d = \frac{\pi}{\lambda} d\,\text{Sin}[\phi]; \qquad \alpha_d = \frac{\pi}{\lambda} d\,\text{Sin}[\theta];$$

D - 27

A Fraunhofer diffraction pattern is formed with a plane diffracting screen having an aperture of any shape. Let the incident beam be normal to the screen, and let the coordinates at any point p on the rim of the aperture be (x_p, y_p). Show that if the aperture is deformed by changing x_p to hx_p for all points p, where h is a constant (so for example, a circle becomes an ellipse), then the irradiance at (x_o, y_o) after the deformation is h^2 times that at (hx_o, y_o) before the deformation.

Solution

$$i[x_o, y_o] = \text{Abs}\left[\frac{1}{\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i[x, y]\, e^{-i k \frac{x x_o}{z}}\, e^{-i k \frac{y y_o}{z}}\, dx\, dy\right]^2$$

We will now deform the aperture so x goes to h x.

$$i_h[x_o, y_o] = \text{Abs}\left[\frac{1}{\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i\left[\frac{x}{h}, y\right] e^{-i k \frac{x x_o}{z}}\, e^{-i k \frac{y y_o}{z}}\, dx\, dy\right]^2$$

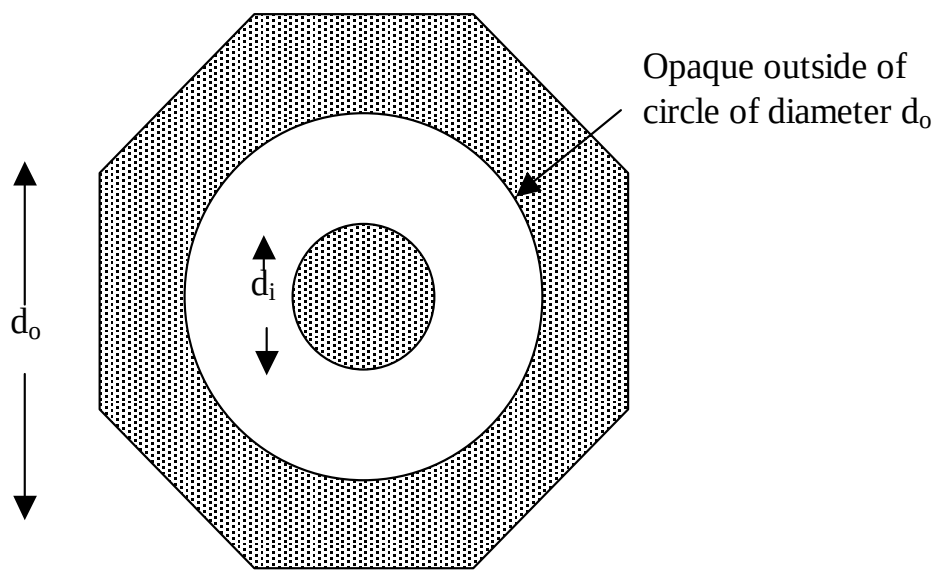
Let xnew = $\frac{x}{h}$.

$$i_h[x_o, y_o] = \text{Abs}\left[\frac{1}{\lambda z}\, h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_i[\text{xnew}, y]\, e^{-i k \frac{\text{xnew}\,(h x_o)}{z}}\, e^{-i k \frac{y y_o}{z}}\, d\text{xnew}\, dy\right]\text{^}2$$

Thus the irradiance at (x_o, y_o) after the deformation is h^2 times that at (hx_o, y_o) before the deformation. This makes sense since increasing an aperture dimension by a factor of h increases the total transmitted flux by a factor h and it concentrates that power into a diffraction pattern that is narrower by an additional factor of h.

D - 28

a) Find an expression for the irradiance distribution in the Fraunhofer diffraction pattern of the aperture shown below. Assume unit-amplitude, normally incident plane-wave illumination of wavelength λ. The diffraction pattern is observed in the focal plane of a lens having focal length f. The aperture is circular and has a circular central obscuration of diameter d_i. The region outside the diameter d_o circular region is opaque.



b) Repeat part a for the situation where the inner circular aperture of diameter d_i is transparent, but introduces a 180 degree phase change to the light transmitted through it.

Solution

- **a)**

Use Babinet's principle.

$$i = \frac{1}{4 r^2} \left(d_o J_1\left[\frac{2\pi}{\lambda} \frac{d_o}{2 f} r \right] - d_i J_1\left[\frac{2\pi}{\lambda} \frac{d_i}{2 f} r \right] \right)^2$$

- **b)**

Use Babinet's principle again, but now center is 180° out of phase.

$$i = \frac{1}{4 r^2} \left(d_o J_1\left[\frac{2\pi}{\lambda} \frac{d_o}{2 f} r \right] - 2 d_i J_1\left[\frac{2\pi}{\lambda} \frac{d_i}{2 f} r \right] \right)^2$$

D - 29

Two telescopes are being used to image a star too small to be resolved by either system. One telescope has a circular aperture of diameter W, while the second has a square aperture of width W. Give the relative on-axis irradiance for the two telescopes.

Solution

On-axis irradiance goes as area of aperture squared.

$$\frac{\text{On-axis irradiance circular aperture}}{\text{On-axis irradiance square aperture}} = \frac{(\pi (w/2)^2)^2}{(w^2)^2} = \frac{\pi^2}{16}$$

D - 30

A circular aperture yields a Fraunhofer diffraction pattern with the first dark ring of radius r_1. Give an explanation as to whether the radius of the first dark ring increases or decreases when a circular central obscuration is inserted inside the circular aperture?

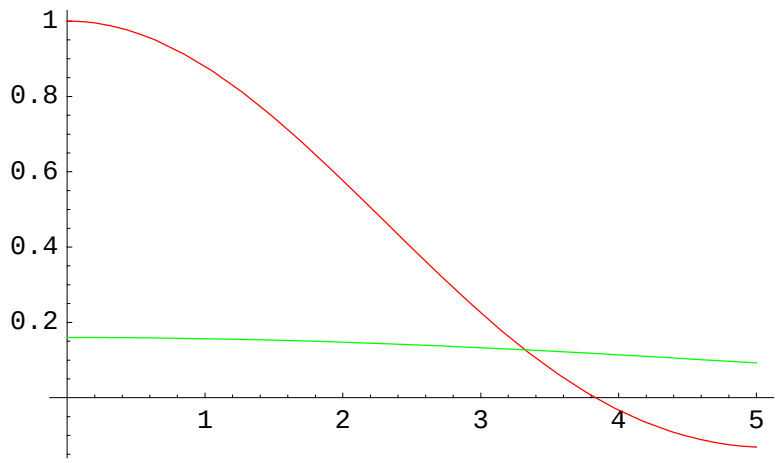
Solution

We will use Babinet's principle.

The resulting amplitude will be the difference between the amplitude for the entire aperture and the amplitude for the central obscuration portion. Since the diffraction pattern for the smaller aperture will have a lower peak and it will be more spread out than the diffraction pattern for the larger aperture, the plot for the smaller aperture will cross the plot of the larger aperture before the plot for the larger aperture goes to zero. Since we are finding the difference between the two amplitudes, the radius of the first dark ring will decrease when a circular central obscuration is inserted inside the circular aperture.

See the plots below for an obscuration ratio of 0.4.

```
Plot[{2 BesselJ[1, x]/x, 2 (0.4) BesselJ[1, x (0.4)]/x},
  {x, 0, 5}, PlotStyle -> {Red, Green}, Background -> White];
```



D - 31

An aberration free lens having a focal length f and a square aperture of width d is used to focus a collimated beam of coherent light. A phase mask is placed across the aperture of the lens to cause the light in one-half the aperture to be out of phase 120 degrees with respect to the other half. Derive an expression for the resultant irradiance in the focal plane of the lens.

Solution

We can almost guess the answer. The irradiance will have a Sinc^2 in y. In x we will also get a Sinc^2 from each half of the aperture. The Sinc^2 in x will be twice as wide as the Sinc^2 in y. The two halves of the aperture will interfere to gives us a Cos^2 function. There will be a phase factor in the Cos function due to the 120 degree phase difference.

The amplitude of the disturbance is given by

$$\mathbf{u = FullSimplify}\left[c \int_{\frac{-d}{2}}^{0} \int_{-d/2}^{d/2} e^{-i k y \frac{\eta}{f}} e^{-i k x \frac{\xi}{f}} \, d\eta \, d\xi + e^{i \frac{2\pi}{3}} c \int_{0}^{\frac{d}{2}} \int_{-d/2}^{d/2} e^{-i k y \frac{\eta}{f}} e^{-i k x \frac{\xi}{f}} \, d\eta \, d\xi\right]$$

$$-\frac{1}{k^2 x y} \left(c\, e^{-\frac{i d k (x+y)}{2 f}} \left(-1 + e^{\frac{i d k x}{2 f}}\right) \left((-1)^{2/3} + e^{\frac{i d k x}{2 f}}\right) \left(-1 + e^{\frac{i d k y}{f}}\right) f^2\right)$$

```
irradiance = FullSimplify[u Conjugate[u], {d, k, f, x, y, c} ∈ Reals]
```

$$\frac{1}{k^4 x^2 y^2} \left(16 c^2 f^4 \operatorname{Sin}\left[\frac{d k x}{4 f}\right]^2 \left(2 - \operatorname{Cos}\left[\frac{d k x}{2 f}\right] + \sqrt{3} \operatorname{Sin}\left[\frac{d k x}{2 f}\right]\right) \operatorname{Sin}\left[\frac{d k y}{2 f}\right]^2\right)$$

But

$$\mathbf{FullSimplify}\left[\mathbf{TrigExpand}\left[4 \operatorname{Cos}\left[\frac{\pi}{3} - \frac{d k x}{4 f}\right]^2\right]\right]$$

$$2 - \operatorname{Cos}\left[\frac{d k x}{2 f}\right] + \sqrt{3} \operatorname{Sin}\left[\frac{d k x}{2 f}\right]$$

Therefore,

$$\text{irradiance} = \frac{64 c^2 f^4 \operatorname{Sin}\left[\frac{d k x}{4 f}\right]^2 \left(\operatorname{Cos}\left[\frac{\pi}{3} - \frac{d k x}{4 f}\right]^2\right) \operatorname{Sin}\left[\frac{d k y}{2 f}\right]^2}{k^4 x^2 y^2}$$

and

$$i = i_0 \operatorname{Cos}\left[\frac{\pi}{3} - \frac{d k x}{4 f}\right]^2 \frac{\operatorname{Sin}\left[\frac{d k x}{4 f}\right]^2}{\left(\frac{d k x}{4 f}\right)^2} \frac{\operatorname{Sin}\left[\frac{d k y}{2 f}\right]^2}{\left(\frac{d k y}{2 f}\right)^2};$$

D - 32

Imagine that you are looking through a piece of square woven cloth at a point source of wavelength 600 nm a distance of 20 meters away. If you see a square arrangement of bright spots located about the point source, each separated by an apparent nearest-neighbor distance of 12 cm, how close together are the strands of cloth?

Solution

For small angles the grating equation is written as $d\,\theta = \lambda$.

$$\mathbf{d} = \frac{\lambda}{\theta} = \frac{0.6\ \mu\mathrm{m}}{\frac{12}{20 \times 10^2}} = 100\ \mu\mathrm{m}$$

D - 33

The major difference in the approximations for Fresnel and Fraunhofer diffraction can be thought of in terms of the shape of the waves leaving the aperture. What is the shape of the waves leaving the aperture for

a) Fresnel diffraction?

b) Fraunhofer diffraction?

Solution

- **a)**

Parabolic wavefronts.

- **b)**

Plane waves.

D - 34

I am looking through a window screen made up of a square wire mesh of wires at a point source 10 meters away. If the mesh has wires 0.2 mm apart and the wavelength is 500 nm, what is the spacing of the square arrangement of bright spots that I see located about the point source? What determines the relative intensities of the bright spots?

Solution

We know that if

```
d = 0.2 mm;  L = 10^4 mm;
```

then

$$d \frac{\text{spotSeparation}}{L} = \lambda$$

```
spotSeparation = L/d λ /. λ → 500 10^-6 mm
25. mm
```

Relative intensities determined by duty cycle of the grating.

D - 35

The Fraunhofer diffraction pattern of an aperture has the condition that I(x,y) ≠ I(-x,-y). What do we know about the aperture?

Solution

There is a phase variation across the pupil such that $\phi[x,y] \neq \phi[-x,-y]$. A simple example is to think of wavefront tilt across the pupil.

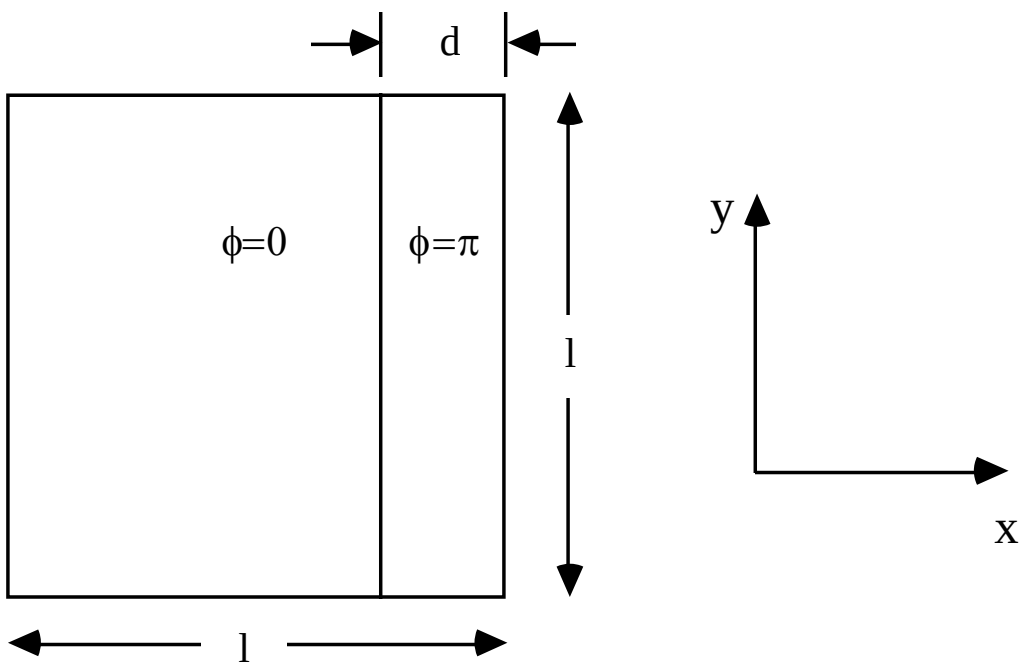
D - 36

An optical system with the square aperture shown below is used to image an incoherently illuminated object. The pupil function has a 180° phase step as shown in the figure.

a) If l = 0.5 meter, d = 0.1 meter, and the wavelength is 500 nm, what is the cutoff frequency in units of lines/radian in the x and y directions?

b) Give equations for the MTF as a function of spatial frequency in the x and y directions.

c) Sketch the MTF in the x and y directions.



Solution

a)

```
length = 0.5;  d = 0.1;  λ = 500 10^-9;
```

$$\nu cutoff = \frac{length}{\lambda}$$

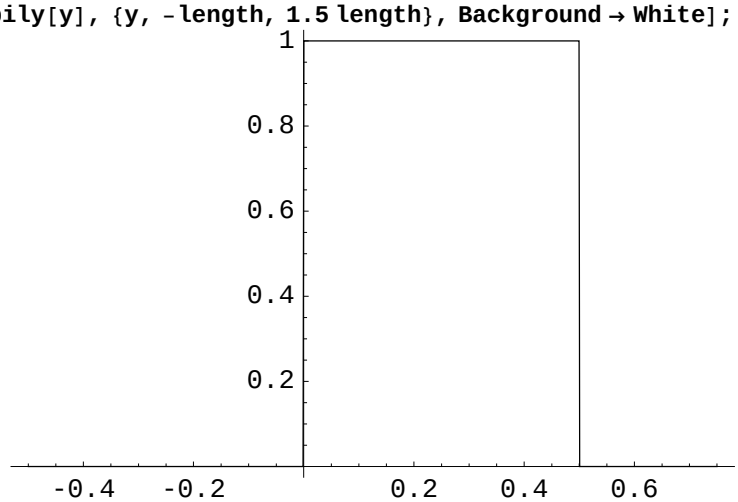
$1. \times 10^{6}$

b)

y-direction

```
pupily[y_ /; y < 0 ] := 0
pupily[y_ /; y ≥ 0 && y ≤ length] := 1
pupily[y_ /; y > length ] := 0
```

```
Plot[pupily[y], {y, -length, 1.5 length}, Background → White];
```



$$mtfy[\nu\_] := Abs\left[\int_0^{length} pupily[y]\ Conjugate\left[pupily\left[y - \frac{length}{\nu cutoff}\ \nu\right]\right] dy\right] \Big/ Abs\left[\int_0^{length} pupily[y]\ Conjugate[pupily[y]]\ dy\right]$$

Or we could write

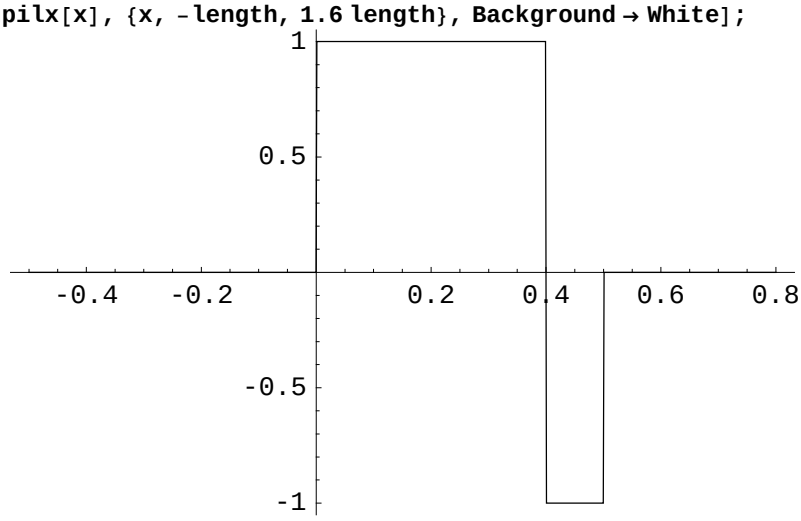
$$mtfy[\nu\_ /; Abs[\nu] \le \nu cutoff] := 1 - Abs\left[\frac{\nu}{\nu cutoff}\right]$$

```
mtfy[ν_ /; Abs[ν] > νcutoff] := 0
```

x-direction

```
pupilx[x_ /; x < 0] := 0
pupilx[x_ /; x ≥ 0 && x ≤ length - d] := 1
pupilx[x_ /; x > length - d && x ≤ length] := e^(i π)
pupilx[x_ /; x > length] := 0
```

```
Plot[pupilx[x], {x, -length, 1.6 length}, Background → White];
```



$$\text{mtfx}[\nu\_] := \text{Abs}\left[\int_0^{\text{length}} \text{pupilx}[x]\ \text{Conjugate}\left[\text{pupilx}\left[x - \frac{\text{length}}{\nu\text{cutoff}}\ \nu\right]\right] dx\right] \Big/ \text{Abs}\left[\int_0^{\text{length}} \text{pupilx}[x]\ \text{Conjugate}[\text{pupilx}[x]]\ dx\right]$$

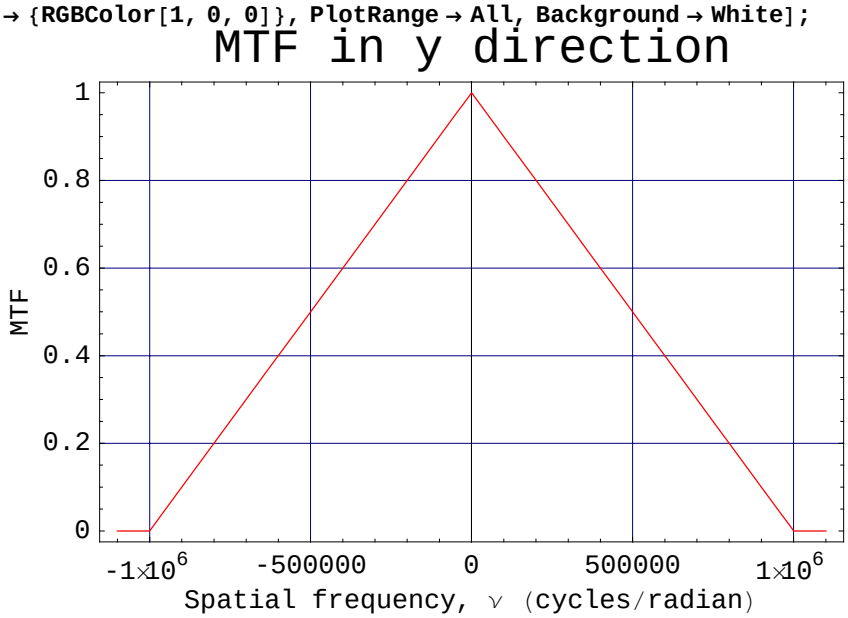
■ c

We will turn off some warning messages.

```
Off[NIntegrate::ploss, NIntegrate::ncvb, NIntegrate::slwcon]
```

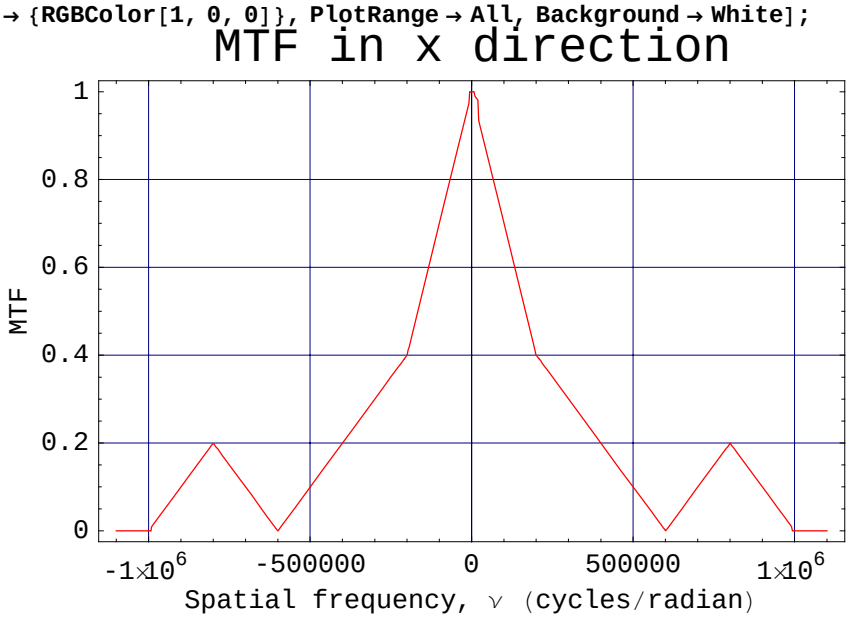
y-direction

```
Plot[mtfy[ν], {ν, -1.1 νcutoff, 1.1 νcutoff}, Frame → True,
  PlotLabel → StyleForm["MTF in y direction", FontSize → 18, FontWeight → Bold],
  FrameLabel → {"Spatial frequency, ν (cycles/radian)", "MTF"}, GridLines → Automatic,
  PlotStyle → {RGBColor[1, 0, 0]}, PlotRange → All, Background → White];
```



x-direction

```
Plot[mtfx[ν], {ν, -1.1 νcutoff, 1.1 νcutoff}, Frame → True,
  PlotLabel → StyleForm["MTF in x direction", FontSize → 18, FontWeight → Bold],
  FrameLabel → {"Spatial frequency, ν (cycles/radian)", "MTF"}, GridLines → Automatic,
  PlotStyle → {RGBColor[1, 0, 0]}, PlotRange → All, Background → White];
```



D - 37

A 20 mm diameter lens having a 200 mm focal length operating at a wavelength of 500 nm is used to image a target. Assuming no aberration, what is the cutoff frequency of the modulation transfer function in image space if

a) the target is at infinity?

b) the target is 400 mm from the lens?

c) repeat a and b for the case where the lens has 2 waves of third-order spherical aberration.

Solution

For incoherent light

$$f_c = \frac{1}{\lambda f^{\#}}$$

- **a)**

$$f^{\#} = \frac{200}{20} = 10; \; f_c = \frac{1}{.5 \times 10^{-3}\,\text{mm}\,(10)} = \frac{200}{\text{mm}}$$

- **b)**

$$f^{\#} = \frac{400}{20} = 20; \; f_c = \frac{1}{.5 \times 10^{-3}\,\text{mm}\,(20)} = \frac{100}{\text{mm}}$$

- **c**

Cutoff frequency does not change.

D - 38

An incoherent light source of approximately 500 nm wavelength is used with a 10 mm diameter, 200 mm focal length lens. The lens images a sinusoidal test target that is located hundreds of meters from the lens.

a) What is the modulation transfer function cutoff frequency in units of lines/radian in object space?

b) What is the modulation transfer function cutoff frequency in units of lines/mm in image space?

c) Repeat part a and b for a 5 mm diameter, 100 mm focal length lens?

d) What do we know about the wavefront aberration if the phase of the OTF is directly proportional to spatial frequency and the modulus of the OTF is equal to the unaberrated value.

Solution

$$\nu_{cutoff} = \frac{1}{\lambda f^{\#}} = \frac{d}{\lambda z}$$

- **a)**

$$\frac{d}{\lambda} = \frac{10 \times 10^3\ \mu m}{.5\ \mu m} = 20000\ \text{lines / radian}$$

- **b)**

$$\frac{d}{\lambda z} = \frac{10 \times 10^3\ \mu m}{.5\ \mu m\ (200\ mm)} = \frac{100}{mm}$$

- **c)**

$$\frac{d}{\lambda} = \frac{5 \times 10^3\ \mu m}{.5\ \mu m} = 10000\ \text{lines / radian}$$

$$\frac{d}{\lambda z} = \frac{5 \times 10^3\ \mu m}{.5\ \mu m\ (100\ mm)} = \frac{100}{mm}$$

- **d)**

Wavefront tilt.

D - 39

A lens is used to image an incoherently illuminated sinusoidal amplitude diffraction grating whose transmission function is given by a(1 + 0.5 sin[2π x/(2 mm)]). Take the wavelength of the illuminating light to be 500 nm and the distance between the lens and the grating be 25 meters. What must be the approximate diameter of the lens if the image has a contrast of 25 % and

a) the lens has a focal length of 10 cm?

b) the lens has a focal length of 20 cm?

If you want you can approximate the MTF of the circular lens as the MTF of a square lens.

Solution

The original object has a contrast of 0.5. Therefore, the MTF must be 0.5 to obtain an image having a contrast of 25%. If we approximate the MTF of the circular lens as the MTF of a square lens, then if we want a MTF of 0.5 we are operating at a spatial frequency equal to $\frac{1}{2}$ the cutoff frequency. The frequency of the object is $\frac{1}{2\,\text{mm}}$, so the cutoff frequency is $\frac{1}{\text{mm}}$.

$$\nu_c = \frac{1}{\lambda \frac{z}{d}}; \quad d = \lambda z \nu_c$$

- **a)**

$$d = (0.5 \times 10^{-3}\,\text{mm})\,(25 \times 10^{3}\,\text{mm})\left(\frac{1}{\text{mm}}\right)$$

12.5 mm

- **b)**

The diameter is independent of the focal length so the diameter is 12.5 mm.

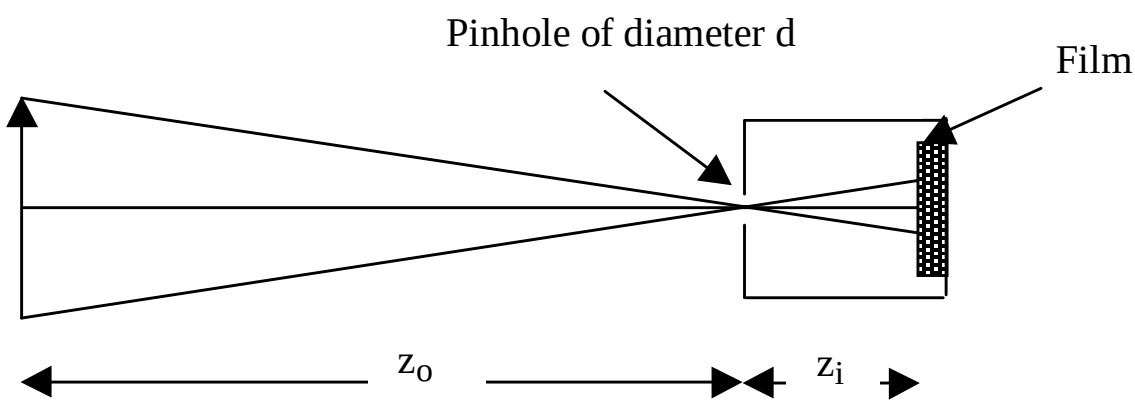
D - 40

Consider the pinhole camera shown below. Assume the object is incoherent and nearly monochromatic, the distance z_o from the object is so large that it can be treated as infinite, and the pinhole is circular with diameter d.

a) Under the assumption that the pinhole is large enough to allow a purely geometrical-optics estimation of the point-spread function, find the optical transfer function of this camera. If we define the "cutoff frequency" of the camera to be the frequency where the first zero of the OTF occurs, what is the cutoff frequency under the above geometrical-optics approximation?

b) Again calculate the cutoff frequency, but this time assuming that the pinhole is so small that Fraunhofer diffraction by the pinhole governs the shape of the point-spread function.

c) Considering the above, estimate the optimum size of the pinhole in terms of the various parameters of the system.



Solution

The Optical Transfer function is the Fourier transform of the point spread function.

- **a)**

PSF is uniform circular spot of diameter d. The OTF is given as the Fourier transform of the PSF, or

$$\text{otf}[\nu] = \frac{2\, J_1[\pi\, d\, \nu]}{\pi\, d\, \nu}$$

$$\nu_{\text{cutoff}} = \frac{1.22}{d}$$

- **b)**

$$\nu_{\text{cutoff}} = \frac{d}{\lambda\, z_i}$$

This is the normal incoherent ν_{cutoff} for aperture of diameter d and focal distance z_i.

- **c)**

There are two approaches to solving this problem. One approach is to have the geometrical blur equal the diffraction blur. The second approach is to have the cutoff frequency for the geometrical case equal the cutoff frequency for the diffraction case.

Geometrical blur equal to diffraction blur.

$$d \approx 2.44\, \frac{\lambda\, z_i}{d} \quad \text{or} \quad d \approx \sqrt{2.44\, \lambda\, z_i}$$

Geometrical cutoff frequency equal to diffraction cutoff frequency.

$$\frac{1.22}{d} \approx \frac{d}{\lambda\, z_i} \quad \text{or } d \approx \sqrt{1.22\, \lambda\, z_i}$$

As can be seen, the two approaches give approximately the same result.

D - 41

Assume that the radius of curvature and the width of a Gaussian beam of wavelength λ = 1 μm at some point on the beam axis are R_1 = 1 m and W_1 = 1 mm, respectively. Determine the beam width and the radius of curvature at a distance d = 10 cm to the right.

Solution

```
R1 = 1 m; λ = 10^-6 m; w1 = 10^-3 m;
```

$$w := wo\sqrt{1 + \left(\frac{z}{zo}\right)^2}\ ;\ R := z\left(1 + \left(\frac{zo}{z}\right)^2\right);$$

So we need to find z, zo, and wo.

$$z = \frac{R1}{1 + \left(\frac{\lambda R1}{\pi w1^2}\right)^2};$$

$$wo = \frac{w1}{\sqrt{1 + \left(\frac{\pi w1^2}{\lambda R1}\right)^2}};$$

$$zo = \frac{\pi}{\lambda} wo^2;$$

```
R // N
1. m

w // N
0.001 m
```

D - 42

Assuming that the width W and the radius of curvature R of a Gaussian beam are known at some point on the beam axis, show that the beam waist is located at a distance

$$z = \frac{R}{\left(1 + \left(\frac{\lambda R}{\pi w^2}\right)^2\right)}$$

to the left and the waist radius is

$$wo = \frac{w}{\sqrt{1 + \left(\frac{\pi w^2}{\lambda R}\right)^2}}$$

Solution

We will start with the three equations

$$w = wo\sqrt{1 + \left(\frac{z}{zo}\right)^2}; \quad R = z\left(1 + \left(\frac{zo}{z}\right)^2\right); \quad wo = \sqrt{\frac{\lambda\, zo}{\pi}};$$

Hence

$$wo = \frac{w}{\sqrt{1 + \left(\frac{z}{zo}\right)^2}} \quad \text{and} \quad R = \frac{1}{z}\left(z^2 + zo^2\right)$$

But

$$w^2 = \frac{\lambda\, zo}{\pi}\left(1 + \frac{z^2}{zo^2}\right) = \frac{\lambda}{\pi\, zo}\left(z^2 + zo^2\right)$$

Thus, $\left(z^2 + zo^2\right) = zo\,\frac{\pi w^2}{\lambda}$

$$R = \frac{zo}{z}\,\frac{\pi w^2}{\lambda} \quad \text{or} \quad \frac{z}{zo} = \frac{\pi w^2}{\lambda R}$$

so

$$\boxed{wo = \frac{w}{\sqrt{1 + \left(\frac{\pi w^2}{\lambda R}\right)^2}}}$$

From above

$$z = \frac{R}{\left(1 + \left(\frac{zo}{z}\right)^2\right)}$$

so

$$\boxed{z = \frac{R}{\left(1 + \left(\frac{\lambda R}{\pi w^2}\right)^2\right)}}$$