Aspheric Surface Testing

AS-1

A scatterplate interferometer is used to perform a null test of a parabolic mirror.

a) Sketch the test setup.

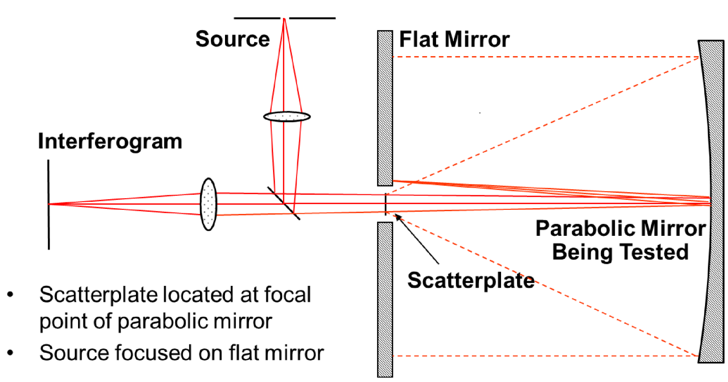
b) How does lateral displacement of the scatterplate change the interference fringes?

c) How does longitudinal displacement of the scatterplate change the interference fringes?

d) How does a scatterplate differ from a simple piece of ground glass?

Solution

a)



b)

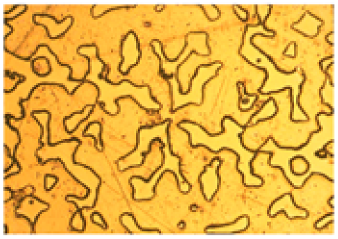
Tilt will be introduced.

c)

Defocus will be introduced.

d)

The scatterplate has inversion symmetry.



AS-2

One way of testing an aspheric surface is to use a computer-generated hologram.

a) Sketch the interferometric setup used to test an aspheric mirror using a CGH. Make sure to note which orders are used.

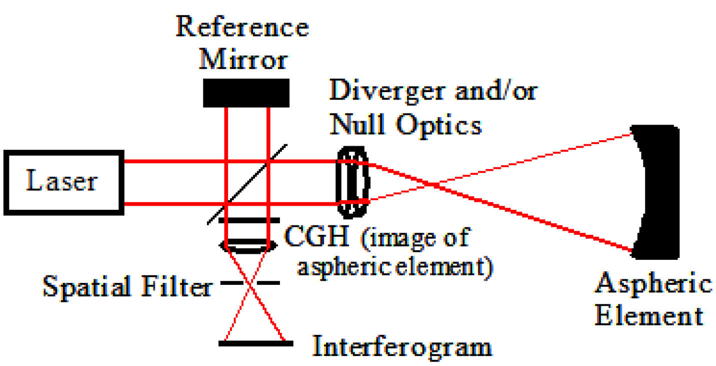
b) How would you test the quality of a particular plotter used to generate a CGH? Include diagram.

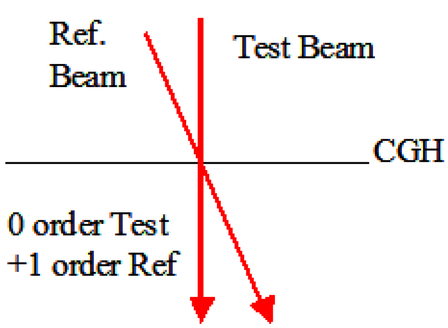
c) An important quantity in determining how large of an aberration that can be measured is the space-bandwidth product. Why is it important?

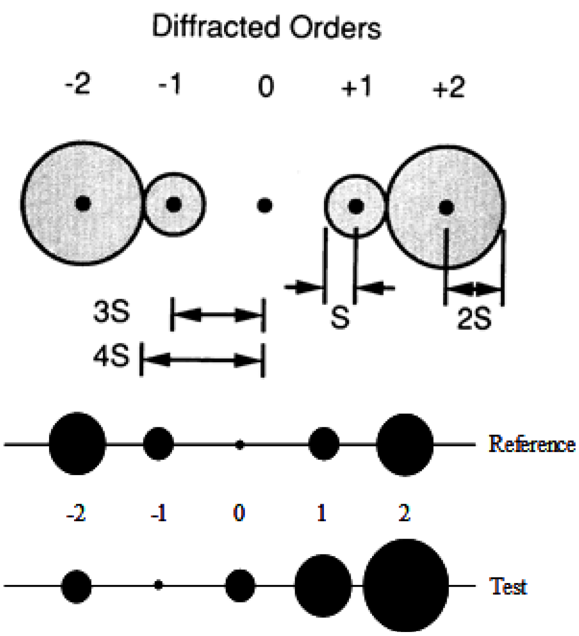
d) Consider an e-beam plotter with a plotting resolution of 0.5 μm over a 1 cm aperture. What is the smallest figure error that can be detected while testing an aspheric mirror with a slope departure of 500 waves/radius from a reference sphere?

Solution

a)

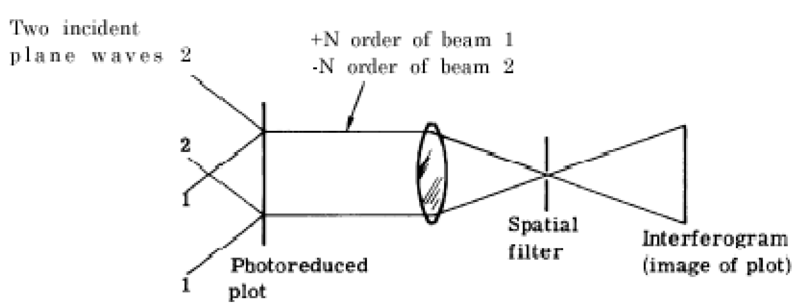






Several combinations of orders can be used. One option would be the zero order of the test beam and the +1 order of the reference beam. A second option would be the zero order of the reference beam and -1 order of the test beam. An advantage of this second method is that the focused spots are smaller since ideally there is no aberration in these two wavefronts and the tilt required to separate the orders of interest would be less.

b)



A plot is made of straight lines and this plot is put into the above setup. The +N order of beam 1 and -N order of beam 2 are interfered. Each fringe of error in the resulting interferogram corresponds to an error in the plot of the line spacing divided by 2N. (See **APPLIED OPTICS,** Vol. 13, page 1549, July 1974.)

c)

The space-bandwidth product is a measure of how much information can be stored in a hologram. In making CGHs we are interested in the total number of distortion-free plotter resolution points, which is equal to the number of distortion-free points/mm times the size of the plot in millimeters. This will tell us the accuracy of the wavefront produced by the CGH as a function of the maximum slope difference between the test beam and the reference beam.

d)

$$\text{NumberOf Distortion Free Points} = P = \frac{10^4\ \mu\text{m}}{0.5\ \mu\text{m}} = 2\times10^4$$

Assuming we want to separate the first and second orders, the maximum difference between the slope of the reference wavefront and the tilted plane wave is 4S waves per hologram radius and

$$\textbf{Error} = \frac{4\,S}{P} = \frac{4\ (500)}{2\times10^4} = \frac{1}{10}\ \textbf{wave}$$

The figure error would be half of this or $\frac{1}{20}$wave.

Assuming we want to separate the zero and first orders and interfere the zero order of the reference beam and -1 order of the test beam the maximum difference between the slope of the reference wavefront and the tilted plane wave is 2S waves per hologram radius and

$$\textbf{Error} = \frac{2\,S}{P} = \frac{2\ (500)}{2\times10^4} = \frac{1}{20}\ \textbf{wave}$$

The figure error would be half of this or $\frac{1}{40}$wave.

AS-3

A computer generated hologram is being made to test a concave aspheric surface having the following parameters:

r = -2.700105 inch

K = -.279229

A_4 = 0.011789 (in) $^{-3}$

A_6 = 0.002166 (in) $^{-5}$

A_8 = 0.000670 (in)$^{-7}$

A_{10} = -.001344 (in)$^{-9}$

The semi-diameter of the optic is 1.07 in. The plotter used to make the CGH has 2000 X 2000 distortion free resolution points.

a) To within a factor of 2, what is the minimum peak error in the measurement of the surface shape introduced by the finite number of plotter resolution points if in making the hologram we introduce enough tilt to separate the +1 diffraction order from the other orders?

b) To within a factor of 2, what is the minimum peak error in the measurement of the surface shape introduced by the finite number of plotter resolution points if in making the hologram we do not introduce tilt to separate the first diffraction order?

Solution

We must calculate the maximum slope of the aspheric departure. To minimize the maximum slope we need to add a defocus term $A_2\ s^2$. A_2 is selected such that the magnitude of the largest positive slope equals the magnitude of the largest negative slope. Equivalently, we can change the radius of the reference sphere that we measure with respect to.

Input known parameters

- Vertex radius of curvature

```
r = -2.700105;
```

- Conic constant

```
k = -.279229;
```

- Aspheric coefficients

```
a4 = .011789;  a6 = .002166;  a8 = .000670;  a10 = -.001344;
```

- Semi-diameter

```
semidia = 1.07;
```

- Wavelength

```
lambda = 0.6328;
```

Convert from inches to waves

$$\text{waves} = \frac{25.4\times 10^{3}}{\text{lambda}};$$

OPD Calculations

- Normalize s so x and y range from -1 to 1

$$s[x\_, y\_] := \text{semidia}\sqrt{x^2 + y^2}$$

- Calculate aspheric departure

$$\text{sagBaseSphere}[x\_, y\_, rb\_] := \frac{s[x, y]^2}{rb\left(1 + \sqrt{1 - \left(\frac{s[x,y]}{rb}\right)^2}\right)}$$

$$\text{sagConic}[x\_, y\_, rc\_, kk\_] := \frac{s[x, y]^2}{rc\left(1 + \sqrt{1 - (kk + 1)\left(\frac{s[x,y]}{rc}\right)^2}\right)}$$

$$\text{asphericTerms}[x\_, y\_] := a4\, s[x, y]^4 + a6\, s[x, y]^6 + a8\, s[x, y]^8 + a10\, s[x, y]^{10}$$

```
zopd[x_, y_, rc_, rb_, kk_] :=
 sagConic[x, y, rc, kk] + asphericTerms[x, y] - sagBaseSphere[x, y, rb]
```

- Calculate slope of aspheric departure

$$\text{slope}[x\_, y\_, rc\_, rb\_, kk\_] = \partial_x \text{zopd}[x, y, rc, rb, kk];$$

- Find radius of curvature to minimize the maximum slope

We will now find the reference radius of curvature, r2, that gives the minimum maximum slope. The minimum maximum slope is obtained when the absolute value of the maximum slope is equal to the absolute value of the minimum slope.

```
For[{r2 = r, s1 = 1, s2 = 2.5}, Abs[s1 - s2] > 0.00001, r2 = r2 - (s2 - s1) r2,
    {slo = Table[slope[x, 0, r, r2, k], {x, 0, 1, 0.05}], s1 = Abs[Max[slo]],
        s2 = Abs[Min[slo]]}];
```

We will now print out the results

```
Print["Reference Radius   ", "Maximum Slope   ", "Minimum Slope"];
Print["   ", r2, "          ", Min[slo], "       ", Max[slo]];
```

```
Reference Radius   Maximum Slope   Minimum Slope

   -3.04346          -0.0156507      0.0156575
```

The following gives the OPD in units of waves, the slope in units of waves/radius, and the minimum slope in units of waves/radius.

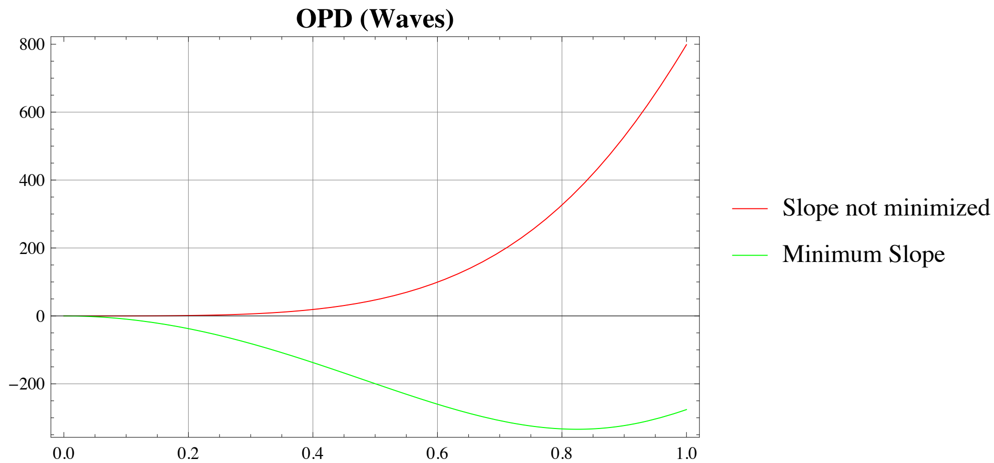
```
TableForm[Table[{x, waves zopd[x, 0, r, r2, k], waves slope[x, 0, r, r, k],
            waves slope[x, 0, r, r2, k]}, {x, 0, 1, 0.1}],
 TableHeadings -> {None, {"x", "waves OPD",   "slope", "Minimum slope"}},
 TableSpacing -> {1, 5}]
```

x	waves OPD	slope	Minimum slope
0.	0.	0.	0.
0.1	-9.53932	2.86285	-189.556
0.2	-37.414	23.1129	-364.165
0.3	-81.3451	79.2131	-507.892
0.4	-137.368	191.811	-602.798
0.5	-199.592	384.526	-628.257
0.6	-259.952	683.305	-561.733
0.7	-308.194	1112.06	-383.302
0.8	-332.666	1680.77	-87.741
0.9	-322.908	2360.4	290.066
1.	-275.792	3036.48	628.351

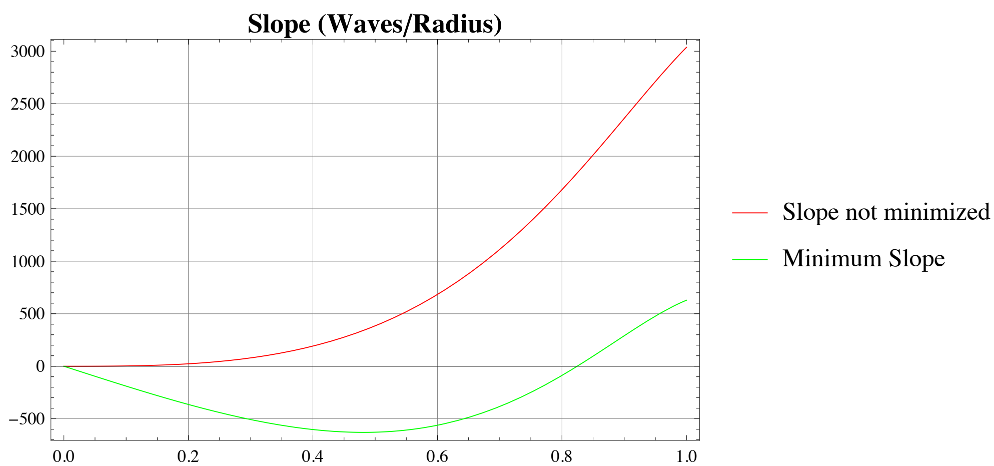
- Plots

The OPD and slope are plotted below. For one curve the OPD is measured relative to the vertex radius of curvature and for the second curve the spherical reference surface is selected to minimize the maximum value of the slope.

```
Plot[{waves zopd[x, 0, r, r, k], waves zopd[x, 0, r, r2, k]},
 {x, 0, 1}, PlotLegends → {"Slope not minimized", "Minimum Slope"},
 PlotLabel → Style["OPD (Waves)", FontSize → 14, FontWeight → Bold],
 AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic, PlotStyle → {Red, Green}]
```




```
Plot[{waves slope[x, 0, r, r, k], waves slope[x, 0, r, r2, k]},
 {x, 0, 1}, PlotLegends → {"Slope not minimized", "Minimum Slope"},
 PlotLabel → Style["Slope (Waves/Radius)", FontSize → 14, FontWeight → Bold],
 AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic, PlotStyle → {Red, Green}]
```



a)

To separate the 1^{st} and 2^{nd} orders the slope of the reference plane wave must be 3 times the maximum slope. Thus, the maximum slope for the fringes will be 4 times the maximum slope. The minimum value of the maximum slope was found above to be approximately 630 waves/radius. If S is the minimum value of the maximum slope and P is the number of distortion free resolution points, then

$$\text{peakError} = \frac{4\,S}{P}\ \text{waves} = \frac{4\,(630)}{2000}\ \text{waves} = 1.26\ \text{waves}$$

b)

Now the maximum slope difference is S.

$$\text{peakError} = \frac{S}{P}\ \text{waves} = \frac{(630)}{2000}\ \text{waves} = 0.32\ \text{waves}$$

Above are peak errors in measuring the surface, not peak errors in measuring the wavefront.

AS-4

I am testing a wavefront having 40 waves of third-order spherical aberration. I want to make a CGH to test this wavefront.

a) How many waves of defocus should be added to minimize the geometrical spot size?

b) What is the minimum number of waves of tilt across the radius of the pupil required to just separate the zero and first orders produced by the hologram?

c) Minimum number of waves of tilt to separate the first and second orders?

Solution

a)

To minimize the blur size an amount of defocus equal to -1.5 times the third-order spherical is required. Thus, we need -60 waves of defocus. The wavefront is then (40 waves) $\left(\rho^4 - 1.5\,\rho^2\right)$.

b)

The slope is now (40 waves/radius) $\left(4\,\rho^3\text{-}3\,\rho\right)$. The maximum slope is thus 40 waves/radius. 40 waves of tilt across the radius of the pupil are required to separate the zero and first orders.

c)

To separate the first and second orders 3 times this, or 120 waves of tilt across the radius of the pupil are required.

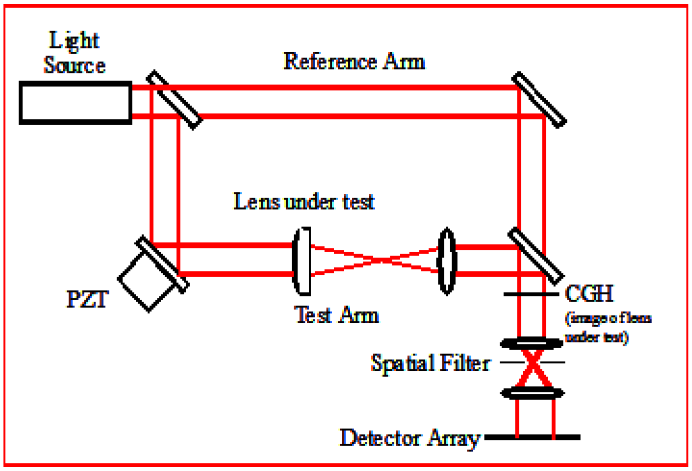
AS-5

A customer wants to use a CGH to test a 3 inch (76.2 mm) diameter, 0.4 inch (10.16 mm)thick, BK7 plano convex lens having a focal length of 10 inches (254 mm). Sketch an interferometer for testing the lens in single pass at infinite conjugates. The plotter used to make the CGH has 1000 X 1000 distortion free resolution points.

a) To within a factor of 2, what is the minimum peak error in the measurement introduced by the finite number of plotter resolution points if in making the hologram we introduce enough tilt to separate the +1 diffraction order from the other orders? (Consider only third-order aberrations.)

b) Devise a good method for using both the +1 and -1 diffraction orders produced by the CGH to test the lens whereby I have to put only 1/2 the aberration into the CGH that would be required if I use the 0 and +1 orders as is normally the case. Compare the error introduced by the finite number of plotter resolution points for cases a) and b).

Solution



a)

For a thin lens

$$\Delta w_{sph} = \frac{h^4 \Phi^3}{32\, n_o}\left(\left(\frac{n}{n-1}\right)^2 + \frac{(n+2)}{n\,(n-1)^2}\left(B + \frac{2\,(n^2-1)}{n+2}\,c\right)^2 - \frac{n}{n+2}\,c^2\right)\rho^4;$$

```
h = semidiameterOfPupil;
nₒ = refractiveIndexOfMaterialSurroundingLens;
n = refractiveIndexOfLens;
Φ = power;
B = shapeFactor;
c = conjugateVariable;
```

$$B = 1;\ \ c = -1\ ;;\ n_o = 1;\ \ n = 1.515;\ \Phi = \frac{1}{254\ \times 10^3\ \mu m};\ h = \left(\frac{76.2}{2}\right)\times 10^3\ \mu m;$$

Δw_{sph}

$35.4744\ \mu m\ \rho^4$

For minimum slope

$$\Delta w = 35.4744\ (\rho^4 - 1.5\,\rho^2)\ \mu m\ ;$$

$$S = 35.4744\ \mu m\ /\ radius$$

If the wavelength is 0.633 μm then

$$S = \frac{35.4744\ \mu m}{radius}\ \frac{1\ \lambda}{0.6328\ \mu m} = \frac{56.1\ \lambda}{radius}$$

$$error = \frac{4\,S}{P} = \frac{4\ (56.1\ )\ \lambda}{1000} = \frac{\lambda}{4.5}$$

b)

Normally we could use the +1 order of the reference beam and the zero order of the test beam or the zero order of the reference beam and the -1 order of the test beam to form the final interferogram. We would have to put only one-half as much aberration into the hologram if we used the +1 order of the reference beam and the -1 order of the test beam. In this way we are subtracting half the aberration from the test beam and adding half the aberration to the reference beam. Unfortunately, the errors are the same as for part a).

AS-6

Computer generated holograms are used to test a mirror producing an aspheric wavefront of 20 ρ^4, in units of waves. ρ is normalized to be between 0 and 1.

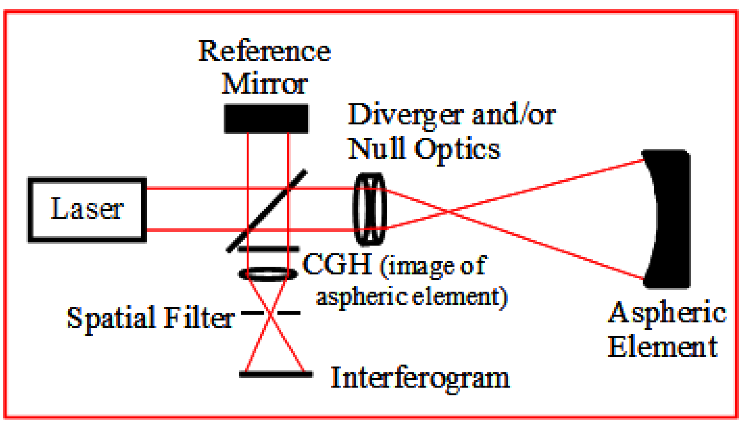
a) Sketch a reasonable setup being careful to show the planes that are conjugate to one another.

b) How many fringes of defocus should be added to the wavefront the hologram produces so errors due to plotting are minimized?

c) Which diffraction orders should be used for the test and reference beams?

Solution

a)



The aspheric mirror is imaged on the CGH and the CGH is imaged on the final interferogram plane.

b)

We want to minimize the slope (i.e., minimize the blur size). Therefore, for each wave of third-order spherical we want to subtract 1.5 waves of defocus, which in this case is -30 waves of defocus.

c)

The reference beam order should be one larger than the test beam order. For example, +1 order for reference beam and 0 order for test beam, or 0 order for reference beam and -1 order for test beam.

AS-7

A computer-generated hologram is a convenient way of testing an aspheric wavefront of the form $20\lambda\,\rho^4$.

a) How much defocus should be added to reduce the amount of wavefront tilt that must be introduced in the making of the CGH to separate the +1 order from the +2 order? Given this amount of defocus, what wavefront tilt must be introduced to separate the +1 and +2 orders?

b) The CGH is produced using a plotter having a distortion-free resolution of 0.5 micron over a 1 cm aperture. If the largest allowable error due to plotter resolution is 1/10 fringe, what is the largest allowable slope difference between the aspheric wavefront and the tilted plane wave used in the making of the CGH? Give the slope in units of waves/radius.

c) Let the hologram diameter be 1 cm. If a wavefront having a slope of 200 waves/radius is being tested, what is the maximum error introduced by translating the CGH sideways a distance of 10 microns from the optimum position?

Solution

a)

$$\text{defocus} = -30\,\lambda\,\rho^2$$

$$\text{wavefrontSlope} = 20\,\lambda\,(4-3)\,/\,\text{radius} = 20\,\lambda\,/\,\text{radius}$$

$$\text{Tilt} \geq 60\,\lambda\,/\,\text{radius}$$

b)

$$0.5\,\mu\text{m} = \frac{1}{10}\,\text{closestFringeSpacing}$$

$$\text{closestFringeSpacing} = 5\,\mu\text{m}$$

Hologram radius is 0.5 cm = 5000 μm

$$\text{slope} = \frac{5\,\mu\text{m}}{5000\,\mu\text{m}} = \frac{1}{1000}\ \text{or}\ 1000\,\frac{\text{waves}}{\text{radius}}$$

c)

Radius is 5000 μm.

$$200\,\frac{\text{waves}}{\text{radius}}\left(\frac{10}{5000}\,\text{radius}\right) = 0.4\ \text{waves}$$

AS-8

I am using the two-wavelength holography technique to test a 20-cm diameter, 40 cm focal length parabolic mirror at center of curvature. A 2-cm diameter, f/2, diverger lens is used in the interferometer.

The hologram is recorded using a wavelength of 514.5 nm, and it is read out using a wavelength of 488 nm. The hologram is recorded on a photographic plate. Between exposure and play back the photographic plate is removed from the interferometer for processing. What is the approximate tolerable error in lateral positioning of the plate (hologram) such that the error in the measurement of the surface of the parabolic mirror is less than 5 microns?

Solution

$$\mathbf{OPDRelativeToSphere} = \frac{\mathbf{s}^4}{\mathbf{8\,r}^3}$$

$$\text{Slope} = \frac{s^3}{2\,r^3} = \frac{(10\text{ cm})^3}{2\,(80\text{ cm})^3} = 0.00098$$

(We could reduce the slope by a factor of 4 if we defocus.)

Slope of wavefront ~ 2×10^{-3}.

Image of parabola is 1 cm diameter, 1/20 original size. Therefore, slope increased by factor of 20.

$$\lambda_{eq} = \frac{\lambda_1\,\lambda_2}{|\,\lambda_1 - \lambda_2\,|} = 9.47\,\mu\text{m} \approx 10\,\mu\text{m}$$

We are looking for 5 micron surface error which is 1 fringe error in wavefront.

$\text{Slope}\,(\Delta x) \sim 0.5\,\mu\text{m}$

$$\boxed{\Delta x = \frac{0.5\,\mu\text{m}}{\text{slope}} = \frac{0.5\,\mu\text{m}}{20\,\left(2\times 10^{-3}\right)} = 12.5\,\mu\text{m}}$$

If we introduce defocus to reduce slope by factor of 4 then $\Delta x \sim 50\,\mu$m.

Note that each visible fringe of change between recording and playback looks like a change of one λ_{eq}.

AS-9

I am using two-wavelength holography to test a concave aspheric mirror. The two wavelengths being used are 488 and 514.5 nm. If the diverger lens has no spherical aberration at 488 nm and 1 wave of spherical aberration (single pass) at 514.5 nm, how much error will result in the measurement of the surface of the aspheric mirror?

Solution

The resulting error will be one wave at the $\lambda_{\text{equivalent}}$ wavelength.

$$\boldsymbol{\lambda_{eq}} = \frac{\mathbf{0.488\,(0.5145)}}{\mathbf{0.5145 - 0.488}}\,\boldsymbol{\mu}\mathbf{m = 9.47}\,\boldsymbol{\mu}\mathbf{m}$$

AS-10

Two-wavelength holography is used to test an aspheric mirror. Phase-shifting interferometry is used and the detector has 256 x 256 detector elements. For this problem you can assume the detectors are point detectors.

a) What is the equivalent wavelength, in units of microns, for the test if one wavelength is 500 nm and the second one is 520nm?

b) How should the defocus in the wavefront being tested be selected such that the steepest asphere can be tested?

c) What is the maximum slope (in units of microns per radius) of the wavefront that can be measured using a single wavelength of 500 nm and phase-shifting interferometry? State any assumptions you make.

d) Repeat part c for the two-wavelength holography test and phase-shifting interferometry.

e) Air turbulence produces 50 nm of error in the making of the hologram. How much error, in units of nm, does this introduce into the measurement of the aspheric wavefront in the two-wavelength holography test?

Solution

a)

$$\lambda_{eq} = \frac{\lambda_1\,\lambda_2}{\mathbf{Abs}[\lambda_2 - \lambda_1]} \;/.\; \{\lambda_1 \to .5\,\mu\mathrm{m},\ \lambda_2 \to .52\,\mu\mathrm{m}\} = 13\,\mu\mathrm{m}$$

b)

Defocus should be added to minimize the largest value of the slope.

c)

We need at least 2 detectors per fringe, unless we have some additional information about the wavefront being measured. I am assuming that the wavefront slope is small enough that the light gets back to the detector alright. Since we have 128 detectors across the radius, the maximum slope would be 64 fringes/radius = 64 (.5 microns)/radius = 32 microns/radius.

d)

The answer is still 64 fringes/radius, but now fringes having a wavelength equal to the equivalent wavelength, 13 microns, so the maximum slope is 64 (13 microns)/radius = 832 microns/radius.

e)

If we are using a wavelength of 500 nm, the error will be 1/10 fringe. The problem is that now a fringe corresponds to 13 microns, so the error is 1.3 microns or 1300 nm.

AS-11

I want to contour an essentially flat object which has some sharp steps in it between 1 mm and 2 mm deep and other sharp steps between 5 and 10 microns deep. I want to measure both of these steps to within an accuracy of 1% of the step height. Someone tells me that I can perform these measurements using a Twyman-Green interferometer and direct-phase measurement interferometry if I use two or three different visible wavelengths. Describe how this would work. Be sure to give reasonable wavelengths and make a realistic estimate as to how well you can and need to measure the phase of the resulting interference pattern.

Solution

The interferogram must have shifts less than 1/2 fringe. This means the step depth must be less than $\lambda_{eq} / 4$. Therefore

For 5 – 10 micron deep	$\lambda_{eq} > 40$ microns
1 – 2 mm deep	$\lambda_{eq} > 8$ mm

$$\lambda_{eq} = \frac{\lambda_1 \lambda_2}{| \lambda_1 - \lambda_2 |}$$

Can use either two-wavelength holography or simply use phase-shifting interferometry at 2 wavelengths and then use equivalent wavelength to solve for phase discontinuities. Need to measure to 1/100 fringe or better if λ_{eq} are minimum values.

AS-12

Corning has sent me a piece of glass having dimensions 0.75 X 2 X 2 inches, which is made up of three separate pieces of glass 0.25 X 2 X 2 inches which were fused together. They want to know how good the block is in transmission when viewed through the 0.75 X 2 inch surface. They do not care about surface flatness. My problem is that when I test the block in transmission in a Mach-Zehnder interferometer I see fringe jumps between the fused surfaces and I cannot tell how large the fringe jumps are because of the discontinuity. How would you propose solving this problem?

Solution

I would measure the blocks using either two-wavelength holography or two-wavelength interferometry where the equivalent wavelength is long enough that the fringe jumps are less than one-half fringe at the equivalent wavelength.

AS-13

Moiré interferometry is used to contour a surface. The surface is illuminated with two plane waves, and the normalized intensity distribution incident on the film in the camera is

$$\frac{1}{2} \left(1 + \mathrm{Cos}\left[2 \pi \frac{(\mathrm{Tan}[\alpha] + \mathrm{Tan}[\beta])\, f[x, y] - y}{d}\right]\right)$$

Assume the photographic recording process is linear in the sense that after development the intensity transmittance of the film is proportional to the exposing intensity. The film is replaced back into the setup in the exact same position it occupied during exposure. The object which had a height distribution $Z = f[x, y]$ is replaced with an object for which $Z = g[x, y]$.

a) What is the resulting moiré pattern? Show that you obtain a so called "beat term" which gives us $f[x, y] - g[x, y]$.

b) It is commonly believed that similar moiré results are obtained if the two intensity patterns are added, i.e., double exposure of the two distributions on one piece of film. Show that superimposing the two intensity distributions does not give the "beat term" if the recording process is linear.

c) Comment upon whether a "beat term" would be obtained if the recording were non-linear.

d) Under proper conditions the naked eye can see the "beat term" by viewing directly two superimposed (added) sinusoidal intensity distributions. What property of the eye makes this possible? Does the eye have to resolve the individual sinusoidal frequencies to see the "beat term" if the patterns are i) added, ii) multiplied?

Solution

a)

We must multiply the intensity falling on the film times the intensity transmission function for the film.

$$\text{output} = \text{i}[\text{x}, \text{y}]\ \text{t}[\text{x}, \text{y}] = \frac{1}{2}\left(1 + \text{Cos}\left[2\pi\,\frac{(\text{Tan}[\alpha] + \text{Tan}[\beta])\ \text{f}[\text{x}, \text{y}] - \text{y}}{\text{d}}\right]\right)$$
$$\frac{1}{2}\left(1 + \text{Cos}\left[2\pi\,\frac{(\text{Tan}[\alpha] + \text{Tan}[\beta])\ \text{g}[\text{x}, \text{y}] - \text{y}}{\text{d}}\right]\right)$$

This is of the form

$$\textbf{output} = \frac{1}{2}\ (1 + \text{Cos}[\text{a}])\ \frac{1}{2}\ (1 + \text{Cos}[\text{b}])\ //\ \text{TrigReduce}$$

$$\frac{1}{8}\ (2 + 2\ \text{Cos}[\text{a}] + \text{Cos}[\text{a} - \text{b}] + 2\ \text{Cos}[\text{b}] + \text{Cos}[\text{a} + \text{b}])$$

Therefore we have

$$\textbf{output} =$$
$$\textbf{output} \ /.\ \left\{\text{a} \text{->} 2\pi\,\frac{(\text{Tan}[\alpha] + \text{Tan}[\beta])\ \text{f}[\text{x}, \text{y}] - \text{y}}{\text{d}},\ \text{b} \text{->} 2\pi\,\frac{(\text{Tan}[\alpha] + \text{Tan}[\beta])\ \text{g}[\text{x}, \text{y}] - \text{y}}{\text{d}}\right\}$$

$$\frac{1}{8}\left(2 + 2\,\text{Cos}\left[\frac{2\pi\,(-\text{y} + \text{f}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}}\right] + 2\,\text{Cos}\left[\frac{2\pi\,(-\text{y} + \text{g}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}}\right] + \right.$$
$$\text{Cos}\left[\frac{2\pi\,(-\text{y} + \text{f}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}} - \frac{2\pi\,(-\text{y} + \text{g}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}}\right] +$$
$$\left.\text{Cos}\left[\frac{2\pi\,(-\text{y} + \text{f}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}} + \frac{2\pi\,(-\text{y} + \text{g}[\text{x}, \text{y}]\ (\text{Tan}[\alpha] + \text{Tan}[\beta]))}{\text{d}}\right]\right)$$

We can simplify this.

$$\text{output} = \frac{1}{8}\left(2 + 2\,\text{Cos}\left[\frac{2\pi(-y + f[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right] + 2\,\text{Cos}\left[\frac{2\pi(-y + g[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right] + \text{Cos}\left[\text{Factor}\left[\frac{2\pi(-y + f[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d} - \frac{2\pi(-y + g[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right]\right] + \text{Cos}\left[\text{FullSimplify}\left[\text{Factor}\left[\frac{2\pi(-y + f[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d} + \frac{2\pi(-y + g[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right]\right]\right]\right)$$

$$\frac{1}{8}\left(2 + \text{Cos}\left[\frac{2\pi(f[x,y] - g[x,y])(\text{Tan}[\alpha] + \text{Tan}[\beta])}{d}\right] + 2\,\text{Cos}\left[\frac{2\pi(-y + f[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right] + 2\,\text{Cos}\left[\frac{2\pi(-y + g[x,y]\,(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right] + \text{Cos}\left[\frac{2\pi(-2y + (f[x,y] + g[x,y])(\text{Tan}[\alpha] + \text{Tan}[\beta]))}{d}\right]\right)$$

The first Cos term is the term of interest because it shows that the beat term gives the difference between the two surfaces.

b)

$$i_1[x, y] = \frac{1}{2}(1 + \text{Cos}[\omega_1 x]);$$

$$i_2[x, y] = \frac{1}{2}(1 + \text{Cos}[\omega_2 x]);$$

```
i1[x, y] + i2[x, y] // TrigReduce
```

$$\frac{1}{2}(2 + \text{Cos}[x\,\omega_1] + \text{Cos}[x\,\omega_2])$$

There is no beat term while if we multiply there is a beat term.

```
i1[x, y] i2[x, y] // TrigReduce
```

$$\frac{1}{8}(2 + 2\,\text{Cos}[x\,\omega_1] + 2\,\text{Cos}[x\,\omega_2] + \text{Cos}[x\,\omega_1 - x\,\omega_2] + \text{Cos}[x\,\omega_1 + x\,\omega_2])$$

c)

Now we will have a non-linear detector we will get a beat term.

$$a\,(i_1[x, y] + i_2[x, y]) + b\,(i_1[x, y] + i_2[x, y])^2 \;//\; \text{TrigReduce}$$

$$\frac{1}{8}(8a + 10b + 4a\,\text{Cos}[x\,\omega_1] + 8b\,\text{Cos}[x\,\omega_1] + b\,\text{Cos}[2x\,\omega_1] + 4a\,\text{Cos}[x\,\omega_2] + 8b\,\text{Cos}[x\,\omega_2] + b\,\text{Cos}[2x\,\omega_2] + 2b\,\text{Cos}[x\,\omega_1 - x\,\omega_2] + 2b\,\text{Cos}[x\,\omega_1 + x\,\omega_2])$$

d)

The eye response is non linear

i) added - if we can resolve the individual frequencies we will obtain a beat term as given in part c.

ii) multiplied - eye only needs to resolve the beat term.