Waves and Polarization

WP - 1

Show that $E[r, t] = \frac{A}{r}\,\mathrm{Cos}[k\,r - \omega\,t]$ is a solution to the wave equation.

Solution

3-D Wave Equation:

$$\nabla^2 \mathrm{E} - \frac{1}{\mathrm{c}^2}\,\frac{\partial^2 \mathrm{E}}{\partial \mathrm{t}^2} = 0 \qquad (1)$$

In polar coordinates:

$$\nabla^2 E = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial E}{\partial r}\right) + \frac{1}{r^2\,\mathrm{Sin}[\theta]}\,\frac{\partial}{\partial \theta}\left(\mathrm{Sin}[\theta]\,\frac{\partial E}{\partial \theta}\right) + \frac{1}{r^2\,\mathrm{Sin}[\theta]^2}\,\frac{\partial^2 E}{\partial \phi^2} \qquad (2)$$

For spherical waves there is no dependence on θ and ϕ so we have

$$\nabla^2 E[r, t] = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial E[r, t]}{\partial r}\right). \text{ Thus} \qquad (3)$$

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right)E[r, t] - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\,E[r, t] = 0 \qquad (4)$$

The problem is to show that

$$\mathrm{E}[r, t] = \frac{A}{r}\,\mathrm{Cos}[k\,r - \omega\,t] \qquad (5)$$

is a solution to the above equation if certain conditions concerning k, ω, and c are satisfied. I will use *Mathematica* to do the algebra.
Let

```
e[r_, t_] = a/r Cos[k r - ω t];
```

Then, substituting Eq. (5) into Eq. (4) yields

```
FullSimplify[1/r^2 ∂r (r^2 ∂r e[r, t]) - 1/c^2 ∂t,t e[r, t]]
```

$$\frac{a\,(-c^2\,k^2 + \omega^2)\,\mathrm{Cos}[k\,r - t\,\omega]}{c^2\,r}$$

Thus, E[r,t]=$\frac{A}{r}$ Cos[$k\,r - \omega\,t$] is a solution if $(-c^2\,k^2 + \omega^2)$ =0, or

$$k^2 = \frac{\omega^2}{c^2}.$$

WP - 2

Suppose that a reflecting surface has a complex reflection coefficient $\rho = \rho_o\, e^{i\delta}$. This means that if a normal incident electric field is given by

E_i(x,t)=A cos(kx - ωt + ϕ)

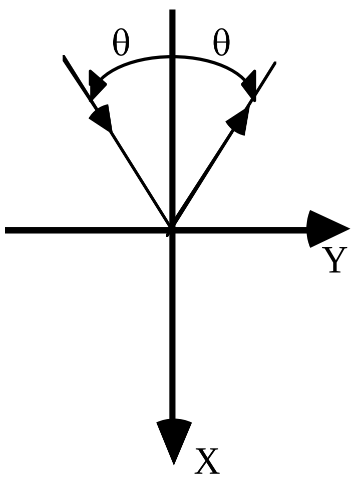
the reflected wave is given by

$E_r(x,\,t) = \rho_o$ A cos(-kx - ωt + ϕ + δ)

Let a plane wave be incident at an angle to the plane reflecting surface. Determine E_i(x,y,t) and E_r(x,y,t). Show that the total field can be written as

E_{tot} =A $(1 - \rho_o)$ cos[k (xcosθ +y sinθ) - ωt + ϕ] + 2 ρ_o A cos[ky sinθ - ωt + ϕ + δ/2] cos[kx cosθ - δ/2].

Interpret this result for the following four cases and find the spatial frequency of the standing wave component: i) θ = 0, ρ_o = 1, ii) θ = 0, $\rho_o \neq 1$, iii) $\theta \neq 0$, ρ_o = 1, iv) $\theta \neq 0$, $\rho_o \neq 1$.



Solution

```
Eᵢ = a Cos[k (x Cos[θ] + y Sin[θ]) - ω t + ϕ];
Eᵣ = ρₒ a Cos[k (-x Cos[θ] + y Sin[θ]) - ω t + ϕ + δ];
```

$\mathbf{E_{tot} = E_i + E_r}$

$a\,\mathrm{Cos}[\phi - t\,\omega + k\,(x\,\mathrm{Cos}[\theta] + y\,\mathrm{Sin}[\theta])] + a\,\mathrm{Cos}[\delta + \phi - t\,\omega + k\,(-x\,\mathrm{Cos}[\theta] + y\,\mathrm{Sin}[\theta])]\,\rho_o$

While this is the correct answer, it is not in the form we want.

If we let
α = k y Sin[θ] - ω t + ϕ + δ/2 and
β = k x Cos[θ] - δ/2 then

$\mathbf{E_{tot} = a\,Cos[\alpha + \beta] + \rho_o\,a\,Cos[\alpha - \beta]}$

$a\,\mathrm{Cos}[\alpha + \beta] + a\,\mathrm{Cos}[\alpha - \beta]\,\rho_o$

Subtracting and adding (a ρ_o Cos[$\alpha + \beta$]) yields

$\mathbf{E_{tot} = Factor[a\,(Cos[\alpha + \beta] - \rho_o\,Cos[\alpha + \beta])] + TrigExpand[\rho_o\,a\,(Cos[\alpha - \beta] + Cos[\alpha + \beta])]}$

$-a\,\mathrm{Cos}[\alpha + \beta]\,(-1 + \rho_o) + 2\,a\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\beta]\,\rho_o$

$\mathbf{E_{tot} = E_{tot}\ /.\ \{\alpha \to k\,y\,Sin[\theta] - \omega\,t + \phi + \delta/2,\ \beta \to k\,x\,Cos[\theta] - \delta/2\}}$

$-a\,\mathrm{Cos}[\phi - t\,\omega + k\,x\,\mathrm{Cos}[\theta] + k\,y\,\mathrm{Sin}[\theta]]\,(-1 + \rho_o) + 2\,a\,\mathrm{Cos}\left[\frac{\delta}{2} - k\,x\,\mathrm{Cos}[\theta]\right]\,\mathrm{Cos}\left[\frac{\delta}{2} + \phi - t\,\omega + k\,y\,\mathrm{Sin}[\theta]\right]\,\rho_o$

which is the result we wanted.

i) $\theta = 0$, $\rho_o = 1$

$\mathbf{E_{tot}\ /.\ \{\theta \to 0,\ \rho_o \to 1\}}$

$2\,a\,\mathrm{Cos}\left[k\,x - \frac{\delta}{2}\right]\,\mathrm{Cos}\left[\frac{\delta}{2} + \phi - t\,\omega\right]$

Normal standing wave having zeros spaced $\lambda/2$.

ii) $\theta = 0$, $\rho_o \neq 1$

$\mathbf{E_{tot}\ /.\ \{\theta \to 0\}}$

$-a\,\mathrm{Cos}[k\,x + \phi - t\,\omega]\,(-1 + \rho_o) + 2\,a\,\mathrm{Cos}\left[k\,x - \frac{\delta}{2}\right]\,\mathrm{Cos}\left[\frac{\delta}{2} + \phi - t\,\omega\right]\,\rho_o$

Same standing wave pattern as in i), but reduced amplitude. Also, have travelling wave in +x direction.

iii) $\theta \neq 0$, $\rho_o = 1$

$\mathbf{E_{tot}\ /.\ \{\rho_o \to 1\}}$

$2\,a\,\mathrm{Cos}\left[\frac{\delta}{2} - k\,x\,\mathrm{Cos}[\theta]\right]\,\mathrm{Cos}\left[\frac{\delta}{2} + \phi - t\,\omega + k\,y\,\mathrm{Sin}[\theta]\right]$

We have a standing wave in the x direction. The phase of the standing wave varies as a sinusoid in the y direction.

- **iv) $\theta \neq 0$, $\rho_o \neq 1$**

$\mathbf{E_{tot}}$

$$-a \operatorname{Cos}[\phi - t\,\omega + k\,x \operatorname{Cos}[\theta] + k\,y \operatorname{Sin}[\theta]]\,(-1 + \rho_o) + 2\,a \operatorname{Cos}\left[\frac{\delta}{2} - k\,x \operatorname{Cos}[\theta]\right] \operatorname{Cos}\left[\frac{\delta}{2} + \phi - t\,\omega + k\,y \operatorname{Sin}[\theta]\right] \rho_o$$

We have a superposition of case iii) with a plane wave travelling along the incident direction with reduced amplitude.

WP - 3

A given argon laser has a cw output of 1 watt. The output beam has a Gaussian amplitude distribution which is truncated at the point where the amplitude falls to 1/e its axial value. The resulting beam diameter is 2 mm.

a) What is the on-axis electric field?

b) What would the on-axis electric field be if we had a uniform amplitude distribution across the beam? (1 watt total power)

Solution

- **a)**

The Poynting vector is given by

$$< \vec{S} >= \frac{1}{2}\,\epsilon_o\,c\,E^2 \quad ; \quad E = E_o\,e^{-r^2/r_o^2}$$

where r_o = 1 mm.

The total power is given by

$$P = \int < \vec{S} > \,dA = \frac{1}{2}\,\epsilon_o\,c \int_0^{r_o}\int_0^{2\pi} E_o^2\,e^{-2\,r^2/r_o^2}\,r\,dr\,d\theta$$

$$P = \pi\,\epsilon_o\,c\,E_o^2 \int_0^{r_o} e^{-2\,r^2/r_o^2}\,r\,dr = \pi\,\epsilon_o\,c\,E_o^2\left(\frac{1}{-4/r_o^2}\right)(e^{-2} - 1)$$

Solving for the square of the on-axis electric field yields

$$E_o{}^2 = \frac{4\,P}{\pi\,\epsilon_o\,c\,r_o{}^2\,(1 - e^{-2})}$$

$$E_o = \frac{2}{r_o}\sqrt{\frac{P}{\pi\,\epsilon_o\,c\,(1 - e^{-2})}} = \frac{2}{10^{-3}}\sqrt{\frac{1}{\pi\,8.85\,10^{-12}\,(3\,10^8)\,(1 - e^{-2})}}\;\frac{\text{volt}}{m}$$

The on-axis electric field is then

$$E_o = 23550.4\,\frac{\text{volt}}{m}$$

- **b)**

$$\frac{1}{2}\,\epsilon_o\,c\,E_o{}^2\;\pi\,r^2 = 1\,\text{watt}$$

$$E_o = \sqrt{\frac{2}{\epsilon_o\,c\,\pi}}\;\frac{1}{10^{-3}} = \sqrt{\frac{2}{8.85\,10^{-12}\,(3\,10^8)\,\pi}}\;\frac{1}{10^{-3}}\;\frac{\text{volt}}{m} = 15485\,\frac{\text{volt}}{m}$$

WP - 4

I have two electric fields

$$\overrightarrow{E_1} = E_1\,\hat{a}_1\,e^{i\phi_1}\,e^{i(\mathrm{kz}-\omega \mathrm{t})}$$

and

$$\overrightarrow{E_2} = E_2\,\hat{a}_2\,e^{i\phi_2}\,e^{i(\mathrm{kz}-\omega \mathrm{t})}$$

where $\hat{a}_1$ and $\hat{a}_2$ are unit vectors in the direction of oscillation of the electric field. By breaking the electric field into components in the x and y directions show that the time average irradiance is proportional to

$$E_1^2 + E_2^2 + 2\,E_1\,E_2(\hat{a}_1\cdot\hat{a}_2)\,\mathrm{Cos}[\phi_2 - \phi_1].$$

Solution

First we need to break the electric field into x and y components. Let one electric field be at an angle θ_1 with respect to the x-axis and the second be at an angle θ_2 with respect to the x-axis. Then.

$$\mathbf{e1} = \begin{pmatrix} \mathbf{e_1\,Cos[\theta_1]}\,e^{i\,\phi_1}\,e^{i\,(k\,z-\omega\,t)} \\ \mathbf{e_1\,Sin[\theta_1]}\,e^{i\,\phi_1}\,e^{i\,(k\,z-\omega\,t)} \end{pmatrix};$$

and

$$\mathbf{e2} = \begin{pmatrix} \mathbf{e_2\,Cos[\theta_2]}\,e^{i\,\phi_2}\,e^{i\,(k\,z-\omega\,t)} \\ \mathbf{e_2\,Sin[\theta_2]}\,e^{i\,\phi_2}\,e^{i\,(k\,z-\omega\,t)} \end{pmatrix};$$

Next we need to add the two x-components and the two y-components.

```
e3 = e1 + e2;
MatrixForm[e3]
```

$$\begin{pmatrix} e^{i(kz-t\omega)+i\phi_1}\,\mathrm{Cos}[\theta_1]\,e_1 + e^{i(kz-t\omega)+i\phi_2}\,\mathrm{Cos}[\theta_2]\,e_2 \\ e^{i(kz-t\omega)+i\phi_1}\,\mathrm{Sin}[\theta_1]\,e_1 + e^{i(kz-t\omega)+i\phi_2}\,\mathrm{Sin}[\theta_2]\,e_2 \end{pmatrix}$$

The irradiance is obtained by multiplying each component by its complex conjugate.

```
e4 = ComplexExpand[e3 Conjugate[e3]];
MatrixForm[Simplify[e4]]
```

$$\begin{pmatrix} \mathrm{Cos}[\theta_1]^2\,e_1^2 + 2\,\mathrm{Cos}[\theta_1]\,\mathrm{Cos}[\theta_2]\,\mathrm{Cos}[\phi_1-\phi_2]\,e_1\,e_2 + \mathrm{Cos}[\theta_2]^2\,e_2^2 \\ \mathrm{Sin}[\theta_1]^2\,e_1^2 + 2\,\mathrm{Cos}[\phi_1-\phi_2]\,\mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]\,e_1\,e_2 + \mathrm{Sin}[\theta_2]^2\,e_2^2 \end{pmatrix}$$

The total irradiance is given by the sum of the irradiance for the x-component and the irradiance for the y-component.

```
Simplify[Plus @@ e4]
```

$$\{e_1^2 + 2\,\mathrm{Cos}[\theta_1-\theta_2]\,\mathrm{Cos}[\phi_1-\phi_2]\,e_1\,e_2 + e_2^2\}$$

Since Cos[$\theta_1 - \theta_2$] is equal to the dot product of the two unit vectors $\hat{a}_1$ and $\hat{a}_2$, we have the desired result.

WP - 5

Prove using Jones calculus that a retarder at 45^o between vertical and horizontal quarter-wave plates is equivalent to a rotator. (Orientation of sandwich not important.)

Solution

First we need to define rotation matrices and the Jones matrices for quarter-wave plates and retarders.

- **Quarter-wave plate with fast axis vertical**

$$\text{qfavJones} = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix};$$

- **Quarter-wave plate with fast axis horizontal**

$$\text{qfahJones} = e^{-i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix};$$

- **Retarder with fast axis vertical**

$$\text{rfavJones}[\phi\_] := e^{i\phi/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Rotation Matrix

```
rotJones[θ_] := ( Cos[θ]   Sin[θ] )
                ( -Sin[θ]  Cos[θ] )
```

The retarder must be rotated so it is at 45 degrees.

```
ret45Jones = rotJones[-45 Degree].rfavJones[ϕ].rotJones[45 Degree];
```

Put the retarder between the vertical and horizontal quarter-wave plates to show that we end up with a rotation matrix.

```
rotation = FullSimplify[qfahJones.ret45Jones.qfavJones];
MatrixForm[rotation]
```

$$\begin{pmatrix} \text{Cos}[\frac{\phi}{2}] & \text{Sin}[\frac{\phi}{2}] \\ -\text{Sin}[\frac{\phi}{2}] & \text{Cos}[\frac{\phi}{2}] \end{pmatrix}$$

Thus, we have rotation matrix of angle $\phi/2$.

Now we want to rotate the quarter-wave plate, retarder, quarter-wave plate sandwich, to show that it's orientation does not change it's effect. We will rotate it an angle α.

```
MatrixForm[FullSimplify[rotJones[-α].rotation.rotJones[α]]]
```

$$\begin{pmatrix} \text{Cos}[\frac{\phi}{2}] & \text{Sin}[\frac{\phi}{2}] \\ -\text{Sin}[\frac{\phi}{2}] & \text{Cos}[\frac{\phi}{2}] \end{pmatrix}$$

Thus, the orientation of the sandwich is not important.

WP - 6

Use Jones calculus to show that a half-wave plate converts right handed circularly polarized light into left handed circularly polarized light and the phase of the light can be changed by rotating the half-wave plate.

Solution

Right handed circular polarization can be represented by

$$\text{rcJones} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix};$$

A half wave plate with the fast axis horizontal can be written as

$$\text{rfahJones} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

The rotation matrix can be written as

```
rotJones[θ_] := ( Cos[θ]   Sin[θ]
                 -Sin[θ]   Cos[θ] )

outputPolarization = rotJones[-θ].rfahJones.rotJones[θ].rcJones;

TrigToExp[TrigFactor[outputPolarization]] // MatrixForm
```

$$\begin{pmatrix} -\frac{i e^{-2 i \theta}}{\sqrt{2}} \\ \frac{e^{-2 i \theta}}{\sqrt{2}} \end{pmatrix}$$

But this is equal to

$$-\frac{i e^{-2 i \theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Thus, right handed circular has been changed to left handed circular and the phase is changed at twice the rotation rate of the half-wave plate.

WP - 7

Derive a Mueller matrix for a half-wave plate having a vertical fast axis. Utilize your result to convert a right-handed state into a left-handed state. Verify that the same wave plate will convert a left-handed state to a right-handed state. Advancing or retarding the relative phase by π should have the same effect. Check this by deriving the matrix for a horizontal fast axis half-wave plate.

Solution

- ### Mueller matrix for a half-wave plate having a vertical fast axis

The Mueller matrix for a quarter-wave plate having the fast axis vertical is

$$\text{qfavMueller} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

Thus, the Mueller matrix for a half-wave plate having the fast axis vertical is

```
hfavMueller = qfavMueller. qfavMueller; hfavMueller // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Stokes vectors for right and left handed circular polarization are given by

```
rcStokes = {{1},{0},{0},{1}}; lcStokes = {{1},{0},{0},{-1}};
```

$$\text{rcStokes} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \ \text{lcStokes} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix};$$

Convert right to left

```
hfavMueller . rcStokes // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Convert left to right

```
hfavMueller . lcStokes // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- **Mueller matrix for a half-wave plate having a horizontal fast axis**

The Mueller matrix for a quarter-wave plate having the fast axis horizontal is

$$\text{qfahMueller} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

Thus, the Mueller matrix for a half-wave plate having the fast axis horizontal is

```
hfahMueller = qfahMueller. qfahMueller;
```

```
hfahMueller // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus the Mueller matrix for a half-wave plate having the fast axis horizontal is the same as the Mueller matrix for a half-wave plate having the fast axis vertical.

WP - 8

a) Assume I want to rotate the polarization of a plane polarized wavefront 90^o by using only "perfect" linear polarizers.What is the maximum flux transmission I can obtain using only two polarizers? What is the minimum number of perfect polarizers required to have a flux transmission greater than 95%?

b) The indices of refraction for quartz for the sodium yellow line are n_o = 1.544 and n_e = 1.553. Calculate the thickness of a quarter-wave plate made from quartz.

Solution

■ a)

From the Malus Law

$$\texttt{trans[n\_] := } \mathrm{Cos}\left[\frac{90\,°}{n}\right]^{2n}$$

If there are two polarizers,

$$\texttt{trans[2]} = \frac{1}{4}$$

Determining n for a flux transmission greater than 95%

```
For[n = 1, trans[n] ≤ 0.95, n++];
Print[n, "  ", N[trans[n]]]
```

```
49  0.950883
```

Thus, we need 49 perfect polarizers.

■ b)

$$\texttt{Solve}\left[(1.553 - 1.544)\, t == \frac{0.589\,\mu m}{4},\ t\right]$$

$$\{\{t \to 16.3611\,\mu m\}\}$$

Can also have a thickness of (4 m+1) 16.3611 μm, where m is an integer since (4) 16.3611 μm is a wave plate.

WP - 9

Use Jones calculus to show that two half-wave plates at angle θ between them are equivalent to a rotator through angle 2θ.

Solution

The matrix for a retarder with the fast axis vertical is

$$\texttt{rfavJones}[\phi\_] := e^{i\phi/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

The rotation matrix is given by

```
rotJones[θ_] := ( Cos[θ]   Sin[θ]
                 -Sin[θ]   Cos[θ] )
```

The matrix for two half-wave plates at an angle θ between them is given by

```
MatrixForm[TrigReduce[rotJones[-θ].rfavJones[π].rotJones[θ].rfavJones[π]]]
```

$$\begin{pmatrix} -\text{Cos}[2\,\theta] & \text{Sin}[2\,\theta] \\ -\text{Sin}[2\,\theta] & -\text{Cos}[2\,\theta] \end{pmatrix}$$

Since this is minus the rotation matrix for a rotation of -2 θ we have the result we want.

WP - 10

Let $E_x = A_x$ cos(kz-ωt) and $E_y = A_y$ cos(kz-ωt+ϕ). Show that

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\left(\frac{E_x}{A_x}\right)\left(\frac{E_y}{A_y}\right)\text{Cos}[\phi] = \text{Sin}[\phi]^2$$

Solution

```
ex = ax Cos[k z - ω t];  ey = ay Cos[k z - ω t + ϕ];
```

$$\text{FullSimplify}\left[\left(\frac{e_x}{a_x}\right)^2 + \left(\frac{e_y}{a_y}\right)^2 - 2\left(\frac{e_x}{a_x}\right)\left(\frac{e_y}{a_y}\right)\text{Cos}[\phi]\right]$$

$$\text{Sin}[\phi]^2$$

Therefore,

$$\left(\frac{e_x}{a_x}\right)^2 + \left(\frac{e_y}{a_y}\right)^2 - 2\left(\frac{e_x}{a_x}\right)\left(\frac{e_y}{a_y}\right)\text{Cos}[\phi] = \text{Sin}[\phi]^2$$

WP - 11

Show that in terms of the Stokes parameters the degree of polarization can be written as

$$V = \frac{(s_1^2 + s_2^2 + s_3^2)^{1/2}}{s_o}$$

Solution

$$\text{Degree of Polarization} = V = \frac{\text{Polarized Light}}{\text{Total Amount of Light}}$$

Let the total amount of light be s_o

For the unpolarized portion of the light

$$\begin{pmatrix} s_o - \sqrt{s_1^2 + s_2^2 + s_3^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For the polarized portion of the light

$$\begin{pmatrix} \sqrt{s_1^2 + s_2^2 + s_3^2} \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

Therefore,

$$V = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_o}$$

WP - 12

A linearly polarized laser source is used with a Twyman-Green interferometer. A 50-50 non-polarization sensitive beamsplitter is used in the interferometer. The flat mirror in the reference arm of the interferometer is tilted slightly to give straight equi-spaced fringes in the output. A quarter-wave plate is placed in the test arm of the interferometer. The fast axis of the quarter-wave plate makes an angle θ with respect to the direction of polarization of the light incident upon the quarter-wave plate. A perfect polarizer is placed in the output of the interferometer. The transmission axis of the polarizer is in the direction of polarization of the light coming from the reference arm.

a) Calculate the irradiance of the interference pattern as a function of θ. What is the fringe visibility as a function of θ?

b) Assume the polarizer is non-perfect in that it transmits 95% of the light having a polarization along the transmission axis of the polarizer and 5% of the light having the orthogonal polarization. What is the observed fringe visibility as a function of θ?

Solution

Since the light goes through the quarter-wave plate twice it will act as a half-wave plate and it will rotate the direction of polarization by an angle 2θ.

a)

The irradiance of the interference pattern is given by

$i = i_1 + i_2 + 2\sqrt{i_1 i_2}\,\mathrm{Cos}[\phi]$, where $i_2 = i_1\,\mathrm{Cos}[2\,\theta]^2$

$$i = i_1 \left(1 + \text{Cos}[2\,\theta]^2 + 2\,\text{Cos}[2\,\theta]\,\text{Cos}[\phi]\right)$$

$$\text{visibility} = \frac{i_{\text{max}} - i_{\text{min}}}{i_{\text{max}} + i_{\text{min}}} = \frac{2\,\text{Cos}[2\,\theta]}{1 + \text{Cos}[2\,\theta]^2}$$

■ **b)**

$$i = i_1 \left(\left(1 + \text{Cos}[2\,\theta]^2 + 2\,\text{Cos}[2\,\theta]\,\text{Cos}[\phi]\right)(0.95) + \text{Sin}[2\,\theta]^2\,(0.05)\right)$$

$$\text{visibility} = \frac{0.95\,(2\,\text{Cos}[2\,\theta])}{0.95\,(1 + \text{Cos}[2\,\theta]^2) + (0.05)\,\text{Sin}[2\,\theta]^2}$$

WP - 13

a) What are the four filters we associate with the four Stokes parameters?

b) Suppose that an ideal polarizer is rotated at a rate ω between a similar pair of stationary crossed polarizers. Give the ratio of the modulation frequency of the emergent flux density to the rotation frequency of the polarizer.

c) I am working on a problem involving partially polarized light. Which should I use, Jones or Mueller matrices? Explain (briefly).

d) I am working on a problem using completely polarized light where I want to keep track of the phase of the beam. Should I use Jones or Mueller matrices? Explain (briefly).

Solution

■ **a)**

All filters transmit 50% of natural radiation.

i) isotropic filter

ii) linear horizontal polarization filter

i) linear polarization at 45° filter

i) right-handed circular polarization filter

■ **b)**

As the polarizer rotates 360° no light will be transmitted at 4 orientations. Therefore, the ratio is 4.

- **c)**

Mueller matrices should be used since Jones matrices can be used only with completely polarized light.

- **d)**

Jones matrices should be used since they keep track of the phase.

WP - 14

A linearly polarized source is used with a Young's two-pinhole interferometer. The same amount of light is transmitted through each pinhole. A half-wave plate is placed over one pinhole. The fast axis of the half-wave plate makes an angle θ with respect to the direction of polarization of the light incident upon the half-wave plate. A perfect polarizer is placed in the output of the interferometer. The transmission axis of the polarizer is in the direction of polarization of the light coming from the pinhole without the half-wave plate.

a) What is the fringe visibility as a function of θ?

b) Assume the polarizer is non-perfect in that it transmits 95% of the light having a polarization along the transmission axis of the polarizer and 5% of the light having the orthogonal polarization. What is the observed fringe visibility as a function of θ?

Solution

The irradiance of the two-beam interference pattern is given by

$$i = i_1 + i_2 + 2\sqrt{i_1 i_2}\,\mathrm{Cos}[\phi];$$

The half-wave plate rotates the direction of polarization by an angle 2 θ. The irradiance of the beam passing through the half-wave plate and the polarizer is given by

$$i_2 = i_1\,\mathrm{Cos}[2\theta]^2;$$

The irradiance of the two-beam interference pattern is then given by

$$i = i_1\,(1 + \mathrm{Cos}[2\theta]^2 + 2\,\mathrm{Cos}[2\theta]\,\mathrm{Cos}[\phi])$$

- **a)**

The visibility is given by

$$v = \frac{i_{max} - i_{min}}{i_{max} + i_{min}} = \frac{4\,\mathrm{Cos}[2\theta]}{2\,(1 + \mathrm{Cos}[2\theta]^2)} = \frac{2\,\mathrm{Cos}[2\theta]}{1 + \mathrm{Cos}[2\theta]^2}$$

- **b)**

The irradiance of the main interference fringe pattern is reduced by a factor of 0.95 and we have to add in 0.05 times the irradiance of the orthogonally polarized beam.

$$i = i_1 \left(\left(1 + \text{Cos}[2\,\theta]^2 + 2\,\text{Cos}[2\,\theta]\,\text{Cos}[\phi]\right) 0.95 + \text{Sin}[2\,\theta]^2\,(0.05)\right)$$

$$v = \frac{i_{max} - i_{min}}{i_{max} + i_{min}} = \frac{0.95\,(2\,\text{Cos}[2\,\theta])}{0.95\,(1 + \text{Cos}[2\,\theta]^2) + (0.05)\,\text{Sin}[2\,\theta]^2}$$

WP - 15

a) Using less than 25 words define polarization angle.
b) Using less than 25 words define critical angle.
c) What is the approximate power reflectance at normal incidence for common glass?
d) What is the approximate power reflectance at a nearly 90 degree angle of incidence for common glass?

Solution

- **a)**

Angle of incidence for which only the polarization component polarized normal to the incident plane, and therefore parallel to the surface, is reflected

- **b)**

When light passes from a medium of lower index to a medium of higher index the angle of refraction is always less than the angle of incidence. When the incident rays approach an angle of 90° the refracted rays approach a fixed angle called the critical angle.

- **c)**

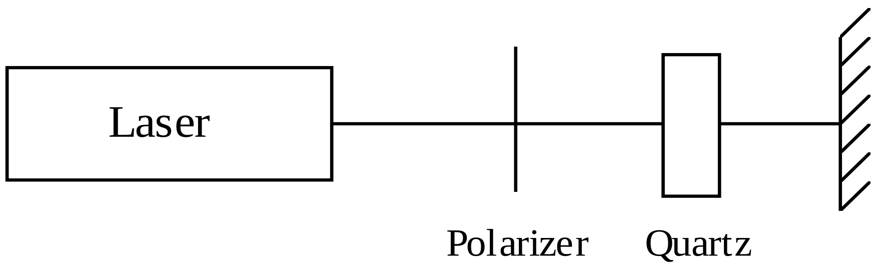
4%

- **d)**

100%

WP - 16

A quartz plate in the setup shown below has its axes at 45 degrees relative to the transmission axis of the linear polarizer. The indices of refraction for quartz are n_o = 1.544 and n_e = 1.553. The laser wavelength is 633 nm.
a) What is the minimum quartz thickness for having no light reflected back to the laser?
b) What is the minimum quartz thickness (other than zero) for having the maximum amount of light reflected back to the laser?



Solution

- **a)**

The quartz plate needs to rotate the direction of polarization by 90 degrees. Thus, we want the quartz plate to act as a half-wave plate. Since the light is going through the quartz plate twice the quartz plate must be a quarter-wave plate.

$$\text{Solve}\left[(1.553 - 1.544)\, t == \frac{0.633\,\mu\text{m}}{4}, t\right]$$

$\{\{t \to 17.5833\,\mu\text{m}\}\}$

- **b)**

We want the quartz plate to act as a wave-plate. Since the light is going through the quartz plate twice the quartz plate must be a half-wave plate.

$$\text{Solve}\left[(1.553 - 1.544)\, t == \frac{0.633\,\mu\text{m}}{2}, t\right]$$

$\{\{t \to 35.1667\,\mu\text{m}\}\}$

WP - 17

A 633 nm wavelength linearly polarized source is used with a Young's two-pinhole interferometer. The same amount of light is transmitted through each pinhole. A 17.58 micron thick quartz plate is placed over one pinhole where the axes of the quartz plate are at 45 degrees relative to the direction of polarization of the incident light. The indices of refraction for quartz are no=1.544 and ne=1.553. If when the quartz plate is removed the fringe contrast is unity, what is the fringe contrast with the quartz plate in place?

Solution

Calculation of retardation of quartz plate in units of the wavelength

$$\frac{\triangle n\, t}{\lambda} = \frac{(1.553 - 1.544)\ 17.58\ \mu\text{m}}{0.633\ \mu\text{m}} = \frac{1}{4}$$

Therefore, one beam has circular polarization. The circularly polarized beam can be thought of as two linearly polarized beams with the amplitude of each beam is $\frac{1}{\sqrt{2}}$ that of the incident beam.

$$\mathbf{e} = \mathbf{e_o}\left(e^{i(\omega t+\phi)} + \frac{1}{\sqrt{2}}\,e^{i(\omega t)}\right)\hat{\mathbf{i}} + \frac{\mathbf{e_o}}{\sqrt{2}}\,e^{i(\omega t)}\,\hat{\mathbf{j}}$$

$$\mathbf{i} = \mathbf{i_o}\left(1 + \frac{1}{2} + \frac{2}{\sqrt{2}}\,\text{Cos}[\phi]\right) + \frac{\mathbf{i_o}}{2}$$

$$\textbf{contrast} = \frac{\mathbf{i_{max}} - \mathbf{i_{min}}}{\mathbf{i_{max}} + \mathbf{i_{min}}} = \frac{2\,\frac{2}{\sqrt{2}}}{4} = \frac{1}{\sqrt{2}}$$