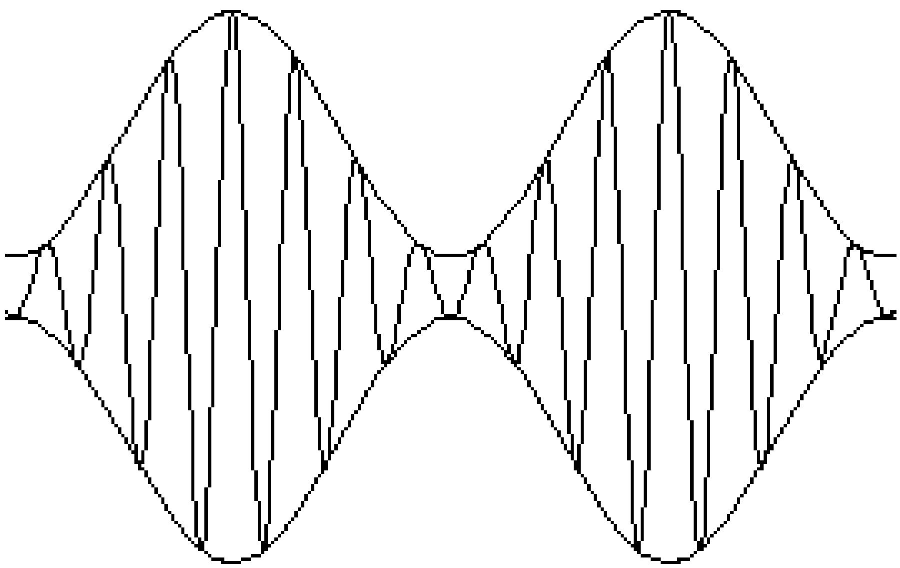
Basic Interference

I - 1

(a) Imagine that we strike two tuning forks, one with a frequency of 340 Hz, the other 342 Hz. What will we hear?

(b) The figure below shows a carrier of frequency ω_c being amplitude-modulated by a sine wave of frequency ω_m, that is E = $E_o(1 + a \cos \omega_m t)\cos \omega_c t$.

Show that this is equivalent to the superposition of three waves of frequency ω_c, $\omega_c + \omega_m$, and $\omega_c - \omega_m$. The $\omega_c + \omega_m$ term constitutes what is called the upper-sideband and the $\omega_c - \omega_m$ term forms the lower sideband. What bandwidth would you need in order to transmit the complete audible range, 20 Hz to 20 KHz?



Solution

- **a)**

We will assume equal amplitudes

```
a = a1 Cos[ω1 t] + a1 Cos[ω2 t];
```

where $\omega 1 = 2\pi\ 340$ and $\omega 2 = 2\pi\ 342$.

```
a = TrigFactor[a]
```

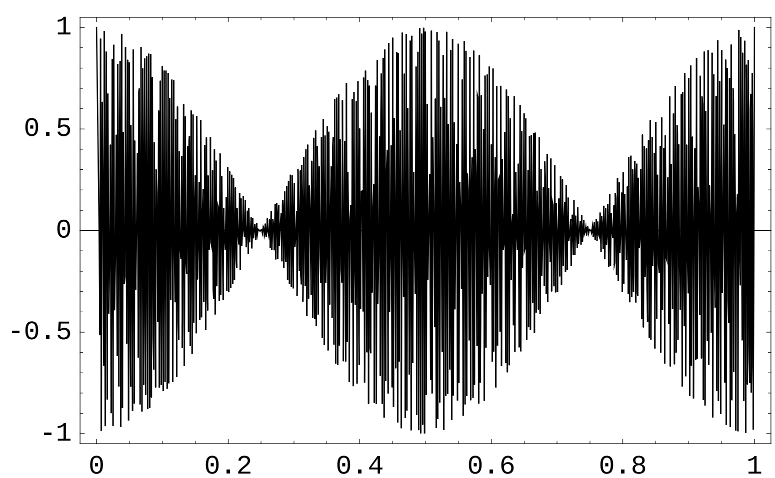
$$2\,a1\,\mathrm{Cos}\left[\frac{t\,\omega 1}{2}-\frac{t\,\omega 2}{2}\right]\,\mathrm{Cos}\left[\frac{t\,\omega 1}{2}+\frac{t\,\omega 2}{2}\right]$$

This can be written as

$$2\,a1\,\mathrm{Cos}\left[\frac{1}{2}\,t\,(\omega 1-\omega 2)\right]\,\mathrm{Cos}\left[\frac{1}{2}\,t\,(\omega 1+\omega 2)\right]$$

We get the cosine of the average frequency multiplied by the cosine of one-half the difference frequency. It is instructive to plot this.

$$\mathrm{Plot}\left[\mathrm{Cos}\left[\frac{1}{2}\,t\,(\omega 1-\omega 2)\right]\,\mathrm{Cos}\left[\frac{1}{2}\,t\,(\omega 1+\omega 2)\right]\;/.\;\{\omega 1\,\text{->}\,680\,\pi,\;\omega 2\,\text{->}\,684\,\pi\},\;\{t,\,0,\,1\},\;\mathrm{Frame}\,\text{->}\,\mathrm{True}\right];$$



We hear the average frequency (341 Hz) which gets loud and soft twice per second (the difference frequency).

■ b)

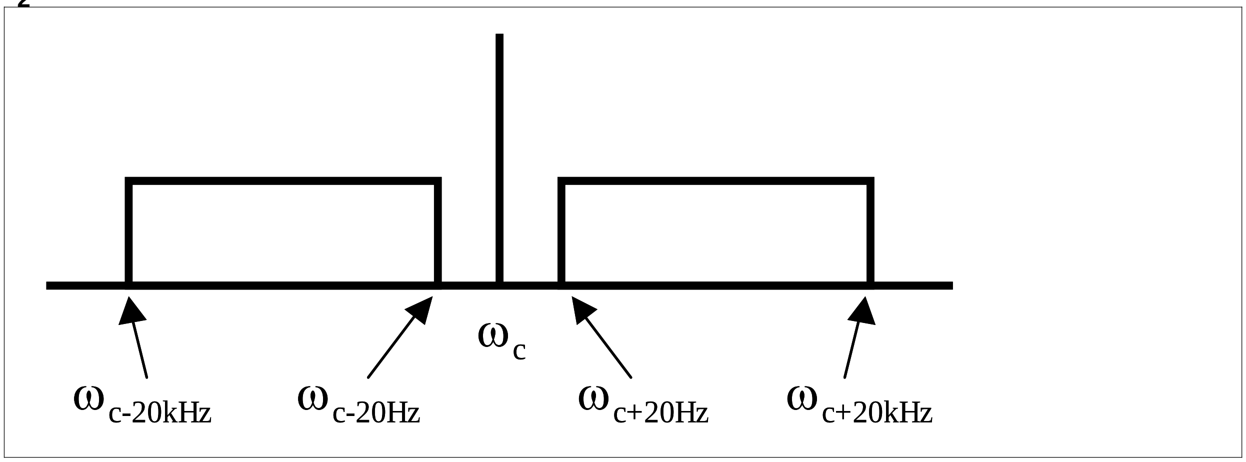
```
e = eo (1 + a Cos[ωm t]) Cos[ωc t];
```

```
Factor[TrigReduce[e]]
```

$$\frac{1}{2}\,eo\,(2\,\mathrm{Cos}[t\,\omega c]+a\,\mathrm{Cos}[t\,\omega c-t\,\omega m]+a\,\mathrm{Cos}[t\,\omega c+t\,\omega m])$$

This can be written as

$$\frac{1}{2}\,\mathrm{eo}\,(2\,\mathrm{Cos}[t\,\omega c] + a\,\mathrm{Cos}[t\,(\omega c - \omega m)] + a\,\mathrm{Cos}[t\,(\omega c + \omega m)])$$



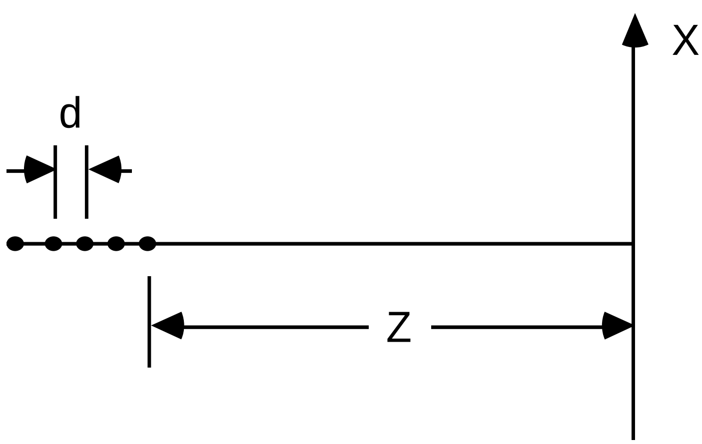
The bandwidth is 40 kHz. If the signal can be filtered, one of the sidebands and the carrier can be suppressed and the signal can be transmitted (single sideband - suppressed carrier) in a bandwidth of 19,980 Hz.

I - 2

You are given five mutually coherent point sources of equal amplitude, each separated a distance d as shown in the drawing. Assume z>>x, y, and d. Also assume that the initial phases of the five spherical wavefronts are equal. Show that the irradiance in the x-y plane goes as

$$\frac{\mathrm{Sin}[\frac{5\phi}{2}]^2}{\mathrm{Sin}[\frac{\phi}{2}]^2}$$

where ϕ is the phase difference for adjacent points. Determine ϕ and the shape of the fringes in the x-y plane.



Solution

```
Clear[e, r, λ, ω, z, n, d, x, y, opd, k]
```

Each of the 5 coherent point sources gives a spherical wave

$$\mathbf{e[n\_] := \frac{a}{r[n]} e^{i\,(k\,rn[n]\,-\,\omega\,t)}}$$

where r[n] is the distance from the point source to an observation point.

$$\mathbf{r[n\_] := \sqrt{(z + n\,d)^2 + x^2 + y^2}\,;}$$

We will factor out the largest term (z + n d).

$$\mathbf{r[n\_] := (z + n\,d)\sqrt{1 + \frac{x^2 + y^2}{(z + n\,d)^2}}\,;}$$

Next we will replace $x^2 + y^2$ with ρ^2.

$$\mathbf{r[n\_] = r[n]\ /.\ x^2 + y^2 \to \rho^2}$$

$$(d\,n + z)\sqrt{1 + \frac{\rho^2}{(d\,n + z)^2}}$$

Let opd[n] be equal to the difference in r[n] for adjacent point sources. That is, opd[n] = r[n+1] - r[n]. We will now expand opd[n] in a binomial expansion to order 2 in ρ.

```
opd[n_] := Simplify[Normal[Series[r[n + 1] - r[n], {ρ, 0, 2}]]];
opd[n]
```

$$d - \frac{d\,\rho^2}{2\,(d\,n + z)\,(d + d\,n + z)}$$

Since z is much larger than d and d n, the opd difference for adjacent point sources, can be approximated as

$$\mathbf{opdDiff = d - \frac{d\,\rho^2}{2\,z^2}\,;}$$

The phase difference, ϕ, for adjacent point sources is given by

$$\mathbf{\phi = k\ opdDiff}$$

$$k\left(d - \frac{d\,\rho^2}{2\,z^2}\right)$$

```
Clear[ϕ]
```

The total field at a point in the observation field is given by

$$\mathbf{etotal = \frac{a}{z} e^{-i\,\omega\,t} e^{i\,k\,z} (1 + e^{i\,\phi} + e^{i\,2\,\phi} + e^{i\,3\,\phi} + e^{i4\,\phi})\,;}$$

where the r in the denominator is approximated as a constant z. We can write

$$\mathbf{amp = 1 + e^{i\,\phi} + e^{i\,2\,\phi} + e^{i\,3\,\phi} + e^{i4\,\phi}\,;}$$

amp can be written as

```
amp = Sum[ⅇ^(ⅈ n ϕ), {n, 0, p}] /. p → 4
```

$$\frac{-1 + e^{5 i \phi}}{-1 + e^{i \phi}}$$

Factoring yields

```
amp = amp /. {-1 + ⅇ^x_ → ⅇ^(x/2) (-ⅇ^(-x/2) + ⅇ^(x/2)), 1/(-1 + ⅇ^x_) → 1/(ⅇ^(x/2) (-ⅇ^(-x/2) + ⅇ^(x/2)))}
```

$$\frac{e^{2 i \phi} \left(-e^{-\frac{5 i \phi}{2}} + e^{\frac{5 i \phi}{2}}\right)}{-e^{-\frac{i \phi}{2}} + e^{\frac{i \phi}{2}}}$$

Converting the exponentials to sins yields

```
amp = (ⅇ^(2 ⅈ ϕ) ExpToTrig[(-ⅇ^(-5 ⅈ ϕ/2) + ⅇ^(5 ⅈ ϕ/2))]) / ExpToTrig[-ⅇ^(-ⅈ ϕ/2) + ⅇ^(ⅈ ϕ/2)]
```

$$e^{2 i \phi} \operatorname{Csc}\left[\frac{\phi}{2}\right] \operatorname{Sin}\left[\frac{5 \phi}{2}\right]$$

The irradiance goes as the square of this or

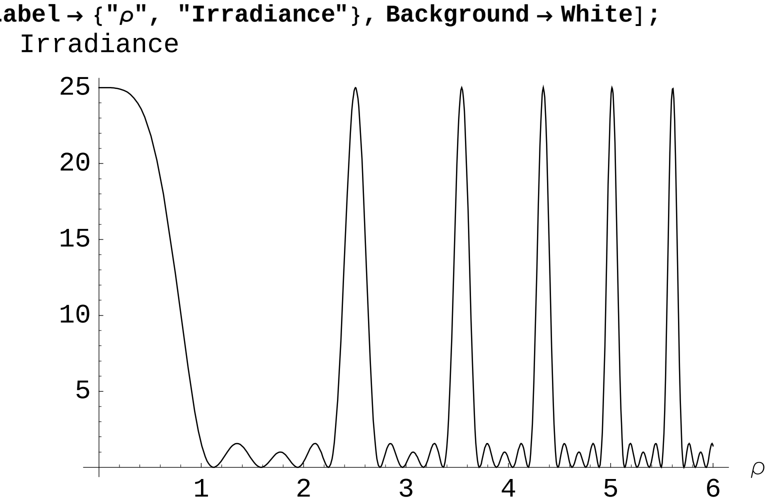
```
irradiance[ϕ_] := Sin[5 ϕ/2]^2 / Sin[ϕ/2]^2;
```

The above result is a common result that we see in interference. If there were N interfering beams, where the phase difference between adjacent beams is a constant, ϕ, the irradiance would go as

$$\frac{\operatorname{Sin}\left[\frac{N\phi}{2}\right]^2}{\operatorname{Sin}\left[\frac{\phi}{2}\right]^2}.$$

Remembering that from above that ϕ goes as ρ^2, it follows that the fringes are circular in the x-y plane. The fringes are widely spaced in the center and become more closely spaced as we move away from the origin. The following shows a profile of the fringes.

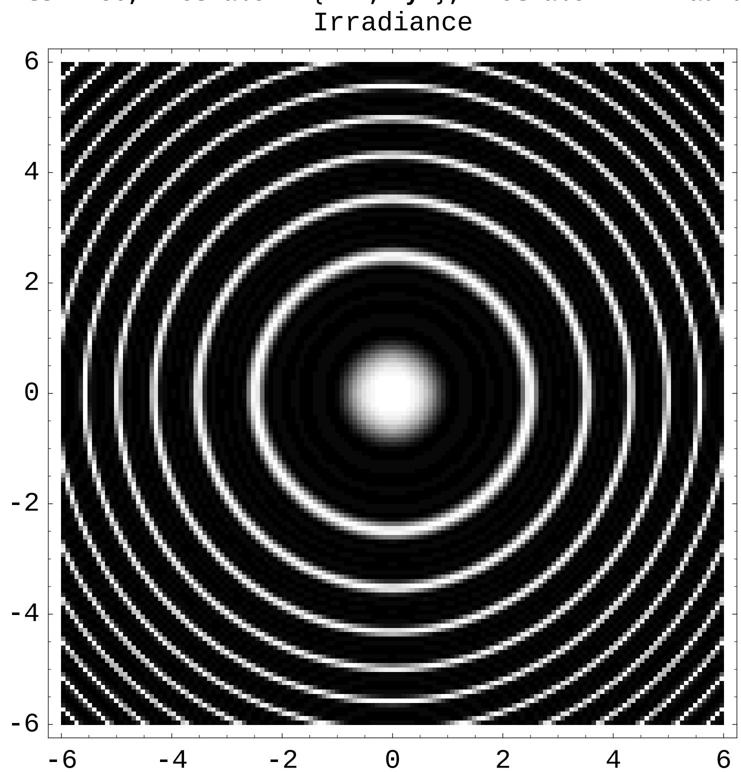
```
Plot[irradiance[ϕ] /. ϕ -> ρ^2, {ρ, 0, 6},
  AxesLabel → {"ρ", "Irradiance"}, Background → White];
```



The fringes get sharper as the number of sources increases. In the above plot d is assumed to be an integer number of wavelengths. If this is not true, the position of the fringes will shift, but the shape will not change.

The following shows a density plot for the irradiance.

```
DensityPlot[irradiance[ϕ] /. ϕ -> x^2 + y^2, {x, -6, 6}, {y, -6, 6}, Mesh → False,
  PlotPoints → 150, AxesLabel → {"x", "y"}, PlotLabel → "Irradiance", Background → White];
```



I - 3

An observation screen is illuminated with two mutually coherent collimated beams of like polarization and equal intensity. The two beams make an angle of ± 1 milli-radian with respect to the normal to the observation screen.
a) What is the irradiance distribution and fringe spacing on the observation screen?
b) Let the interference fringes be observed by use of a photocell, which has an effective area of a slit. The height, H, of the slit, which is parallel to the fringes, is fixed, while the width, w, is variable. Assume that the photocurrent is proportional to the light flux falling on the detector. Give the variation of the current as a function of abscissa x of the slit and show that the observed fringe visibility is given by Sin(πw/s)/(πw/s), where s is the spacing of the fringes. Plot the visibility as a function of w/s.

Solution

a)

$$\text{irradiance} = i_o \left(1 + \text{Cos}\left[\frac{2\pi}{\lambda}\,\text{opd}\right]\right)$$

$$\text{opd} = 2\,y\,\text{Sin}[\theta];\quad \theta = 0.001;$$

Let s = fringe spacing

$$\frac{2\pi}{\lambda}\,2\,s\,(0.001) = 2\pi$$

$$\textbf{Solve}\left[\frac{2\pi}{\lambda}\,2\,s\,(0.001) == 2\pi,\ s\right]$$

$$\{\{s \to 500.\,\lambda\}\}$$

$$\boxed{\text{irradiance} = i_o\left(1 + \text{Cos}\left[2\pi\frac{y}{s}\right]\right)}$$

b)

Let c be a proportionality constant.

$$\textbf{i} = \textbf{c}\int_{y-w/2}^{y+w/2}\left(1 + \text{Cos}\left[2\pi\frac{yy}{s}\right]\right)d\,yy$$

$$c\left(w - \frac{s\,\text{Sin}\left[\frac{2\pi\left(-\frac{w}{2}+y\right)}{s}\right]}{2\pi} + \frac{s\,\text{Sin}\left[\frac{2\pi\left(\frac{w}{2}+y\right)}{s}\right]}{2\pi}\right)$$

$$\textbf{i} = \textbf{TrigExpand}[\textbf{i}]$$

$$c\,w + \frac{c\,s\,\text{Cos}\left[\frac{\pi y}{s}\right]^2\text{Sin}\left[\frac{\pi w}{s}\right]}{\pi} - \frac{c\,s\,\text{Sin}\left[\frac{\pi w}{s}\right]\text{Sin}\left[\frac{\pi y}{s}\right]^2}{\pi}$$

$$\textbf{i} = \textbf{c w} + \textbf{Simplify}\left[\frac{c\,s\,\text{Cos}\left[\frac{\pi y}{s}\right]^2\text{Sin}\left[\frac{\pi w}{s}\right]}{\pi} - \frac{c\,s\,\text{Sin}\left[\frac{\pi w}{s}\right]\text{Sin}\left[\frac{\pi y}{s}\right]^2}{\pi}\right]$$

$$c\,w + \frac{c\,s\,\text{Cos}\left[\frac{2\pi y}{s}\right]\text{Sin}\left[\frac{\pi w}{s}\right]}{\pi}$$

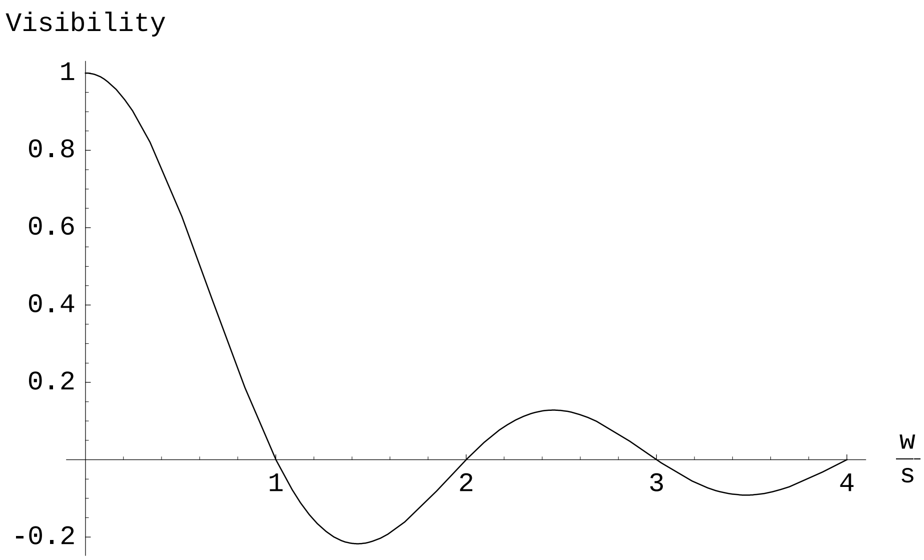
$$\textbf{i} = \textbf{c w}\left(1 + \frac{\text{Cos}\left[\frac{2\pi y}{s}\right]\text{Sin}\left[\frac{\pi w}{s}\right]}{\frac{\pi w}{s}}\right)$$

$$\text{visibility} = \frac{i_{max} - i_{min}}{i_{max} + i_{min}};$$

$$\textbf{visibility} = \frac{\text{Sin}\left[\frac{\pi w}{s}\right]}{\frac{\pi w}{s}};$$

Negative visibility means a phase reversal. That is, light fringes where dark fringes should be and vice versa.

```
Plot[visibility /. w → β s, {β, 0, 4},
  AxesLabel → {"w/s", "Visibility"}, Background → White, PlotRange → All];
```

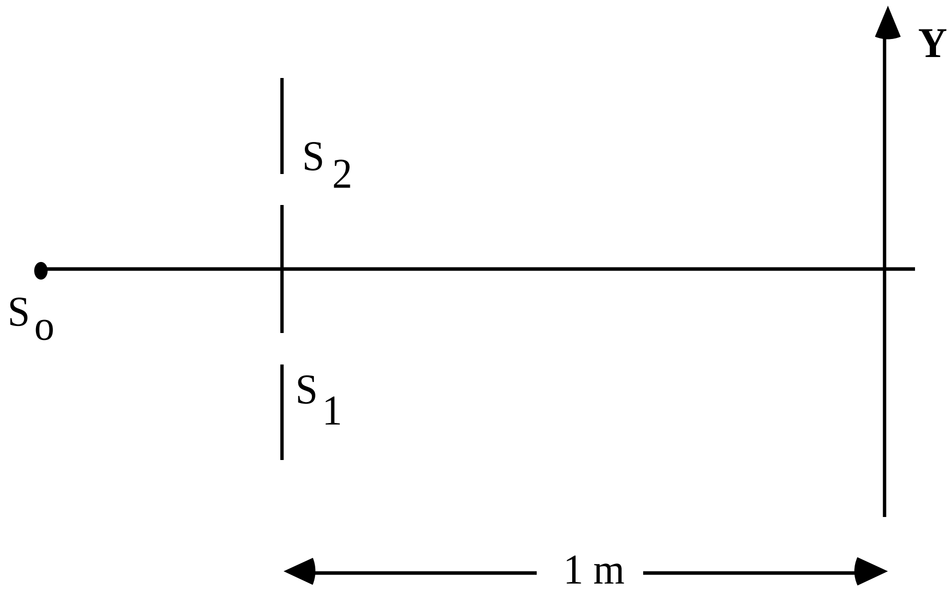


I - 4

A point source S_o illuminates two narrow identical parallel slits S_1 and S_2 ruled vertically in an opaque screen. The wavelength of the light source is 0.5 microns and the slits are separated a distance 2 mm. The interference pattern is observed in a plane parallel to and at a distance of 1 m from the screen.

a) What is the irradiance distribution and fringe spacing on the observation screen?

b) Let the interference fringes be observed by use of a photocell, which has an effective area of a slit. The height, H, of the slit, which is parallel to the fringes, is fixed, while the width, w, is variable. Assume that the photocurrent is proportional to the light flux falling on the detector. Give the variation of the current as a function of ordinate y of the slit and show that the observed fringe visibility is given by Sin(□w/s)/(□w/s), where s is the spacing of the fringes. Plot the visibility as a function of w/s.



Solution

```
Clear[i1, i2]
```

$\lambda = 0.5\,\mu m;\ k = \frac{2\pi}{\lambda};$

The distance, d, between the slits is

$d = 2\,mm;\quad mm = 10^3\,\mu m;$

The distance from the slits to the screen, zo, is

$zo = 1\,m;\quad m = 10^6\,\mu m;$

■ a)

If the irradiances of the two beams are i1 and i2, the resulting irradiance, i, is given by

$i = i1 + i2 + 2\sqrt{i1\,i2}\,\mathrm{Cos}[k\,opd];$

$opd = y\,\frac{d}{zo};$

In going from one fringe to the next fringe (k opd) changes by 2π. Thus, the fringe spacing, s, is given by

$s = \frac{2\pi}{k}\,\frac{zo}{d}$

$250.\,\mu m$

■ b)

```
Clear[i, s, i1, i2, imeasured, visibility];
```

From the above, the irradiance can be written in terms of the fringe spacing, s, as

$$\mathbf{i = i1 + i2 + 2\sqrt{i1\,i2}\,Cos\left[2\pi\frac{y}{s}\right];}$$

$$\mathbf{imeasured = \int_{y'-\frac{w}{2}}^{y'+\frac{w}{2}} i\,dy}$$

$$i1\,w + i2\,w - \frac{\sqrt{i1\,i2}\;s\,Sin\left[\frac{2\pi\left(-\frac{w}{2}+y'\right)}{s}\right]}{\pi} + \frac{\sqrt{i1\,i2}\;s\,Sin\left[\frac{2\pi\left(\frac{w}{2}+y'\right)}{s}\right]}{\pi}$$

A little algebra needs to be done making use of the trig identity Sin[a+b]=Cos[a]Sin[b]+Cos[b]Sin[a]. Using *Mathematica* we get

```
imeasured = FullSimplify[ExpToTrig[Simplify[TrigToExp[imeasured]]]]
```

$$(i1 + i2)\,w + \frac{2\sqrt{i1\,i2}\;s\,Cos\left[\frac{2\pi y'}{s}\right]\,Sin\left[\frac{\pi w}{s}\right]}{\pi}$$

Let i1 = i2 = 1

```
i1 = 1; i2 = 1; imeasured
```

$$2\,w + \frac{2\,s\,Cos\left[\frac{2\pi y'}{s}\right]\,Sin\left[\frac{\pi w}{s}\right]}{\pi}$$

We need to do some factoring so we get the equation into the form of

$$\text{Irradiance} = \text{constant}\left(1 + \text{visibility}\,\text{Cos}\left[\frac{2\pi\,y'}{s}\right]\right)$$

$$\mathbf{imeasured = imeasured\ /.\ a\_\,w + b\_ \to a\,w\left(1 + \frac{b}{a\,w}\right)}$$

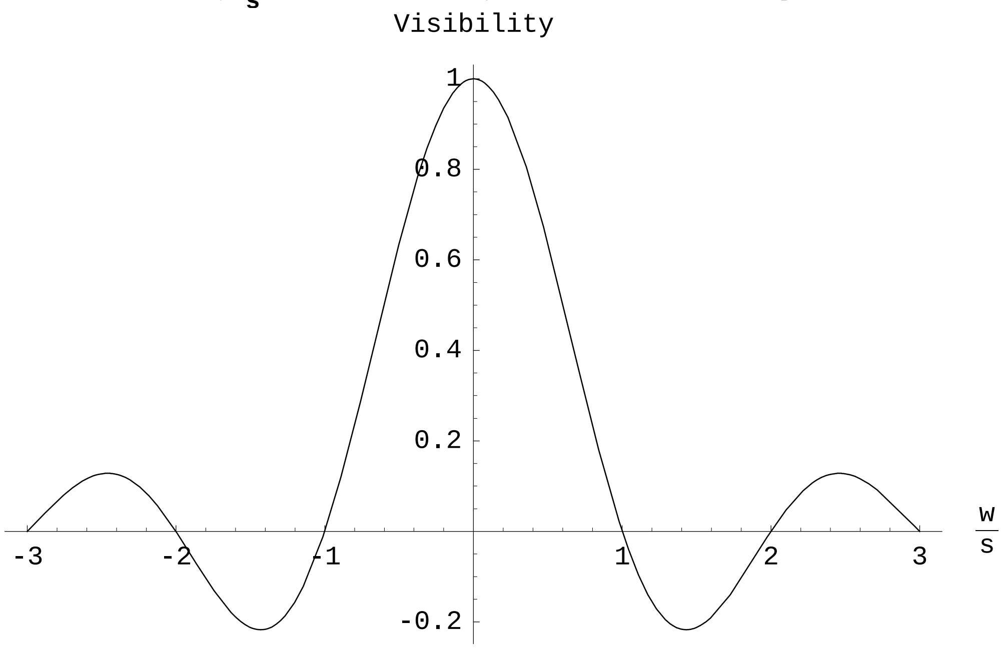
$$2\,w\left(1 + \frac{s\,Cos\left[\frac{2\pi y'}{s}\right]\,Sin\left[\frac{\pi w}{s}\right]}{\pi\,w}\right)$$

Thus,

$$\mathbf{visibility = \frac{Sin\left[\frac{\pi w}{s}\right]}{\frac{\pi w}{s}};}$$

Let $\frac{w}{s} = a$ so we can easily plot,

```
Plot[visibility /. {w → a , s → 1 }, {a, -3, 3},
  AxesLabel -> {"w/s", "Visibility"}, Background -> White];
```



When the visibility goes negative we have a phase reversal in the measured fringe pattern.

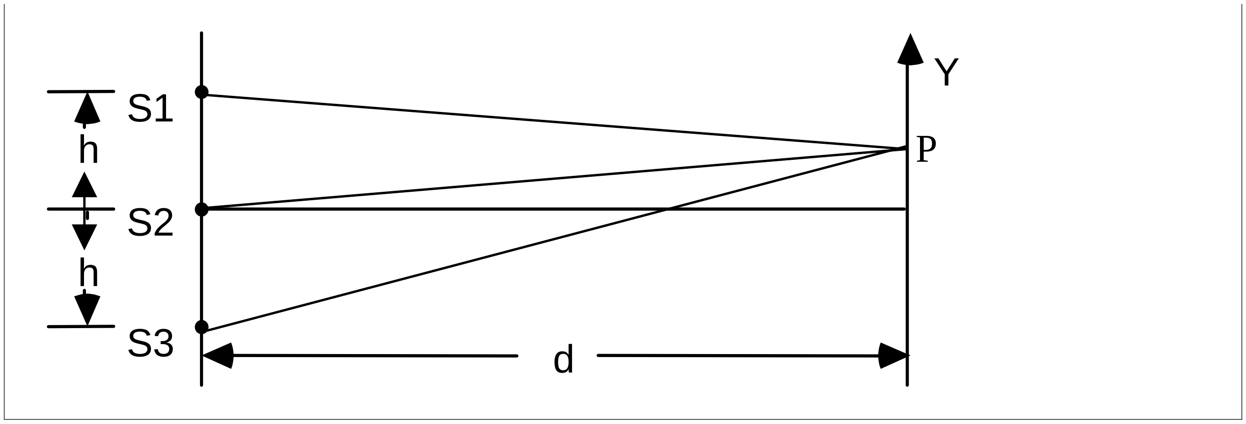
I - 5

You are given 3 mutually coherent point sources of equal amplitude, each separated a distance h, as shown in the figure. The three point sources have the initial phases θ_1, θ_2, and θ_3. Find the irradiance in the y direction if

$D >> y$ and $h << \sqrt{\lambda D}$,

and give a physical explanation if

Case I: $\theta_1 = \theta_2 = \theta_3 = 0$.
Case II: $\theta_1 = \theta_3 = \pi/2$ and $\theta_2 = 0$.



Solution

$$\text{s1p} = \sqrt{(y - h)^2 + d^2}$$

$$\text{s2p} = \sqrt{y^2 + d^2}$$

$$\text{s3p} = \sqrt{(y - h)^2 + d^2}$$

We can approximate these distances as

$$\text{s1p} \approx d + \frac{y^2 - 2\,y\,h + h^2}{2\,d} \approx d + \frac{y^2 - 2\,y\,h}{2\,d}$$

$$\text{s2p} \approx d + \frac{y^2}{2\,d}$$

$$\text{s3p} \approx d + \frac{y^2 + 2\,y\,h + h^2}{2\,d} \approx d + \frac{y^2 + 2\,y\,h}{2\,d}$$

The corresponding phases can be written as

$$\phi 1 = \theta 1 + \frac{2\,\pi}{\lambda}\left(d + \frac{y^2 - 2\,y\,h}{2\,d}\right);$$

$$\phi 2 = \theta 2 + \frac{2\,\pi}{\lambda}\left(d + \frac{y^2}{2\,d}\right);$$

$$\phi 3 = \theta 3 + \frac{2\,\pi}{\lambda}\left(d + \frac{y^2 + 2\,y\,h}{2\,d}\right);$$

Since the sources have the same strengths, the normalized irradiance at point p can be written as

$$i = (e^{i\,\phi 1} + e^{i\,\phi 2} + e^{i\,\phi 3})\,(e^{-i\,\phi 1} + e^{-i\,\phi 2} + e^{-i\,\phi 3});$$

```
irradiance = Simplify[ExpToTrig[i]]
```

$$3 + 2\,\text{Cos}\left[\theta 1 - \theta 3 - \frac{4\,h\,\pi\,y}{d\,\lambda}\right] + 2\,\text{Cos}\left[\theta 1 - \theta 2 - \frac{2\,h\,\pi\,y}{d\,\lambda}\right] + 2\,\text{Cos}\left[\theta 2 - \theta 3 - \frac{2\,h\,\pi\,y}{d\,\lambda}\right]$$

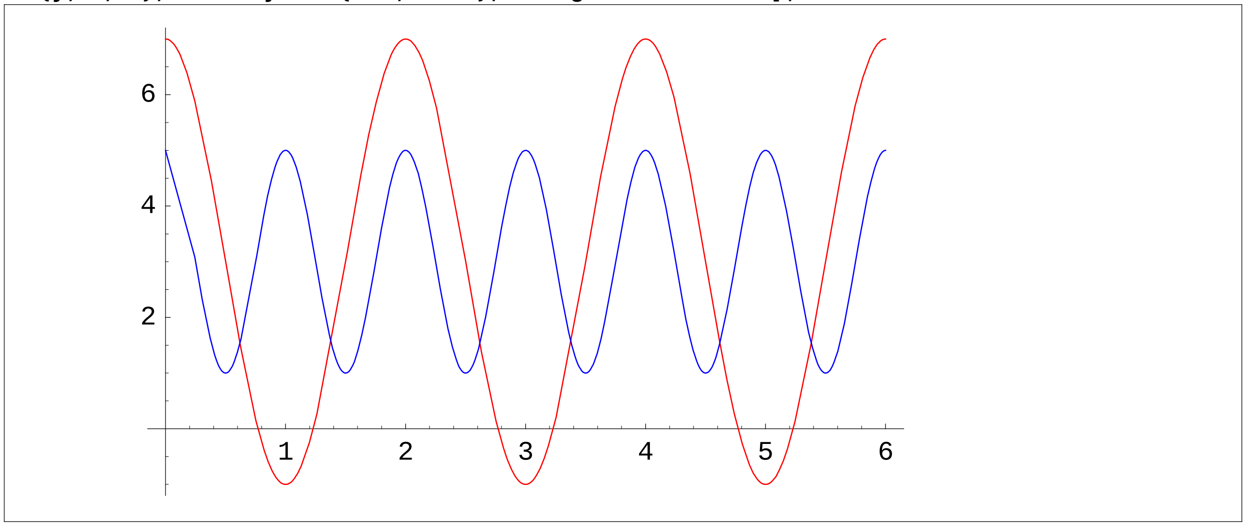
Case I

$$\Theta_1 = \Theta_2 = \Theta_3 = 0$$

```
irradiance /. {θ1 -> 0, θ2 -> 0, θ3 -> 0}
```

$$3 + 4\,\mathrm{Cos}\left[\frac{2\,h\,\pi\,y}{d\,\lambda}\right] + 2\,\mathrm{Cos}\left[\frac{4\,h\,\pi\,y}{d\,\lambda}\right]$$

```
Plot[{3 + 4 Cos[π y], 3 + 2 Cos[π 2 y]},
  {y, 0, 6}, PlotStyle → {Red, Blue}, Background -> White];
```



The double frequency component is due to the interference of s1 and s3. The single frequency component is due to the interference of s2 and s3 and s1 and s2. The total pattern is the superposition of the pattern obtained from all possible pairs of point sources.

Case II

$\Theta_1 = \Theta_3 = \frac{\pi}{2}$ and $\Theta_2 = 0$.

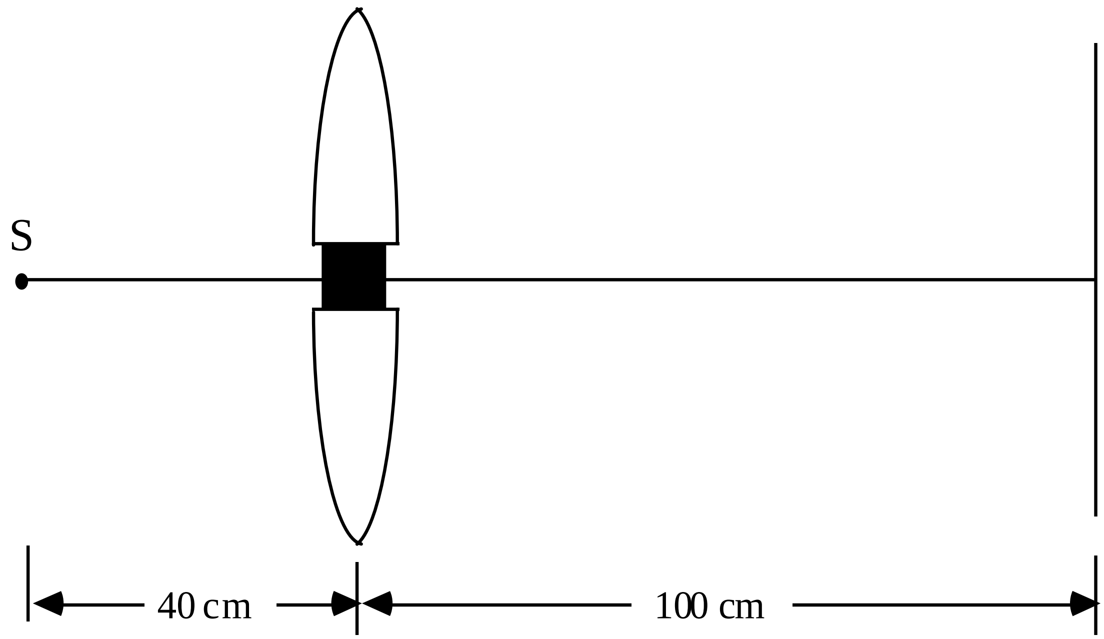
```
irradiance /. {θ1 -> π/2, θ3 -> π/2, θ2 -> 0}
```

$$3 + 2\,\mathrm{Cos}\left[\frac{4\,h\,\pi\,y}{d\,\lambda}\right]$$

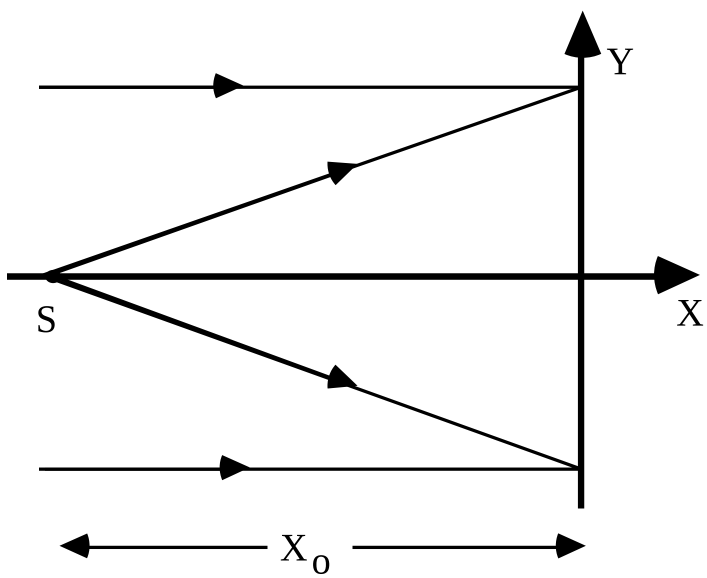
The interference of points s1 and s2 and the interference of points s2 and s3 are out of phase 180° and hence cancel one another out. Thus, the single frequency component is cancelled and we are left with only the double frequency component.

I - 6

a) Billet's split lens. A converging lens of 20 cm focal length is cut in two by means of a plane passing through its optic axis. A source S of monochromatic light lies in this plane 40 cm from the lens, as shown in the figure. As the half-lenses are gradually moved apart, the image of the source splits into two images, acting as coherent sources. The region between the two half-lenses is blocked. Determine the width of the interference fringes observed on the screen at a distance of 100 cm when the lenses are 0.5 mm apart. Assume λ = 500 nm.



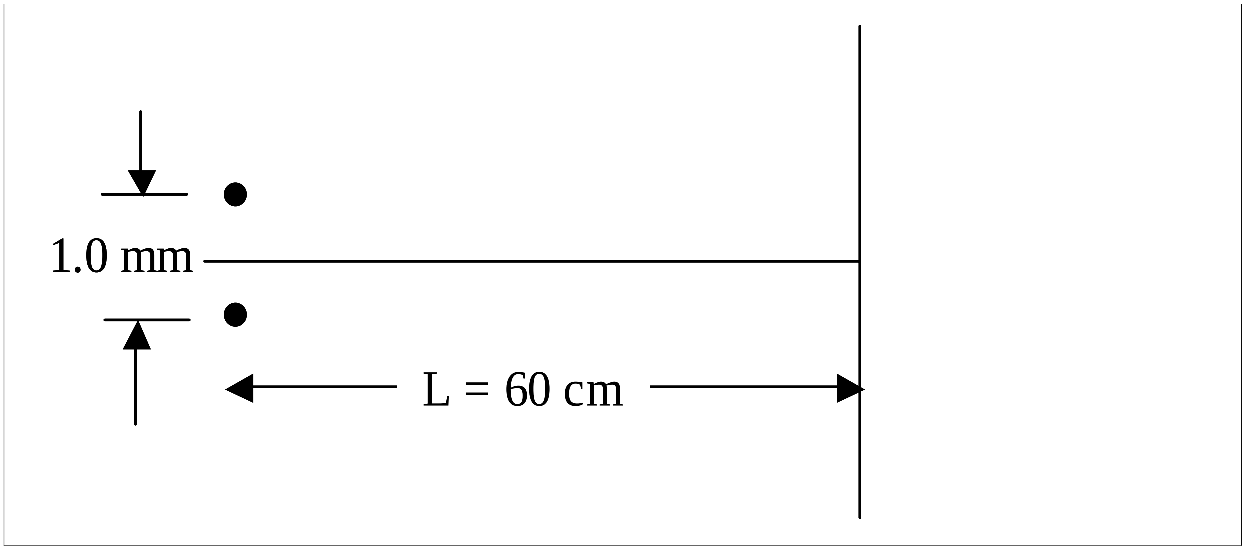
b) A spherical wave coming from a source a distance x_o from an observation screen is interfered with a plane wave propagating normal to the observation screen. Show that the areas of the annular regions between consecutive bright fringes are equal if we assume $|x_o| \gg |y|$ or $|z|$.



Solution

- **a)**

The point source will be imaged to the right of the lens 40 cm. Due to the 0.5 mm separation of the two parts of the lens there will be two images of the point source separated by 1 mm.



Let s be the fringe spacing.

$$(1.0\ \text{mm})\ \frac{s}{600\ \text{mm}} = \lambda$$

$$\mathbf{s = \frac{600\ mm}{1.0\ mm}\ (0.5\ 10^{-3}\ mm) = 0.3\ mm = fringe\ spacing}$$

- **b)**

$$\text{opd} = \frac{r^2}{2\, x_o},\quad \text{where } r = \sqrt{y^2 + z^2}$$

If the opd = m λ then

$$m\,\lambda = \frac{r^2}{2\, x_o}\ \text{ and } r = \sqrt{2\, x_o\, m\, \lambda}$$

The area of an annular zone is $\pi\, r^2$.

$$\text{area} = \pi\,(2\, x_o\, m\, \lambda)$$

Between zones m changes by 1 so the area of each zone is

$$\text{areaOfZone} = 2\,\pi\, x_o\, \lambda,\quad \text{independent of } m$$

I - 7

Sometimes when an airplane flies over my house the picture on my TV fades in and out repeatedly. Explain.

Solution

I am seeing the interference betwwen the direct TV wave and the TV wave reflected off the airplane. The path difference (and phase difference) varies as the plane flies along and hence I have constructive and destructive interference repeadly. I guess I need cable TV.

I - 8

Describe quantitatively the fringes observed when a 10 cm radius of curvature cylindrical surface is placed in contact at one end with an optical flat and at the other end the separation between the optical flat and the 20 cm long cylindrical lens is 0.1 mm. Let the wavelength of the light be 450 nm.

Solution

Let r be the radius of the cylinder. Let the surface of the optical flat be the x-y plane. Let z be the distance between the flat and the cylinder. Let θ be the tilt between the cylinder and the flat plane. $\theta = \frac{0.1}{200}$. z can be approximated by

$$z = \frac{x^2}{2r} + y\,\theta$$

Due to the π phase change upon reflection, the condition for a dark fringe is

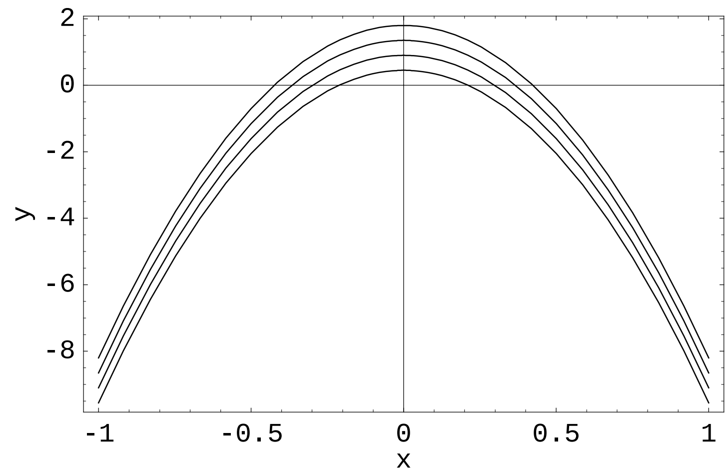
$$2z = m\lambda, \text{ where } m \text{ is an integer.}$$

$$\frac{x^2}{2r} + y\,\theta = m\,\frac{\lambda}{2}$$

$$y = -\frac{x^2}{2r\theta} + \frac{m\lambda}{2\theta}, \quad m = 0, 1, 2, \cdots$$

$$\theta = \frac{0.1}{200};\ r = 100;\ \lambda = 0.45 \times 10^{-3};$$

$$\text{Plot}\left[\left\{-\frac{x^2}{2\,r\,\theta}+\frac{1\,\lambda}{2\,\theta},\ -\frac{x^2}{2\,r\,\theta}+\frac{2\,\lambda}{2\,\theta},\ -\frac{x^2}{2\,r\,\theta}+\frac{3\,\lambda}{2\,\theta},\ -\frac{x^2}{2\,r\,\theta}+\frac{4\,\lambda}{2\,\theta}\right\},\right.$$
$$\left.\{x,-1,1\},\ \text{Frame}\to\text{True},\ \text{FrameLabel}\to\{x,y\},\ \text{Background}\to\text{White}\right];$$



Spacing along y-axis

$$\Delta y = \frac{\lambda}{2\,\theta}\ \text{mm}$$

0.45 mm

Spacing along x-axis

$$y = -\frac{x^2}{2\,r\,\theta} + \frac{m\,\lambda}{2\,\theta},\quad m = 0, 1, 2, \cdots$$

$$x = \sqrt{\left(\frac{m\,\lambda}{2\,\theta} - y\right) 2\,r\,\theta}$$

For a given y, the fringe spacing, Δx, is given by

$$\Delta x = \frac{r\,\lambda}{2\,x}$$

I - 9

Two plane waves having a wavelength of 633 nm are interfered at an angle of 10 degrees.

a) Where are the fringes localized?

b) What is the fringe spacing if one of the plane waves is normal to the observation plane?

c) What is the fringe spacing if the bisector to the two plane waves is normal to the observation plane?

d) What is the fringe spacing (in microns) if the bisector to the two plane waves makes an angle of 25 degrees to the observation plane?

Solution

$\lambda = 0.633\,\mu m;$

$\theta = 10\,\text{Degree};$

a)

Since the source is a point source, the fringes are non-localized and the fringes exist every place the two beams overlap.

b)

The fringe spacing, d, is given by

$$d = \frac{\lambda}{\text{Sin}[\theta]}$$

3.6453 μm

c)

The fringe spacing, d, is given by

$$d = \frac{\lambda}{2\,\text{Sin}[\frac{\theta}{2}]}$$

3.63143 μm

d)

$\theta avg = 25\,\text{Degree};$

The fringe spacing, d, is given by

$$d = \frac{\lambda}{\text{Sin}[\theta avg + \frac{\theta}{2}] - \text{Sin}[\theta avg - \frac{\theta}{2}]}$$

4.00684 μm

Note that this is equivalent to the result for part c divided by Cos[θavg].

$$\frac{\frac{\lambda}{2\,\text{Sin}[\frac{\theta}{2}]}}{\text{Cos}[\theta avg]}$$

4.00684 μm

I - 10

Two plane waves having a wavelength of 500 nm are interfered at an angle of 1 degree.

a) Where are the fringes localized?

b) What is the fringe spacing (in microns) if the bisector to the two plane waves is normal to the observation plane?

c) What is the fringe visibility if the irradiance of the two plane waves is the same and the two plane waves are linearly polarized with an angle of 30 degrees between the direction of polarization for the two waves?

d) What is the fringe visibility if the two waves have parallel electric fields and one plane wave has 5 times the irradiance of the other plane wave?

Solution

■ a)

Two plane waves interfering so the fringes are non-localized.

■ b)

$$\text{fringeSpacing} = \frac{\lambda}{2\,\text{Sin}[0.5\,°]}\ /.\ \lambda \to 0.5\,\mu\text{m}$$

$$28.6483\,\mu\text{m}$$

■ c)

$$\text{visibility} = \text{Cos}[30\,°] = \frac{\sqrt{3}}{2} = 0.866$$

■ d)

$$\text{i} = I_1 + I_2 + 2\sqrt{I_1 I_2}\ \text{Cos}[\phi]$$

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2};\ I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2};$$

$$\text{visibility} = \text{Simplify}\left[\frac{I_{max} - I_{min}}{I_{max} + I_{min}}\right]$$

$$\frac{2\sqrt{\mathbb{i}_1\,\mathbb{i}_2}}{\mathbb{i}_1 + \mathbb{i}_2}$$

$$\text{visibility}\ /.\ I_2 \to 5\,I_1 = \frac{\sqrt{5}}{3}$$

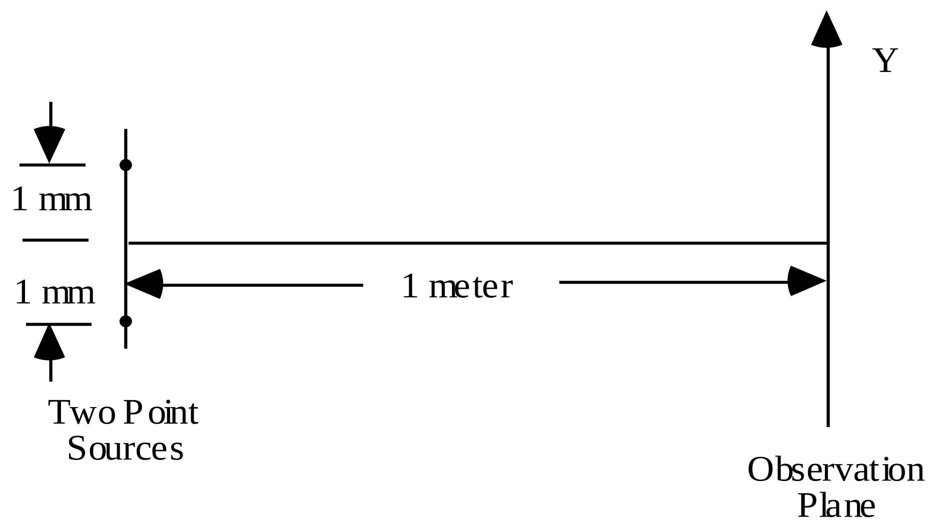
I - 11

Two spherical waves are interfered as shown below.

a) What is the fringe spacing (in microns) if the wavelength is 500 nm?

b) What is the fringe visibility if the relative intensity of the two spherical waves is 9 to 1? State any assumptions you are making.

c) Let the two sources have the same intensity, but now two wavelengths, 500 nm and 510 nm, are present. Give the first value of Y for which the fringe contrast drops to zero.



Solution

■ a)

Let Y_1 = fringe spacing

L = distance from source to screen

d = distance between sources

$$d\left(\frac{Y_1}{L}\right) = \lambda$$

$$Y_1 = \frac{\lambda}{(d/L)} = \frac{0.5\,\mu m}{2\times 10^{-3}} = 250\,\mu m$$

■ b)

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} = \frac{2\sqrt{I_2/I_1}}{1 + I_2/I_1} = \frac{2\sqrt{9}}{1+9} = 0.6$$

- **c)**

$$d\,\theta = m\,\lambda_1 = \left(m + \frac{1}{2}\right)\lambda_2;\quad m\,(\lambda_1 - \lambda_2) = \frac{\lambda_2}{2};\quad m = \frac{\lambda_2}{2\,\Delta\lambda}$$

$$\theta = \frac{\lambda_1\,\lambda_2}{2\,\Delta\lambda\,d} = \frac{0.5\,(510)}{2\,(10)\,2\times10^3} = 0.006375 = \frac{y}{L}$$

$$y = 0.006375\,(10^3\ \text{mm}) = 6.38\ \text{mm}$$

I - 12

Newton's rings are observed with a plano-convex lens resting on a plane glass surface. The radius of curvature of the convex surface is R = 10m. Assume the lens is slowly moved away from the plane glass surface until the separation of the plane glass surface and the convex surface is 0.1 mm.

a) Do the radii of the fringes increase or decrease?

b) How many times does the intensity at the center of the interference fringes go through a maximum if the incident light is monochromatic and has a wavelength 500 nm?

Solution

- **a)**

For a given fringe the OPD is a constant. Therefore, for a convex surface the fringe radii will decrease as the convex surface is moved away from the flat.

- **b)**

When the surfaces are in contact there will be a dark fringe at the center. The first bright fringe at the center will occur when the surface is moved 1/4 wavelength, and hence the OPD changes by 1/2 wavelength. After this a bright fringe will appear every time the surface is moved 1/2 wavelength and the OPD changes by 1 wavelength. The number of bright fringes we go through is given by

$$1 + \frac{0.1\,10^6\ \text{nm} - \frac{\lambda}{4}}{\frac{\lambda}{2}}\ /.\ \lambda \to 500\ \text{nm}$$

400.5

Thus, we go through 400 bright fringes in moving the lens 0.1 mm from the flat.

I - 13

Newton's interference fringes are observed with a plano-convex lens resting on a spherical-concave glass surface. The radius of curvature of the convex lens surface is 15 meters and the radius of curvature of the spherical-concave glass surface is 30 meters. The source has a wavelength of 500 nm.

a) If the two spherical surfaces are touching in the center, what is the radius of the 4th dark fringe from the center?

b) Do the radii of the fringes increase or decrease as the lens is slowly moved away from the concave spherical glass surface.

c) If the source has two wavelengths, 500 and 510 nm, of equal strength, the fringe visibility drops to zero for the mth dark fringe from the center for the 510 nm wavelength. What is the value of m?

Solution

$$\text{sag} = \frac{\rho^2}{2\,r_1} - \frac{\rho^2}{2\,r_2};\ \text{opd} = 2\,\text{sag};$$

$$r_1 = 15\,\text{m};\ \ r_2 = 30\,\text{m};\ \ \lambda = 0.5\times 10^{-6}\,\text{m};$$

■ a)

For the 4th dark fringe opd = 4 λ.

$$\text{radius} = \rho\ /.\ \text{Solve}[\text{opd} == 4\,\lambda,\ \rho\ //\ \text{N}]$$

$$\{-0.00774597\,\text{m},\ 0.00774597\,\text{m}\}$$

$$\text{radius}[[2]]\ \frac{10^3\,\text{mm}}{\text{m}}$$

$$7.74597\,\text{mm}$$

■ b)

For a given fringe the OPD is a constant. Therefore, as the separation is increased the fringe radii will decrease.

■ c)

If the fringe visibility drops to zero for the mth dark fringe from the center for wavelength λ_1 then

$$m\,\lambda_1 = \left(m + \frac{1}{2}\right)\lambda_2$$

$$\text{m} = \frac{\lambda_2}{2\,(\lambda_1 - \lambda_2)};$$

```
m /. {λ1 → 510, λ2 → 500}
25
```

I - 14

Newton's rings are observed with a quasi-monochromatic light of wavelength 500 nm. If the 20th bright fringe has a radius of 1 cm, what is the radius of curvature of the lens forming one part of the interfering system?

Solution

$$\text{sag} = \frac{\rho^2}{2\,r};\ \text{opd} = 2\,\text{sag};$$

$$\rho = 10^{-2}\,\text{m};\quad \lambda = 0.5\times 10^{-6}\,\text{m};$$

For the 20th dark fringe opd = 19.5 λ.

```
radiusOfCurvature = r /. Solve[opd == 19.5 λ, r // N]
{10.2564 m}
```

```
radiusOfCurvature[[1]]
```

```
10.2564 m
```

I - 15

a) Where are the fringes localized in a Michelson interferometer if

i) I have circular fringes?

ii) I have essentially straight fringes?

b) I adjust a Michelson interferometer to have white light fringes. Are the fringes circular or straight? Explain.

Solution

- **a)**

The circular fringes are fringes of equal inclination and they are localized at infinity. The straight fringes are fringes of equal thickness and they are localized in a region close to the location of the tilted mirrors in the Michelson.

- **b)**

For white light fringes the OPD must be close to zero. The fringes are fringes of equal thickness and they are nearly straight.

I - 16

a) I have broken the compensator plate in my Michelson interferometer. Why can't I compensate for the change in optical path by moving one of the mirrors an appropriate amount?
b) I shine a red light of wavelength 633 nm into my swimming pool. If the refractive index of the water is 1.33, what is the wavelength of the light under water?

Solution

- **a)**

Due to dispersion the OPD is a function of wavelength. To match paths for all wavelengths we must have the same dispersion in both paths. Therefore, we need the compensator plate.

- **b)**

$$\lambda = \frac{\lambda_o}{n} = \frac{633\,\text{nm}}{1.33} = 475.94\,\text{nm}$$

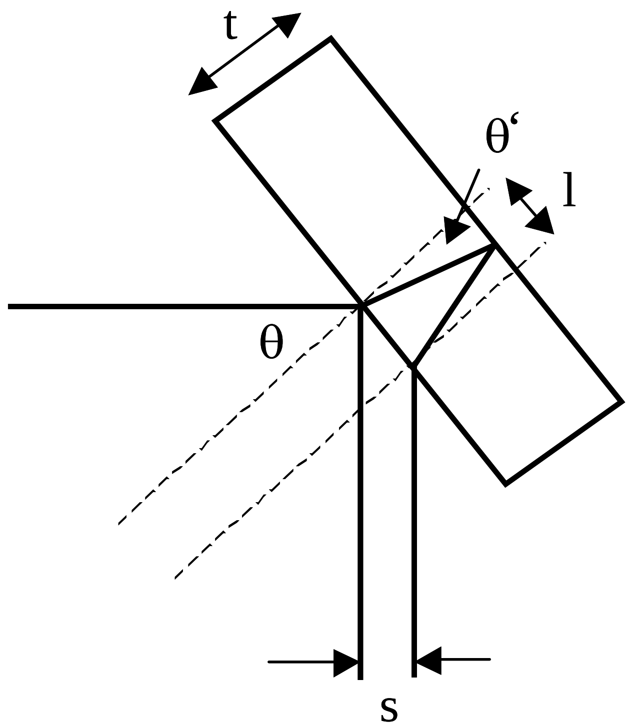
I - 17

Show that for a plane parallel plate (Murty) lateral shear interferometer the shear, S, is given by

$$S = \frac{t\,\text{Sin}[2\,\theta]}{\sqrt{n^2 - \text{Sin}[\theta]^2}}$$

where t is the plate thickness, θ is the angle of incidence, and n is the refractive index.

Solution



$l = 2\, t\, \mathrm{Tan}[\theta']$; $s = l\, \mathrm{Cos}[\theta]$;

$\mathrm{Sin}[\theta] = n\, \mathrm{Sin}[\theta']$; $\mathrm{Cos}[\theta'] = \sqrt{1 - \mathrm{Sin}[\theta']^2} = \dfrac{\sqrt{n^2 - \mathrm{Sin}[\theta]^2}}{n}$

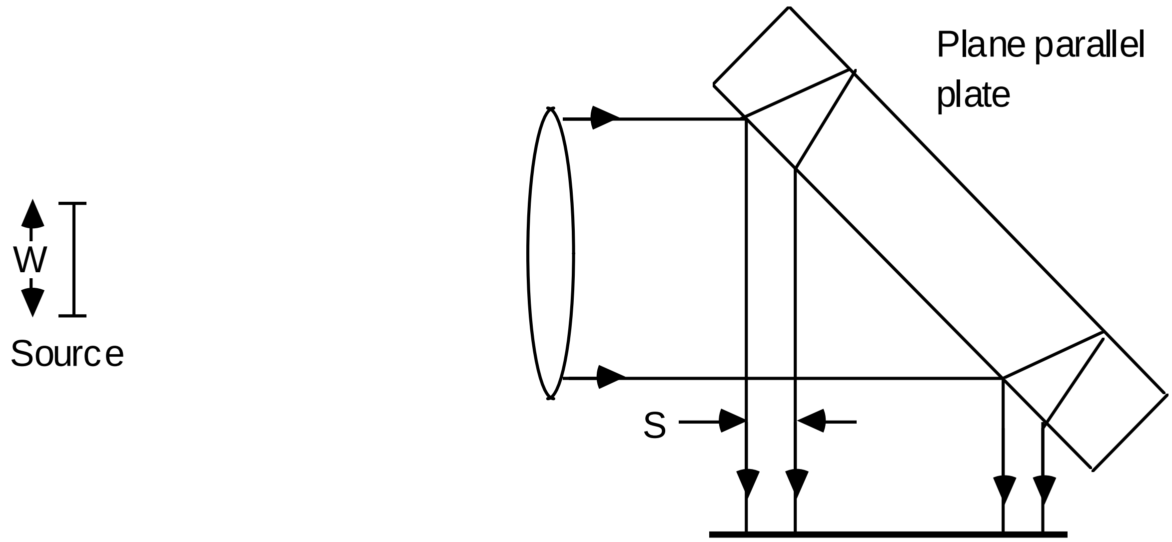
$s = 2\, t\, \mathrm{Tan}[\theta']\, \mathrm{Cos}[\theta] = \dfrac{2\, t\, \mathrm{Sin}[\theta]\, \mathrm{Cos}[\theta]}{\sqrt{n^2 - \mathrm{Sin}[\theta]^2}} = \dfrac{t\, \mathrm{Sin}[2\, \theta]}{\sqrt{n^2 - \mathrm{Sin}[\theta]^2}}$;

I - 18

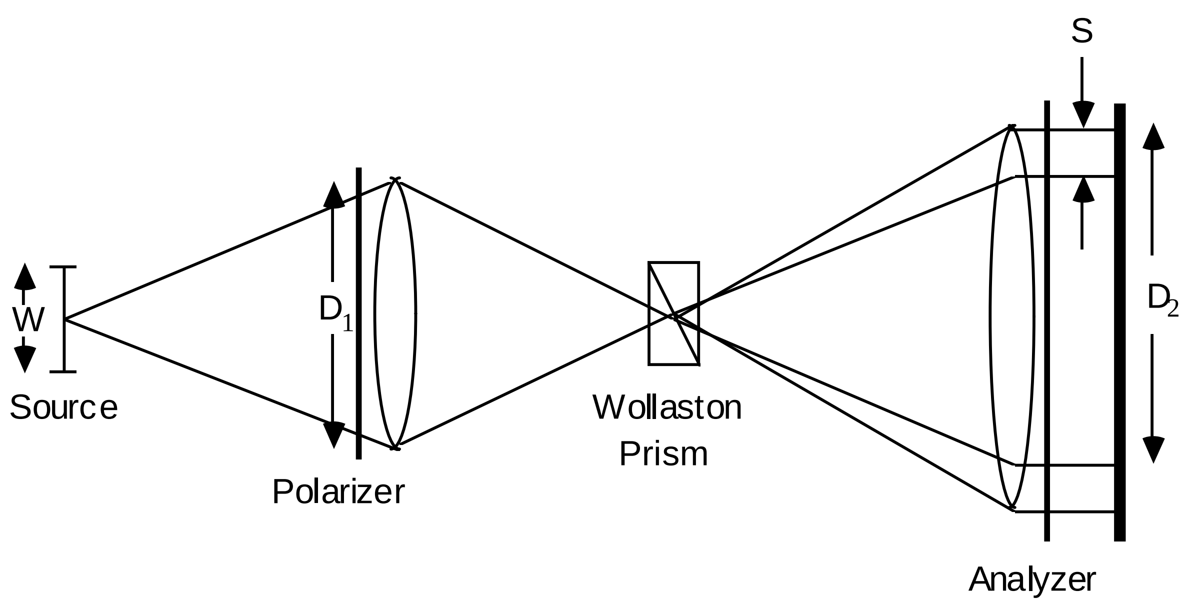
A quasi-monochromatic extended source of wavelength λ is used in a lateral shear interferometer setup as shown below. The source is square with a width W, the focal length of the lens is f, the lens diameter is D, and the shear is S. The interferometer is adjusted to give straight equi-spaced fringes.

a) Without changing the amount of relative shear, how would you adjust the interferometer setup to change the number of straight equi-spaced fringes.

b) If the extended source is replaced with a point source unity visibility fringes are produced. In terms of λ, W, f, D, S, and any other pertinent quantities, what fringe visibility is obtained when the extended source is used in the interferometer? Assume the source size is small compared to f and D.



c) Repeat part b for the following setup.



Solution

■ a)

Change the distance between the source and the lens.

■ b)

Using the Van Cittert Zernike theorem the visibility goes as the Fourier transform of the source distribution. For a slit source we know the Fourier transform is a sinc function.

An alternate way of calculating the fringe visibility is to break the extended source into a collection of point sources and add up the irradiances produce by all the point sources. If f is the focal length of the lens and x is the position along the extended source we can write the irradiance as

$$\mathbf{i} = \int_{-w/2}^{w/2} \left(1 + \mathrm{Cos}\left[\frac{2\pi}{\lambda}\, x\, \frac{s}{f} + \phi\right]\right) dx$$

$$w + \frac{f\,\lambda\,\mathrm{Cos}[\phi]\,\mathrm{Sin}\left[\frac{\pi\, s\, w}{f\,\lambda}\right]}{\pi\, s}$$

Thus, the visibility is given by

$$\text{visibility} = \frac{\mathrm{Sin}\left[\frac{\pi\, s\, w}{f\,\lambda}\right]}{\frac{\pi\, s\, w}{f\,\lambda}}$$

- **c)**

Same result as for b except f is replaced with L, where L is the distance between the source and the first lens.

I - 19

A plate having a wedge of 10 seconds of arc is used to check a 2 cm diameter helium neon laser beam for collimation. The plate produces a shear of 2 mm. Let the plate have a refractive index of 1.5 and let it be oriented such that the shear is horizontal.

a) If collimated light is present, horizontal fringes are obtained. What is the spacing of the fringes?

b) If the beam incident upon the beamsplitter has a radius of curvature of 100 meters, what angle, in degrees, will the fringes make with respect to the horizontal?

Solution

- **a)**

The wedge of the plate in radians is given by

$$\mathbf{wedge} = 10\ \text{seconds} \left(\frac{1\,°}{3600\ \text{seconds}}\right) \frac{\pi}{180\,°};$$

When we have collimated light the fringes are a result of only the wedge and not by the shear. The angle between the two interfering beams 2 n wedge.

```
angle = 2 n wedge /. {n -> 1.5};
```

The fringe spacing is given by

$$\mathbf{fringeSpacing} = \frac{\lambda}{\mathrm{Sin}[\text{angle}]}\ /.\ \lambda \to 0.633 \times 10^{-3}\ \text{mm}$$

4.35219 mm

b)

From above we know the opd due to the wedge is given by 2 n α x, where α is the wedge angle.

```
opdWedge = 2 n α y;
```

The opd due to the shear is

$$\texttt{opdShear} = \frac{x^2}{2r} - \frac{(x-s)^2}{2r} \;\texttt{// Simplify}$$

$$-\frac{s(s-2x)}{2r}$$

Therefore the opd is given by

```
opdWedge + opdShear
```

$$-\frac{s(s-2x)}{2r} + 2ny\alpha$$

For a given fringe this is a constant.

```
Solve[opd == opdWedge + opdShear, y]
```

$$\left\{\left\{y \to -\frac{-2\,\text{opd}\,r - s^2 + 2sx}{4nr\alpha}\right\}\right\}$$

Thus, the equation giving the loci of the fringes is

$$y = -\frac{-2\,\text{opd}\,r - s^2 + 2sx}{4nr\alpha};$$

The slope of the fringes is

$$\texttt{slope} = \frac{s}{2nr\alpha} \;\texttt{/. \{s -> 2 mm, n -> 1.5, r ->}\; 10^5 \;\texttt{mm, α -> wedge\}}$$

```
0.13751
```

The slope in degrees is given by

```
ArcTan[slope] / Degree
7.82963
```

I - 20

A lateral shear interferometer produces a lateral shear given by

$$\vec{S} = a\,\hat{\imath} + b\,\hat{\jmath}$$

Give an expression describing the loci of bright fringes if the phase distribution ϑ(x,y) of the wavefront under test is given by ϑ(x,y) = $A(x^2 + y^2)$, where A is a constant.

Solution

$2\pi m = A\,(x^2 + y^2) - A\,((x-a)^2 + (y-b)^2);$ m is an integer

$\mathbf{2\pi m = -A\,(a^2 + b^2 - 2\,a\,x - 2\,b\,y)}$

I - 21

Newton's fringes are observed with a quasi-monochromatic light of wavelength 500 nm. If the radius of curvature of the lens forming one part of the interfering system is 10 meters, what is the radius of the 20th bright fringe?

Solution

Due to phase change upon reflection the center of the fringe pattern will be a dark fringe. The OPD for the 20th bright fringe is 19.5 λ.

$\mathbf{sag = \frac{y^2}{2\,r};\ opd = 2\,sag;}$

$\mathbf{Solve\left[\frac{y^2}{r} == 19.5\,\lambda,\ y\right]\ /.\ \{r \to 1000\,mm,\ \lambda \to 0.5\times 10^{-3}\,mm\}}$

$\{\{y \to -3.1225\,mm\},\ \{y \to 3.1225\,mm\}\}$

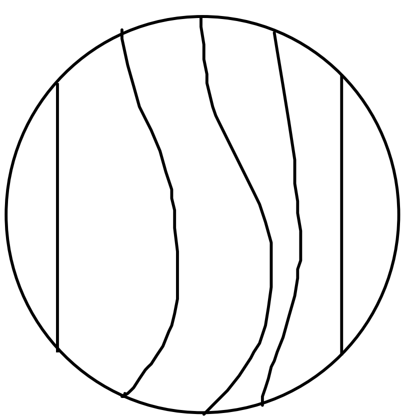
Radius of 20th bright fringe is 3.12 mm

I - 22

The following interferogram was obtained using a Twyman-Green to test in transmission a window having a refractive index of 1.5. The wavelength of the source is 633 nm.

a) What is the maximum thickness variation in units of microns in the window?

b) The same window is tested using a Mach-Zehnder interferometer. In 25 words or less, describe the difference between the Twyman-Green and the Mach-Zehnder interferograms.



Solution

a)

Maximum fringe error is approximately 1 fringe that corresponds to an OPD of 1 wave. In the Twyman-Green interferometer the window is tested in double pass. If Δt is the error in the thickness,

opd=2(n-1) Δt. Thus,

$$\Delta t = \frac{1\,\lambda}{2\,(1.5-1)} \;/.\; \lambda \to 0.633\,\mu m$$

$0.633\,\mu m$

b)

In the Mach-Zehnder the window is tested in single pass, so one-half the aberration is observed. Thus, the peak-valley error will be approximately one-half fringe.

I - 23

A radial shear interferometer giving a maximum shear of 10% of the diameter of the beam is used to test a wavefront having a phase OPD of the form $A(x^2+y^2)+B(x^2+y^2)^2$. A and B are constants. Give an expression describing the loci of the bright fringes.

Solution

If the maximum shear is 10% of the diameter it is 20% of the radius.

Let $r = \sqrt{x^2+y^2}$

$A\,r^2 + B\,r^4 - (A\,(0.8\,r)^2 + B\,(0.8\,r)^4)$

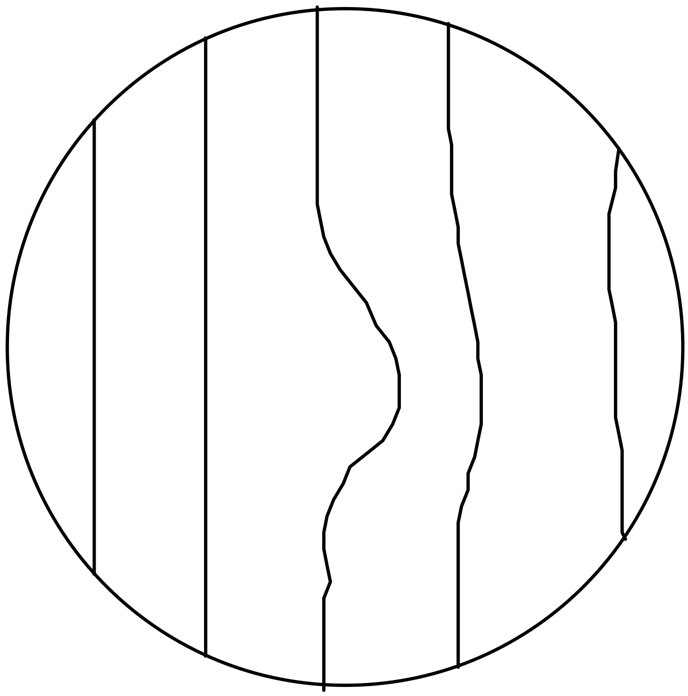
$0.36\,A\,r^2 + 0.5904\,B\,r^4$

I - 24

The following interferogram was obtained testing a nearly flat mirror in a Fizeau interferometer using a wavelength of 500 nm. When the right side of the reference surface is pressed downward toward the sample surface the number of fringes in the interferogram increases.

a) What is the peak-valley error, in units of microns, of the mirror surface?

b) Is the center of the mirror a high point or a low point? Explain.



Solution

■ a)

The error is approximately one-half fringe which would correspond to $\lambda/4$ surface height error or 0.125 μm.

■ b)

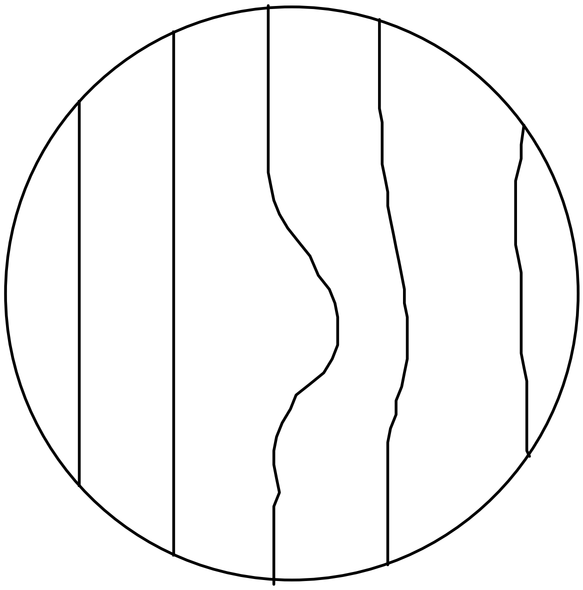
When the right side of the reference surface is pressed downward toward the sample surface the number of fringes in the interferogram increases so the open side of the wedge must be on the left. Since for a given fringe the separation between the two surfaces must be a constant, and the center part of the center fringe moves toward the thin portion of the wedge, the OPD in the middle is too much and we must have a low region in the middle of the mirror.

I - 25

The following interferogram was obtained testing a nearly flat mirror in a Fizeau interferometer using a wavelength of 500 nm. When the right side of the reference surface is pressed downward toward the sample surface the number of fringes in the interferogram increases.

a) What is the peak-valley error, in units of microns,of the mirror surface?

b) Is the center of the mirror a high point or a low point? Explain.



Solution

■ a)

Approximately $\frac{1}{2}$ fringe error so the surface height error is $\frac{1}{4}$ wave or 0.125 microns.

■ b)

The open side of the wedge must be on the left. Since for a given fringe the separation between the two surfaces is a constant the center of the mirror surface being tested must be a low region.

I - 26

A Fizeau interferometer is used to compare a spherical surface of 100 m radius of curvature with a flat mirror. The flat mirror is 100 mm in diameter and the spherical surface is 50 mm in diameter. The wavelength is 500 nm. Let the two surfaces touch at the center of the spherical surface.

a) Is the center fringe bright or dark? Explain.

b) Let there be no tilt between the two surfaces so circular fringes are obtained. If the two surfaces touch in the center of the spherical mirror, how many bright circular fringes are obtained?

c) What angle will the flat mirror have to be tilted relative to the center of the spherical surface so there are no closed fringes?

d) The interferometer is again adjusted for circular fringes. Do the circular fringes collapse to the center or expand from the center if the spherical surface is moved away from the flat surface.

Solution

■ a

The center fringe will be dark. There are two ways of thinking about this. First, if the two surfaces are in optical contact there will be no reflection and the fringe must be dark. Second, if the two surfaces are extremely close, but not in optical contact, the OPD will be essentially zero and one reflection will be from a high index to a low index and the second reflection will be from a low index to a high index. One reflected beam will have a phase change upon reflection of 180 degrees and the second will have a phase change upon reflection of 0 degrees and a dark fringe will result.

■ b

Since the spherical mirror has a long radius of curvature the sag can be approximated as $\frac{\rho^2}{2r}$ where ρ is the radius and r is the radius of curvature. When the surfaces are in contact there is a dark fringe at the center. The first bright fringe occurs when the separation between the two surfaces is 1/4 wavelength, and hence the OPD is 1/2 wavelength. After this a bright fringe will appear every time the separation between the two surfaces increases by 1/2 wavelength and the OPD changes by 1 wavelength. The number of bright circular fringes is given by

$$1 + \frac{\frac{\rho^2}{2\,r} - \frac{\lambda}{4}}{\frac{\lambda}{2}} \;/.\; \{\lambda \to 0.5\,10^{-6}\,\mathrm{m},\ \rho \to 25\,10^{-3}\,\mathrm{m},\ r \to 100\,\mathrm{m}\}$$

```
13.
```

Thus, there will be 13 bright circular fringes.

■ c

To not have any closed fringes the tilt must be large enough that no part of the spherical surface is parallel to the flat. The maximum slope of the sphere is given by

$$\mathrm{maxSlope} = \frac{\rho}{r} \;/.\; \{\rho \to 25\,10^{-3}\,\mathrm{m},\ r \to 100\,\mathrm{m}\} \;//\; N$$

```
0.00025
```

The maximum slope in arc-seconds is given by

$$\mathrm{maxSlope}\,\frac{1\,\mathrm{arcsec}}{5\,10^{-6}}$$

```
50 arcsec
```

Thus, the flat surface must be tilted at least 50 arc-seconds to not have any closed fringes.

- **d**

If the two surfaces touch at the center the spherical surface must be convex. For a given fringe the OPD is a constant. Therefore, for a convex surface the fringe radii will decrease as the convex surface is moved away from the flat.