3.0 Wave Equation

In deriving the wave equation we will first make use of the vector identity

$$\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

However, from Maxwell's equations $(\nabla \cdot \vec{E}) = 0$ so

$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} = -\mu_o \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu_o \frac{\partial}{\partial t} (\nabla \times \vec{H}) = -\mu_o \epsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \mu_o \epsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Letting $c = \sqrt{\frac{1}{\mu_o \epsilon_o}}$ yields the wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

In one dimension we can write

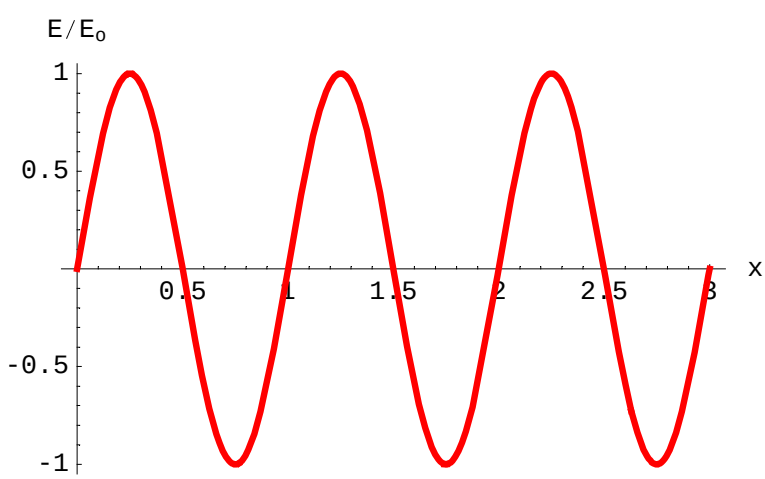
$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0$$

3.1 Transverse waves

One solution to the one-dimensional wave equation is

$$\vec{E}_y[x, t] = E_o \hat{j} \operatorname{Cos}[k x - \omega t]$$

which represents a plane wave travelling in the +x direction.



One way of showing this is a solution is to substitute the solution into the wave equation to see if it checks.

$$-E_o k^2 \text{Cos}[k x - \omega t] + \frac{\omega^2}{c^2} E_o \text{Cos}[k x - \omega t] = 0$$

This is a solution if

$$k = \frac{\omega}{c}$$

We will let $k = \frac{2\pi}{\lambda}$.

3.2 Plane Wave

A solution to the three-dimensional wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

is

$$\vec{E}[x, y, z; t] = \vec{E}_o \text{Cos}[\vec{k} \cdot \vec{r} - \omega t]$$

where the position vector is

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

and the propagation vector which gives the direction of propagation is given by

$$\vec{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$$

Then

$$\vec{k} \cdot \vec{r} - \omega t = k_x x + k_y y + k_z z - \omega t = \text{constant}$$

represents planes in space of constant phase. Hence we call this solution a plane wave.

3.3 Complex Representation

The use of the complex representation, rather than only the real representation, sometimes makes algebraic computations simpler. Thus, instead of using

$$\vec{E}[x, y, z; t] = \vec{E}_o \text{Cos}[\vec{k} \cdot \vec{r} - \omega t]$$

to represent a plane wave it is often easier to use

$$\vec{E}[x, y, z; t] = \vec{E}_o e^{i (\vec{k} \cdot \vec{r} - \omega t)}$$

While the complex expression is itself a solution to the wave equation, it needs to be understood that for physical reasons the real part is the actual quantity being represented.

3.4 Spherical Wave

$\text{Cos}\left[\vec{k}\cdot\vec{r}-\omega t\right]$ and $e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$ have constant have constant values on a sphere at any given radius r at a given time t, however they are not solutions to the wave equation. It can be shown (see homework problems) that $\frac{1}{r}\,\text{Cos}\left[\vec{k}\cdot\vec{r}-\omega t\right]$ and $\frac{1}{r}\,e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$ are solutions to the wave equation and they represent spherical waves propagating outward from the origin.

3.5 Linear Superposition

Since the electric field is a vector quantity a vector summation must be performed to find the electric field resulting from the summation of several electric fields.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

This linear superposition is only approximately true in the presence of matter. Deviations from linearity are observed at high intensities produced by lasers when the electric fields approach the electric fields comparable to atomic fields (non-linear optics). In these notes we will consider only situations where linear superposition is valid.

3.5.1 Sine waves having same frequency, but different amplitude and phase

- **3.5.1.1 Two Sine Waves (same polarization)**

$$E_1[x,\ t] = A_1\,\text{Cos}[k\,x - \omega\,t + \phi_1] = \text{Re}[A_1\,e^{i\,\phi_1}\,e^{i\,(k\,x-\omega\,t)}]$$

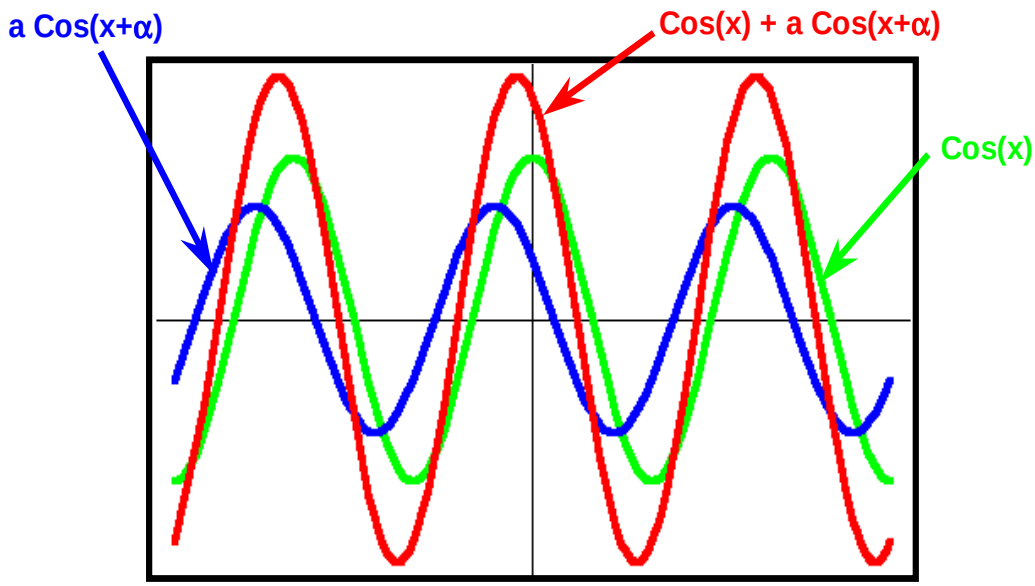
$$E_2[x,\ t] = A_2\,\text{Cos}[k\,x - \omega\,t + \phi_2] = \text{Re}[A_2\,e^{i\,\phi_2}\,e^{i\,(k\,x-\omega\,t)}]$$

From now on we will forget about taking the Real part until the end of the calculation.

$$\begin{aligned} E_o[x,\ t] &= E_1[x,\ t] + E_2[x,\ t] \\ &= \left(A_1\,e^{i\,\phi_1} + A_2\,e^{i\,\phi_2}\right)\left(e^{i\,(k\,x-\omega\,t)}\right) \\ &= A_o\,e^{i\,\phi_o}\,e^{i\,(k\,x-\omega\,t)} \end{aligned}$$

Superposition of two harmonic waves of given frequency produces a harmonic wave of same frequency with a given amplitude and phase.

Waves Having Same Frequency, but Different Amplitude and Phase



Now we will find the amplitude and phase.

$$A_1\, e^{i\,\phi_1} + A_2\, e^{i\,\phi_2} = A_o\, e^{i\,\phi_o}$$

$$A_1 \operatorname{Cos}[\phi_1] + A_2 \operatorname{Cos}[\phi_2] + i\,(A_1 \operatorname{Sin}[\phi_1] + A_2 \operatorname{Sin}[\phi_2]) = A_o \operatorname{Cos}[\phi_o] + i\, A_o \operatorname{Sin}[\phi_o]$$

Since the real and imaginary parts must be equal it follows that the phase is given by

$$\frac{A_o \operatorname{Sin}[\phi_o]}{A_o \operatorname{Cos}[\phi_o]} = \frac{A_1 \operatorname{Sin}[\phi_1] + A_2 \operatorname{Sin}[\phi_2]}{A_1 \operatorname{Cos}[\phi_1] + A_2 \operatorname{Cos}[\phi_2]} = \operatorname{Tan}[\phi_o]$$

Next we must calculate the amplitude.

$$\operatorname{Abs}[A_o\, e^{i\,\phi_o}]^2 = A_o^2 = \operatorname{Abs}[A_1\, e^{i\,\phi_1} + A_2\, e^{i\,\phi_2}]^2$$

This yields the basic equation of two-beam interference.

$$\boxed{A_o^2 = A_1^2 + A_2^2 + 2\, A_1\, A_2 \operatorname{Cos}[\phi_2 - \phi_1]}$$

If $A_1 = A_2$

$$A_o^2 = 2\, A_1^2\, (1 + \operatorname{Cos}[\phi_2 - \phi_1]) = 4\, A_1^2 \operatorname{Cos}\left[\frac{1}{2}\,(\phi_2 - \phi_1)\right]^2$$

Thus we can think of the linear combination giving a cosine intensity distribution, or equivalently a cosine squared distribution.

3.5.1.2 Combination of several sine waves

$$E[x, t] = \sum_{n=1}^{N} A_n \, e^{i \phi_n} \, e^{i (k x - \omega t)} = A_o \, e^{i \phi_o} \, e^{i (k x - \omega t)}$$

$$A_o \, e^{i \phi_o} = \sum_{n=1}^{N} A_n \, e^{i \phi_n}$$

In the same manner as shown above for two sine waves

$$\mathrm{Tan}[\phi_o] = \frac{\sum_{n=1}^{N} A_n \, \mathrm{Sin}[\phi_n]}{\sum_{n=1}^{N} A_n \, \mathrm{Cos}[\phi_n]}$$

Now we need to calculate the amplitude.

$$A_o^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} A_n \, e^{i \phi_n} \, A_m \, e^{-i \phi_m}$$

$$= \sum_{n=1}^{N} A_n^2 + \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} A_n \, A_m \, e^{i (\phi_n - \phi_m)}$$

$$= \sum_{n=1}^{N} A_n^2 + \sum_{n=1}^{N} \sum_{m=1, m < n}^{N} A_n \, A_m \left(e^{i (\phi_n - \phi_m)} + e^{-i (\phi_n - \phi_m)} \right)$$

$$= \sum_{n=1}^{N} A_n^2 + \sum_{n=1}^{N} \sum_{m=1, m < n}^{N} 2 \, A_n \, A_m \, \mathrm{Cos}[\phi_n - \phi_m]$$

This last term is the interference term.

If N is large and ϕ is random the sine waves are incoherent and the add with no apparent interference to yield

$$A_o^2 = \sum_{n=1}^{N} A_n^2$$

If all the A_n are equal

$$A_o^2 = N \, A_n^2$$

For the in-phase coherent case

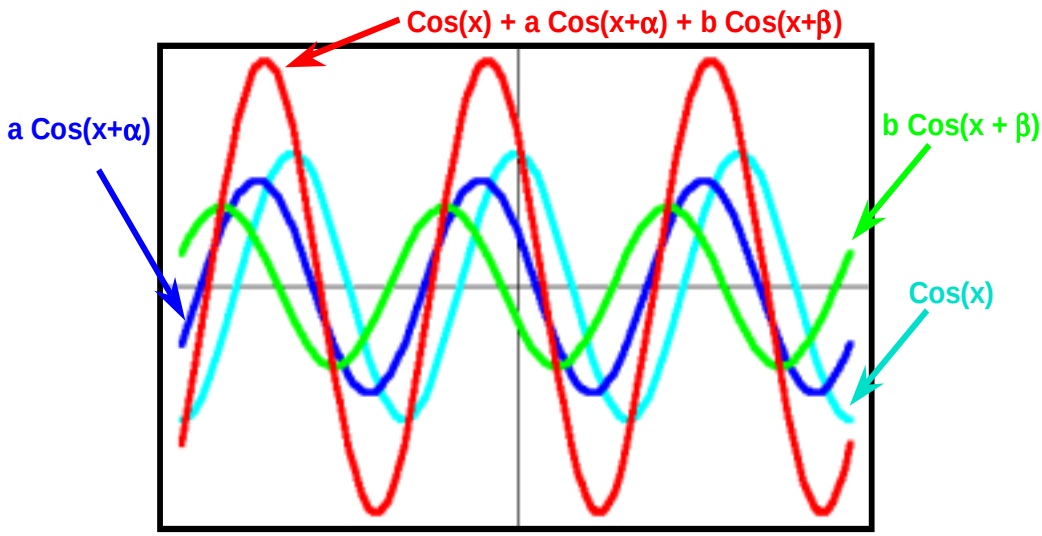
$$A_o^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} A_n \, A_m = \left(\sum_{n=1}^{N} A_n \right)^2$$

If all the A_n are equal

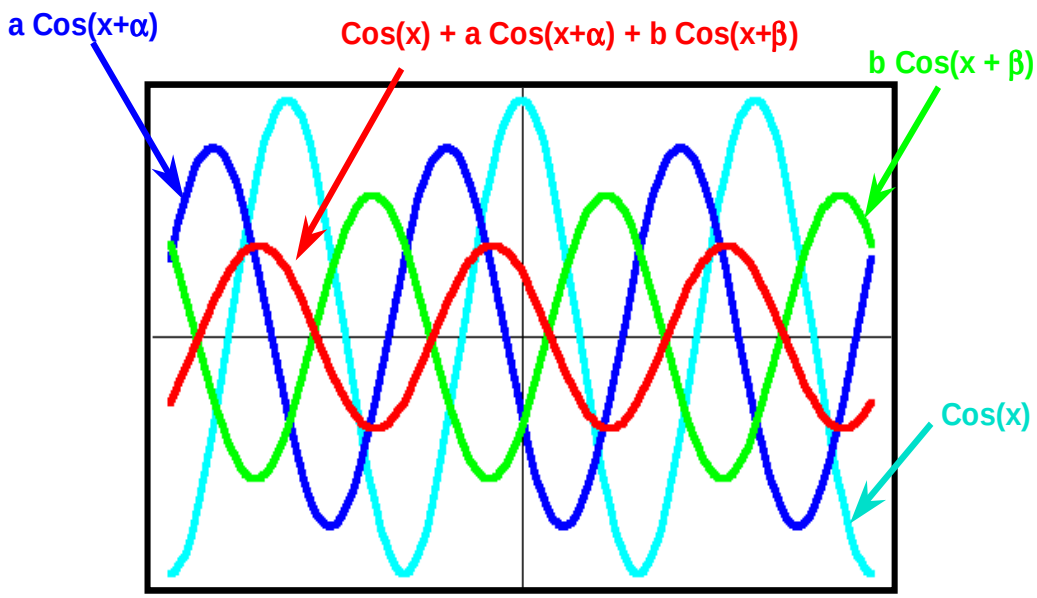
$$A_o^2 = N^2 \, A_n^2$$

Note that for the coherent and incoherent cases the total energy does not change, but the distribution of the energy does change.

Addition of 3 Waves of Given Frequency Gives Another Wave of Same Frequency



Resultant of Adding Waves Not Always Larger Than Individual Waves



3.5.2 Beats

We will now combine waves having different frequencies, but the same amplitude.

$$E_1[x, t] = A \operatorname{Cos}[k_1 x - \omega_1 t + \phi_1]$$

$$E_2[x, t] = A \operatorname{Cos}[k_2 x - \omega_2 t + \phi_2]$$

Adding the two waves yields

$$E = A \left(\operatorname{Cos}[k_1 x - \omega_1 t + \phi_1] + \operatorname{Cos}[k_2 x - \omega_2 t + \phi_2]\right)$$

But

$$\operatorname{Cos}[\alpha] + \operatorname{Cos}[\beta] = 2 \operatorname{Cos}\left[\frac{1}{2}(\alpha + \beta)\right] \operatorname{Cos}\left[\frac{1}{2}(\alpha - \beta)\right]$$

so

$$E = 2 A \operatorname{Cos}\left[\frac{1}{2}\left((k_1 + k_2) x - (\omega_1 + \omega_2) t + (\phi_1 + \phi_2)\right)\right] \operatorname{Cos}\left[\frac{1}{2}\left((k_1 - k_2) x - (\omega_1 - \omega_2) t + (\phi_1 - \phi_2)\right)\right]$$

Let

$$\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2) \qquad \bar{k} = \frac{1}{2}(k_1 + k_2)$$
$$\omega_m = \frac{1}{2}(\omega_1 - \omega_2) \qquad k_m = \frac{1}{2}(k_1 - k_2)$$
$$\alpha = \frac{1}{2}(\phi_1 + \phi_2) \qquad \beta = \frac{1}{2}(\phi_1 - \phi_2)$$

then

$$E = 2\,A\,\mathrm{Cos}[\bar{k}\,x - \bar{\omega}\,t + \alpha]\,\mathrm{Cos}[k_m\,x - \omega_m\,t + \beta]$$

If $\omega_1 \approx \omega_2$ then $\bar{\omega} >> \omega_m$ and the second term changes much more slowly than the first term.

$$E^2 = 4\,A^2\,\mathrm{Cos}[\bar{k}\,x - \bar{\omega}\,t + \alpha]^2\,\mathrm{Cos}[k_m\,x - \omega_m\,t + \beta]^2$$
$$= 2\,A^2\,\mathrm{Cos}[\bar{k}\,x - \bar{\omega}\,t + \alpha]^2\,(1 + \mathrm{Cos}[2\,(k_m\,x - \omega_m\,t + \beta)])$$

Let

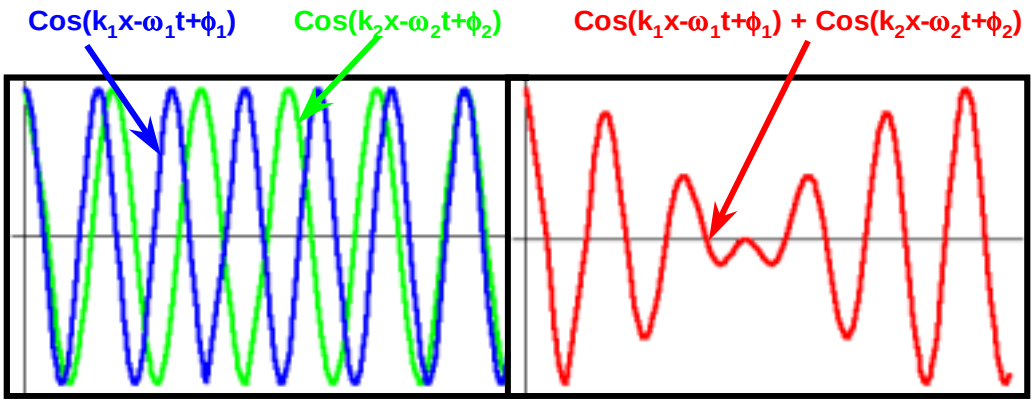
$$k_1 = \frac{2\,\pi}{\lambda_1} = \text{period of one signal and}$$
$$k_2 = \frac{2\,\pi}{\lambda_2} = \text{period of the second signal}$$
$$2\,k_m = (k_1 - k_2) = 2\,\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

Let the period of the modulation signal be λ_{eq}. Then

$$2\,\pi\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\lambda_{eq} = 2\,\pi$$

$$\boxed{\frac{1}{\lambda_{eq}} = \frac{1}{\lambda_1} - \frac{1}{\lambda_2}, \qquad \lambda_{eq} = \frac{\lambda_1\,\lambda_2}{\lambda_2 - \lambda_1}}$$

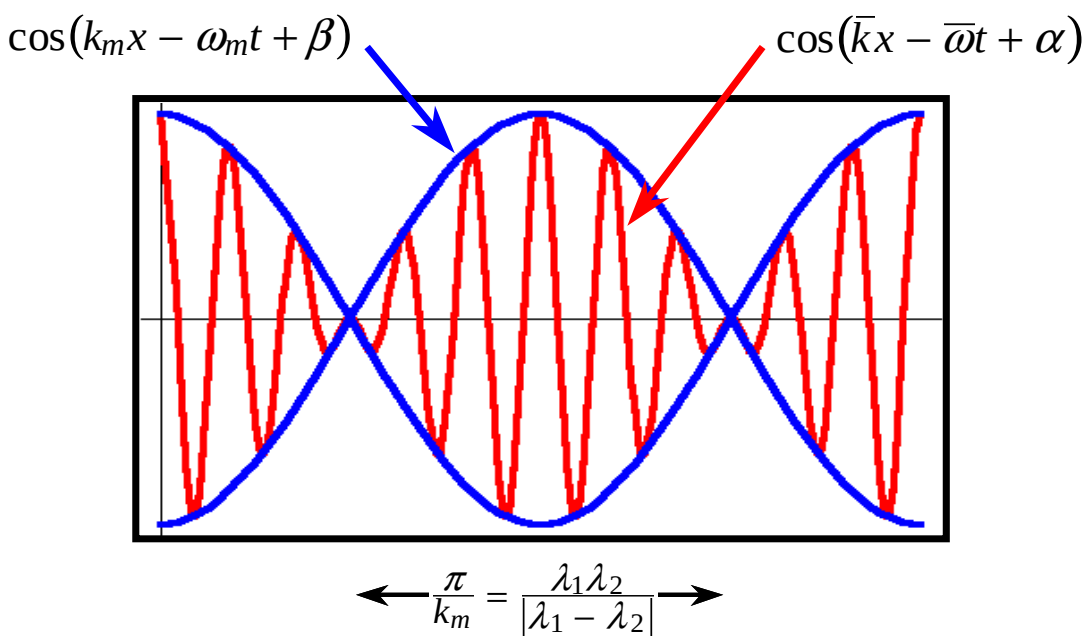
Waves Having Same Amplitude, but Different Frequency (Beats)



$$\boxed{E = 2\cos(\bar{k}x - \bar{\omega}t + \alpha)\cos(k_m x - \omega_m t + \beta)}$$

$$\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2),\ \bar{k} = \frac{1}{2}(k_1 + k_2),\ \alpha = \frac{1}{2}(\phi_1 + \phi_2),\ \omega_m = \frac{1}{2}(\omega_1 - \omega_2),\ k_m = \frac{1}{2}(k_1 - k_2),\ \beta = \frac{1}{2}(\phi_1 - \phi_2)$$

Beats



Must resolve high frequency to see beats

3.5.3 Standing Waves

If we have two waves of same amplitude and frequency travelling in the opposite directions

$$E[x, t] = A\,\mathrm{Cos}[k\,x - \omega\,t + \phi_1] + A\,\mathrm{Cos}[k\,x + \omega\,t + \phi_2]$$

We will now make use of the trig identity

$$\mathrm{Cos}[\alpha \pm \beta] = \mathrm{Cos}[\alpha]\,\mathrm{Cos}[\beta] \mp \mathrm{Sin}[\alpha]\,\mathrm{Sin}[\beta]$$

or

$$\mathrm{Cos}[\alpha - \beta] + \mathrm{Cos}[\alpha + \beta] = 2\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\beta]$$

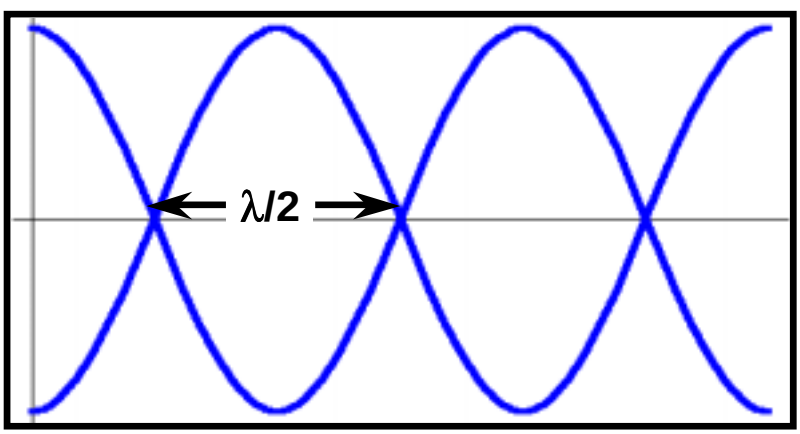
We can let

$$\alpha = k\,x + \frac{\phi_1 + \phi_2}{2} \quad \text{and} \quad \beta = \omega\,t + \frac{\phi_2 - \phi_1}{2}$$

$$\boxed{E[x, t] = 2\,A\,\mathrm{Cos}\left[k\,x + \frac{\phi_1 + \phi_2}{2}\right]\mathrm{Cos}\left[\omega\,t + \frac{\phi_2 - \phi_1}{2}\right]}$$

The first term is space dependent and the second term is time dependent.

Standing Waves



$$\boxed{\begin{aligned} E &= A\cos(kx - \omega t + \phi_1) + A\cos(kx + \omega t + \phi_2) \\ &= 2A\cos\left(kx + \frac{\phi_1 + \phi_2}{2}\right)\cos\left(\omega t + \frac{\phi_1 - \phi_2}{2}\right) \end{aligned}}$$