Direct Phase Measurement

DP-1

Phase shifting interferometry is used with a Twyman-Green interferometer to measure a spherical mirror. Both the reference beam and the beam incident upon the spherical mirror are collimated beams. The wavelength is 633 nm. Let the detector be a 1024 element linear detector array with a pixel spacing of 6 microns. Assuming unit magnification between the spherical mirror and the detector, what is the minimum radius of curvature spherical mirror that can be measured such that there are no 2π phase discontinuities in the measurements?

Solution

Need at least 2 detectors per fringe.

$$\text{Sag} = \frac{y^2}{2\,r}; \quad \text{slope} = \frac{y}{r} < \frac{(0.633\ \mu\text{m})\ /\ 4}{6\ \mu\text{m}} = \frac{\text{max surface height change between pixels}}{\text{pixel spacing}};$$

$$y_{max} = 512\ \text{detectorElement}\left(\frac{6\ \mu\text{m}}{\text{detectorElement}}\right);$$

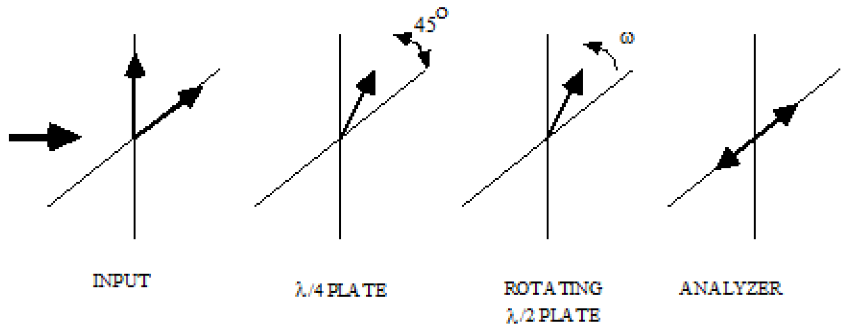
$$r > \frac{6\ \mu\text{m}\ (y_{max})}{0.633\ \mu\text{m}\ /\ 4}\ \frac{1\ \text{mm}}{1000\ \mu\text{m}}$$

$$r > 116.474\ \text{mm}$$

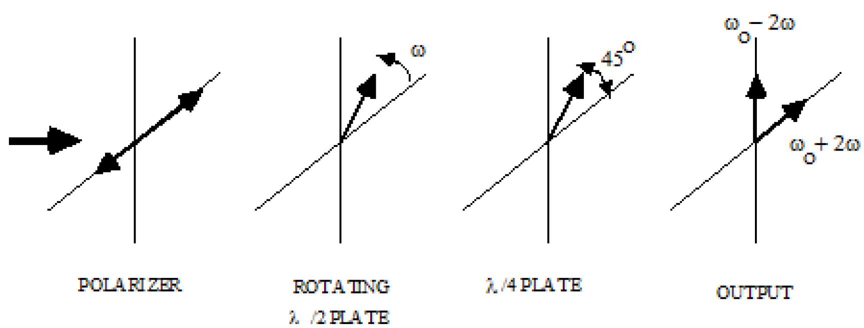
DP-2

Use Jones calculus to show the following:

a) The output of an interferometer can be made to vary sinusoidally in time by using a stationary $\lambda/4$ plate at 45^{O}, a rotating $\lambda/2$ plate at frequency ω, and an analyzer. When the test and reference beams exit the interferometer, they are linearly polarized in orthogonal directions.



b) A $\lambda/2$ plate rotating at frequency ω followed by a stationary $\lambda/4$ plate with its axis at 45° with respect to the direction of the original polarization converts a linearly polarized beam into two linearly polarized beams of orthogonal polarization which differ in optical frequency by an amount 4ω.



Solution

a)

The two output beams from an interferometer have orthogonal polarization and phase difference ϕ. The two beams pass through a quarter wave plate at 45^o and a half wave plate rotating at frequency ω. The beams then pass through a horizontal linear analyzer. We will show that the resulting intensity will vary sinusoidally at frequency 4 ω and have a phase that depends on ϕ.

Input

$$\textbf{input} = \begin{pmatrix} e^{i\phi} \\ 1 \end{pmatrix};$$

Quarter-Wave Plate at 45 degrees

$$\textbf{qwp45} = e^{i\pi/4} \begin{pmatrix} \text{Cos}[45°] & -\text{Sin}[45°] \\ \text{Sin}[45°] & \text{Cos}[45°] \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} . \begin{pmatrix} \text{Cos}[45°] & \text{Sin}[45°] \\ -\text{Sin}[45°] & \text{Cos}[45°] \end{pmatrix};$$

Rotating Half-Wave Plate

$$\textbf{rhwp} = e^{i\pi/2} \begin{pmatrix} \text{Cos}[\omega t] & -\text{Sin}[\omega t] \\ \text{Sin}[\omega t] & \text{Cos}[\omega t] \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} \text{Cos}[\omega t] & \text{Sin}[\omega t] \\ -\text{Sin}[\omega t] & \text{Cos}[\omega t] \end{pmatrix};$$

Linear Analyzer at 0 degrees

$$\textbf{la0} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

Output

```
output = FullSimplify[la0.rhwp.qwp45.input] // MatrixForm
```

$$\begin{pmatrix} \frac{i\left(e^{2 i t \omega}+i e^{i(\phi-2 t \omega)}\right)}{\sqrt{2}} \\ 0 \end{pmatrix}$$

The output looks like the interference of two beams having frequency difference 4 ω and phase difference $\phi+\pi/2$.

b)

Input

$$\texttt{input} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

Rotating Half-Wave Plate (from above)

```
rhwp;
```

Quarter-Wave Plate at 45 degrees (from above)

```
qwp45;
```

Output

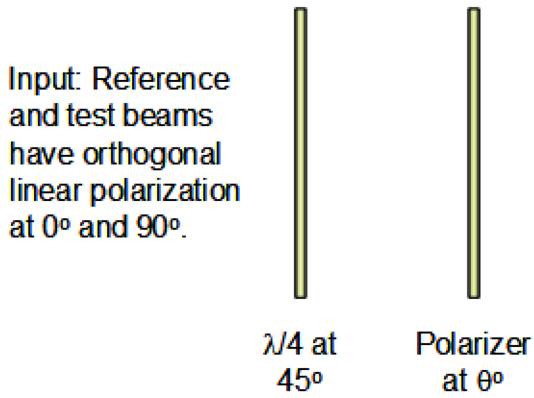
```
output = FullSimplify[qwp45.rhwp.input] // MatrixForm
```

$$\begin{pmatrix} -\frac{e^{-2 i t \omega}}{\sqrt{2}} \\ \frac{i e^{2 i t \omega}}{\sqrt{2}} \end{pmatrix}$$

The output looks like two beams having orthogonal polarization where one beam is frequency shifted up an amount 2ω and the second is frequency shifted down an amount 2ω.

DP-3

The figure below shows a diagram of the phase shifter as it could be used in the output of an interferometer where the reference and test beams have orthogonal linear polarization. A quarter-wave plate converts one of the two interfering beams into right-handed circularly polarized beam and the second interfering beam into a left-handed circularly polarized beam. Show that as a polarizer is rotated an angle θ the phase difference between the test and reference beams changes by 2θ. The polarizer also makes it possible for the two beams to interfere.



Solution

We need the following four Jones matrices.

Horizontal linear polarizer

$$\text{hlpJones} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

Rotation Matrix

$$\text{rot}[\theta\_] := \begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

Retarder with fast axis horizontal

$$\text{rfah}[\phi\_] := e^{-i\,\phi/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\,\phi} \end{pmatrix}$$

Calculation of matrix of a retarder of retardation ϕ having a fast axis at an angle θ from the horizontal

```
rrot[ϕ_, θ_] := rot[-θ].rfah[ϕ].rot[θ];
```

Let the input polarization be given by

$$\text{inputPolarization} = \begin{pmatrix} a\,e^{i\,\phi} \\ b \end{pmatrix};$$

The polarizer is rotated an angle θ.

$$\text{amplitudeOutput} = \text{rot}[-\theta].\text{hlpJones}.\text{rot}[\theta].\text{rrot}\left[\frac{\pi}{2}, \frac{\pi}{4}\right].\text{inputPolarization};$$

```
intensityOutput = FullSimplify[amplitudeOutput Conjugate[amplitudeOutput],
  {a, b, ϕ, θ} ∈ Reals]; MatrixForm[intensityOutput]
```

$$\begin{pmatrix} \frac{1}{2}\text{Cos}[\theta]^2 \left(a^2 + b^2 + 2\,a\,b\,\text{Sin}[2\,\theta - \phi]\right) \\ \frac{1}{2}\text{Sin}[\theta]^2 \left(a^2 + b^2 + 2\,a\,b\,\text{Sin}[2\,\theta - \phi]\right) \end{pmatrix}$$

```
intensity = Extract[Total[intensityOutput], 1] // Simplify
```

$$\frac{1}{2}\left(a^2 + b^2 + 2\,a\,b\,\text{Sin}[2\,\theta - \phi]\right)$$

Thus as the polarizer is rotated an angle θ the phase difference between the two interfering beams changes by $2\,\theta$.

DP-4

In phase-shifting interferometry, the general expression for the intensity of an interferogram with a phase step δ is
$i[x, y, \delta] = i_o[x, y]\,(1 + \gamma\,\mathrm{Cos}[\phi[x, y] + \delta[t]])$.
Rather than stepping the phase in discrete steps, we will integrate the intensity as the phase is linearly changed through a phase interval Δ.

a) Determine I_i[x,y] by integrating over phase values of $\delta_i - \Delta/2 < \delta < \delta_i + \Delta/2$. Make sure to include a normalization factor which keeps the average integrated signal for each interferogram independent of Δ.

b) What is the effect of integrating the interferogram intensity while the phase is changing?

c) Write out the equations for the three bucket technique with $\Delta = \pi/2$,and $\delta_i = \pi/4$, $3\,\pi/4$,and $5\,\pi/4$.

Solution

a)

$$i_i[x, y] = \frac{1}{\Delta}\int_{\delta_i-\Delta/2}^{\delta_i+\Delta/2} i_o[x, y]\,(1 + \gamma\,\mathrm{Cos}[\phi[x, y] + \delta])\,d\delta =$$

$$i_o[x, y]\left(1 + \gamma\left(\frac{\mathrm{Sin}\left[\frac{\Delta}{2}\right]}{\frac{\Delta}{2}}\right)\mathrm{Cos}[\delta_i + \phi[x, y]]\right) =$$

$$i_o[x, y]\left(1 + \gamma\,\mathrm{Sinc}\left[\frac{\Delta}{2}\right]\mathrm{Cos}[\delta_i + \phi[x, y]]\right)$$

b)

The integration changes the visibility from γ to $\gamma\,\mathrm{Sinc}[\Delta/2]$. For $\Delta = \frac{\pi}{2}$the visibility is reduced by $\mathrm{Sinc}\left[\frac{\pi}{4}\right] = 0.9$.

c)

$$\Delta = \pi/2; \qquad \delta_i = \frac{\pi}{4},\quad 3\,\frac{\pi}{4},\quad 5\,\frac{\pi}{4}$$

$$i_1[x, y] = i_o[x, y]\left(1 + 0.9\,\gamma\,\mathrm{Cos}\left[\phi[x, y] + \frac{\pi}{4}\right]\right) =$$

$$i_o[x, y]\left(1 + 0.9\,\gamma\left(\frac{\mathrm{Cos}[\phi[x, y]]}{\sqrt{2}} - \frac{\mathrm{Sin}[\phi[x, y]]}{\sqrt{2}}\right)\right);$$

$$i_2[x, y] = i_o[x, y]\left(1 + 0.9\,\gamma\,\mathrm{Cos}\left[\phi[x, y] + 3\,\frac{\pi}{4}\right]\right) =$$

$$i_o[x, y]\left(1 + 0.9\,\gamma\left(-\frac{\mathrm{Cos}[\phi[x, y]]}{\sqrt{2}} - \frac{\mathrm{Sin}[\phi[x, y]]}{\sqrt{2}}\right)\right);$$

$$\text{i}_3[\text{x},\text{y}] = \text{i}_o[\text{x},\text{y}]\left(1 + 0.9\,\gamma\,\text{Cos}\left[\phi[\text{x},\text{y}] + 5\,\frac{\pi}{4}\right]\right) = \text{i}_o[\text{x},\text{y}]\left(1 + 0.9\,\gamma\left(-\frac{\text{Cos}[\phi[\text{x},\text{y}]]}{\sqrt{2}} + \frac{\text{Sin}[\phi[\text{x},\text{y}]]}{\sqrt{2}}\right)\right);$$

$$\text{Tan}[\phi[\text{x},\text{y}]] = \frac{\text{i}_3[\text{x},\text{y}] - \text{i}_2[\text{x},\text{y}]}{\text{i}_1[\text{x},\text{y}] - \text{i}_2[\text{x},\text{y}]}$$

DP-5

The 4 step technique described by Carré is used to measure phase. If between each readout of the intensity the phase of the reference beam is changed by an amount 2α the 4 irradiance measurements are given by

$A[x,y] = I_1 + I_2\,\text{Cos}[\phi[x,y] - 3\alpha]$
$B[x,y] = I_1 + I_2\,\text{Cos}[\phi[x,y] - \alpha]$
$C[x,y] = I_1 + I_2\,\text{Cos}[\phi[x,y] + \alpha]$
$D[x,y] = I_1 + I_2\,\text{Cos}[\phi[x,y] + 3\alpha]$

Show that tan α and tan ϕ are given by

$$\textbf{Tan}[\alpha] = \sqrt{\frac{3\,(\textbf{B}[\textbf{x},\textbf{y}] - \textbf{C}[\textbf{x},\textbf{y}]) - (\textbf{A}[\textbf{x},\textbf{y}] - \textbf{D}[\textbf{x},\textbf{y}])}{(\textbf{B}[\textbf{x},\textbf{y}] - \textbf{C}[\textbf{x},\textbf{y}]) + (\textbf{A}[\textbf{x},\textbf{y}] - \textbf{D}[\textbf{x},\textbf{y}])}}$$

$$\textbf{Tan}[\phi[\textbf{x},\textbf{y}]] = \left(\frac{(\textbf{A}[\textbf{x},\textbf{y}] - \textbf{D}[\textbf{x},\textbf{y}]) + (\textbf{B}[\textbf{x},\textbf{y}] - \textbf{C}[\textbf{x},\textbf{y}])}{(\textbf{B}[\textbf{x},\textbf{y}] + \textbf{C}[\textbf{x},\textbf{y}]) - (\textbf{A}[\textbf{x},\textbf{y}] + \textbf{D}[\textbf{x},\textbf{y}])}\right)\textbf{Tan}[\alpha]$$

How would you determine ϕ(x,y) modulo 2π?

Solution

The four measurements are

$$\textbf{a} = I_1 + I_2\,\textbf{Cos}[\phi - 3\,\alpha];$$

$$\textbf{b} = I_1 + I_2\,\textbf{Cos}[\phi - \alpha];$$

$$\textbf{c} = I_1 + I_2\,\textbf{Cos}[\phi + \alpha];$$

$$\textbf{d} = I_1 + I_2\,\textbf{Cos}[\phi + 3\,\alpha];$$

Mathematica makes it easy to show that $\text{Tan}[\alpha]^2$ is given by

$$\textbf{Simplify}\left[\frac{3\,(\textbf{b} - \textbf{c}) - (\textbf{a} - \textbf{d})}{(\textbf{b} - \textbf{c}) + (\textbf{a} - \textbf{d})}\right]$$

$$\text{Tan}[\alpha]^2$$

Mathematica also makes it easy to show that $\text{Tan}[\phi]$ is given by

$$\mathbf{Simplify}\left[\left(\frac{(a-d)+(b-c)}{(b+c)-(a+d)}\right)\mathbf{Tan}[\alpha]\right]$$

$\mathrm{Tan}[\phi]$

Determining ϕ(x,y) modulo 2π

numerator = Simplify[(a - d) + (b - c)]

$8\,\mathrm{Cos}[\alpha]^2\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\phi]\,i_2$

denominator = Simplify[(b + c) - (a + d)]

$8\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\alpha]^2\,i_2$

To determine ϕ modulo 2π we must know the signs of sin ϕ and cos ϕ so we know which quadrant ϕ is in. As long as α is between 0 and 180 degrees the sign of sin α is positive and the sign of the numerator gives the sign of sin ϕ. However, without doing an additional measurement we cannot determine the sign of cos ϕ. Thus, we only know ϕ modulo π. This means that the most the phase can change between adjacent detector points is 90 degrees before we run into phase ambiguity problems. If we know the quadrant α is in, then we know the sign of cos α, and we can determine the sign of cos α from the denominator. Hence we would know ϕ modulo 2π. Since 2α is generally close to 90°, we generally have no trouble determining ϕ modulo 2π.

DP-6

The three integrating bucket technique described in class is used to measure the phase distribution across an interferogram. The interferometer is a Twyman-Green with a helium neon laser as the light source. The detector used contains 100 X 100 detector elements. The detector is read out at a rate of 10^6 detector elements per second. In the following assume that the detector can be read out continuously with no dead time.

a) Let the phase shifter be a rotating 1/4 wave plate in the reference arm of the interferometer. How many revolutions per second must the 1/4 wave plate rotate?

b) Let the phase shifter be a moving mirror mounted on a piezoelectric transducer. How fast must the mirror be moving during the taking of the data? Give velocity in units of microns per second.

Solution

a)

It takes $\frac{1}{100}$sec to read out the detector array once. During this time the phase must change by 90°.
The rotating quarter-wave plate changes the phase of the reference beam by 720° for each revolution.
Therefore it must make 100/8 = 12.5 revolutions per second.

b)

The mirror must move $\lambda/8$ each $\frac{1}{100}$ sec.

$$\text{mirrorVelocity} = \frac{0.6328\,\mu\text{m}}{8}\,\frac{100}{\text{sec}} = \frac{7.91\,\mu\text{m}}{\text{sec}}$$

DP-7

a) Give 3 main sources of error for each conventional interferometry and phase-shifting interferometry.

b) Why are at least 3 separate measurements required in phase-shifting interferometry?

Solution

a)

Conventional Interferometry
- Air turbulence
- Errors in interferometer
- Digitization of interferogram
- Film distortion

Phase-Shifting Interferometry
- Air turbulence
- Vibration
- Errors in interferometer
- Digitization of intensity measurement
- Incorrect phase shifts between measurements
- Spurious interference fringes

b)

$$i = i_1 + i_2 + 2\sqrt{i_1\, i_2}\ \text{Cos}[\phi] \quad \text{or} \quad i = a + b\,\text{Cos}[\phi]$$

There are 3 unknowns so we must have 3 measurements.

DP-8

a) Give three advantages of phase-shifting interferometry over simply measuring fringe centers.

b) Phase stepping interferometry is used with a Twyman-Green interferometer to measure a nearly flat mirror. Let the detector be a 1024 element linear detector array. In units of waves/radius, what is the maximum wavefront slope that can be measured such that there are no 2π phase discontinuities in the measurements?

Solution

a)

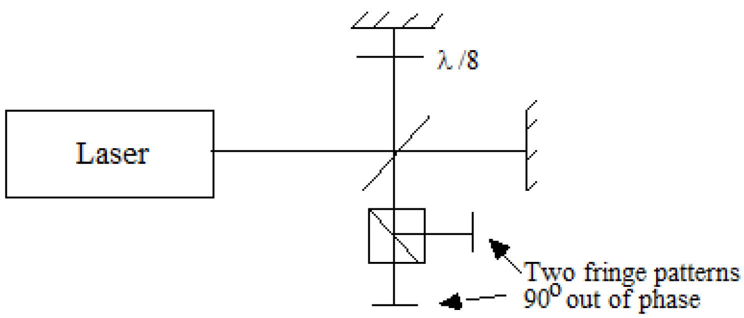
Better accuracy
Rapid measurement
Works with a null fringe
Good results with low contrast fringes
Results independent of intensity variations across the pupil
Phase obtained at a fixed grid of points

b)

We must have less than $\frac{1}{2}$fringe between adjacent pixels, thus the maximum slope is 256 waves/radius.

DP-9

I have a Twyman-Green interferometer whose light source is a linearly polarized laser. I am told that if I place a l/8 wave plate in one arm of the interferometer, and a polarization beamsplitter in the output beam I obtain two fringe patterns 90^o out of phase. Is this right or wrong? Why might I want to obtain two interference patterns 90^o out of phase?



Solution

It is correct that two fringe patterns are obtained 90^{o}out of phase. We will call the beam containing the 1/8 wave plate the reference beam and the second beam the test beam. The plane polarized light can be broken into two components where one component is parallel to the fast axis of the 1/8 wave plate in the reference beam and the second component is perpendicular. After passing through the 1/8 wave plate twice the two polarization components in the reference beam will be 90^{o}out of phase. Since there is no wave plate placed in the test beam the two polarization components of the test beam are in phase. A polarization beamsplitter reflects one polarization component and transmits the second. Due to the 90^{o}phase difference between the two polarization components in the reference beam the two resulting interferograms will be 90^{o}out of phase. Having two interferograms 90^{o}out of phase (sine and cosine) makes it possible for us to determine the quadrant the phase is in and hence the sign of the error (determine whether we are looking at low spots or high spots).

DP-10

I am using phase shifting interferometry to test an optical system. Unfortunately, I am getting a stray reflection off the diverger lens used in the interferometer.

a) Let the stray reflection have 1/25 the intensity of the test beam, the phase of the test beam be 60^{o}, and the phase of the stray reflection be 90^{o}. How much phase error will the stray reflection introduce into the measurement of the test beam?

b) What phase difference between test beam and stray reflection will introduce the maximum error into the phase measurement?

Solution

a)

```
amplitudeMeasured = ⅇ^(ⅈ phaseTest) + amplitudeStray ⅇ^(ⅈ phaseStray);
```

Let the phase difference between the stray and the test beam be δ.

```
δ = phaseStray - phaseTest;
```

```
amplitudeMeasured = ⅇ^(ⅈ phaseTest) (1 + amplitudeStray ⅇ^(ⅈ δ));
```

Then

```
amplitudeErrorFactor = (1 + amplitudeStray ⅇ^(ⅈ δ));
```

```
phaseTest = 60 °;  phaseStray = 90 °;  amplitudeStray = 1/5;
```

The phase error in degrees is

```
phaseError = N[Arg[amplitudeErrorFactor]] / Degree
```

```
4.87192
```

b)

We will solve this for the case where the stray reflection has 1/25 the intensity of the test beam.

```
Clear[phaseTest, phaseStray, δ];   amplitudeStray = 1 / 5;
```

Let ϵ be the phase error and let

```
tanε = Tan[ε];
```

```
tanε = Simplify[Im[amplitudeErrorFactor] / Re[amplitudeErrorFactor] /. phaseStray - phaseTest → δ, {δ ∈ Reals}]
```

$$\frac{\mathrm{Sin}[\delta]}{5 + \mathrm{Cos}[\delta]}$$

We will now take the derivative of tanϵ with respect to δ and set equal to zero and then solve for δ.

```
ans = Solve[D[tanε, δ] == 0, δ] // N
```

```
{{δ → -1.77215}, {δ → 1.77215}}
```

```
(δ /. {ans[[1]], ans[[2]]}) / Degree
```

```
{-101.537, 101.537}
```

DP-11

a) Show that if α is the phase step between consecutive frames in a phase shifting interferometer, and i_0, i_1, i_2,i_3, and i_4 are the intensities of the five measured frames, the phase step α can be obtained from the expression

$$\frac{i_4 - i_0}{i_3 - i_1} = 2\,\mathrm{Cos}[\alpha]$$

b) I am using a direct phase measurement interferometer to test a coated spherical mirror. The diverger lens reflects 1% of the incident light. Give an estimate as to how large an error in measurement of the phase can be introduced by this extraneous reflected light. Other than putting an AR coating on the lens, how would you reduce the error?

Solution

a)

```
i0 = a + b Cos[ϕ]
```

```
i1 = a + b Cos[ϕ + α]
```

```
i3 = a + b Cos[ϕ + 3 α]
```

```
i4 = a + b Cos[ϕ + 4 α]
```

For simplicity we can write
$\theta = \phi + 2\,\alpha$, then

```
i0 = a + b Cos[θ - 2 α] // TrigExpand
```

$a + b\,\mathrm{Cos}[\alpha]^2\,\mathrm{Cos}[\theta] - b\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\alpha]^2 + 2\,b\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\theta]$

```
i1 = a + b Cos[θ - α] // TrigExpand
```

$a + b\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\theta] + b\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\theta]$

```
i3 = a + b Cos[θ + α] // TrigExpand
```

$a + b\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\theta] - b\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\theta]$

```
i4 = a + b Cos[θ + 2 α] // TrigExpand
```

$a + b\,\mathrm{Cos}[\alpha]^2\,\mathrm{Cos}[\theta] - b\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\alpha]^2 - 2\,b\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\theta]$

$\mathbf{i_4 - i_0}$

$-4\, b\, \mathrm{Cos}[\alpha]\, \mathrm{Sin}[\alpha]\, \mathrm{Sin}[\theta]$

$\mathbf{i_3 - i_1}$

$-2\, b\, \mathrm{Sin}[\alpha]\, \mathrm{Sin}[\theta]$

$$\frac{i_4 - i_0}{i_3 - i_1} = 2\, \mathrm{Cos}[\alpha]$$

b)

A common problem in interferometers using lasers as a light source is extraneous interference fringes due to stray reflections. The easiest way of thinking about the effect of stray reflections is that the stray reflection adds to the test beam to give a new beam of some amplitude and phase. The difference between this resulting phase, and the phase of the test beam, gives the phase error. If we express the amplitude of the stray light as a fraction of the amplitude of the test beam we can write

$\mathbf{amplitudeMeasured} = e^{i\, \mathrm{phaseTest}} + \mathbf{amplitudeStray}\, e^{i\, \mathrm{phaseStray}};$

For a given amplitude of the stray light the maximum error will occur when the difference in the phase of the test beam and the reference beam is nearly $\frac{\pi}{2}$. As an example let

$\mathbf{phaseTest} = \phi; \quad \mathbf{phaseStray} = \left(\phi + \frac{\pi}{2}\right); \quad \mathbf{amplitudeStray} = 0.1;$

Then

$\mathbf{phaseMeasured = Simplify[Arg[amplitudeMeasured],\ \phi \in Reals]}$

$\mathrm{Arg}\left[(0.995037 + 0.0995037\, i)\, e^{i\, \phi}\right]$

$\mathbf{phaseError} = \mathrm{Arg}\left[\mathrm{Simplify}\left[\frac{\mathrm{amplitudeMeasured}}{e^{i\, \phi}}\right]\right] \Big/ \mathrm{Degree}$

5.71059

To reduce the effect of the stray reflection we could average two data sets where the phase of the test beam differs by 180 ° between the two measurements. This is assuming that the phase of the stray reflection does not change between the two measurements. That is, we are finding

$$\frac{1}{2}\left(e^{i\, \mathrm{phaseTest}} + \mathrm{amplitudeStray}\, e^{i\, \mathrm{phaseStray}} - \left(-e^{i\, \mathrm{phaseTest}} + \mathrm{amplitudeStray}\, e^{i\, \mathrm{phaseStray}}\right)\right)$$

$e^{i\, \mathrm{phaseTest}}$

DP-12

In the three-step phase-shifting technique for measuring phase where the phase is shifted $\frac{\pi}{2}$ between the first and second measurement and another $\frac{\pi}{2}$ between the second and third measurement the common equation for ϕ is of the form

$$\phi = \text{ArcTan}\left[\frac{i_1 + i_3 - 2\, i_2}{i_1 - i_3}\right];$$

a) Show that the above equation is correct.

b) Show that we can get a similar equation for ϕ if instead of measuring ϕ we measure $\phi \pm \frac{\pi}{4}$. (You tell me whether it is $\frac{+\pi}{4}$ or $\frac{-\pi}{4}$.) There is still a $\frac{\pi}{2}$ phase shift between consecutive intensity measurements.

c) How will this $\frac{\pi}{4}$ phase shift influence our measurement of the phase distribution across the exit pupil of the system under test?

Solution

a)

$i_1 = a + b\,\text{Cos}[\phi];$

$i_2 = a + b\,\text{Cos}\left[\phi + \frac{\pi}{2}\right]$

$a - b\,\text{Sin}[\phi]$

$i_3 = a + b\,\text{Cos}[\phi + \pi]$

$a - b\,\text{Cos}[\phi]$

$i_1 + i_3 - 2\, i_2 = 2\, b\,\text{Sin}[\phi]$

$i_1 - i_3 = 2\, b\,\text{Cos}[\phi]$

Therefore,

$$\frac{i_1 + i_3 - 2\, i_2}{i_1 - i_3} = \text{Tan}[\phi]$$

b)

First we will try $\frac{+\pi}{4}$.

$i_1 = a + b\,\text{Cos}\left[\phi + \frac{\pi}{4}\right]$ // TrigExpand

$$a + \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} - \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$i_2 = a + b\,\text{Cos}\left[\phi + \frac{\pi}{2} + \frac{\pi}{4}\right]$ // TrigExpand

$$a - \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} - \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_3 = a + b\,\text{Cos}\left[\phi + \pi + \frac{\pi}{4}\right] \text{ // TrigExpand}$$

$$a - \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} + \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_3 - i_2 = \frac{2\,b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_1 - i_2 = \frac{2\,b\,\text{Cos}[\phi]}{\sqrt{2}}$$

Therefore

$$\frac{i_3 - i_2}{i_1 - i_2} = \text{Tan}[\phi]$$

Now we will try $\frac{-\pi}{4}$.

$$i_1 = a + b\,\text{Cos}\left[\phi - \frac{\pi}{4}\right] \text{ // TrigExpand}$$

$$a + \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} + \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_2 = a + b\,\text{Cos}\left[\phi + \frac{\pi}{2} - \frac{\pi}{4}\right] \text{ // TrigExpand}$$

$$a + \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} - \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_3 = a + b\,\text{Cos}\left[\phi + \pi - \frac{\pi}{4}\right] \text{ // TrigExpand}$$

$$a - \frac{b\,\text{Cos}[\phi]}{\sqrt{2}} - \frac{b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_1 - i_2 = \frac{2\,b\,\text{Sin}[\phi]}{\sqrt{2}}$$

$$i_2 - i_3 = \frac{2\,b\,\text{Cos}[\phi]}{\sqrt{2}}$$

Therefore

$$\frac{i_1 - i_2}{i_2 - i_3} = \text{Tan}[\phi]$$

Therefore both $\frac{+\pi}{4}$and $\frac{-\pi}{4}$phase shifts can simplify the calculation of the phase.

c)

The $\frac{\pi}{4}$ phase shift will make no difference because we are only interested in phase variations across the beam.

DP-13

Four-bucket phase-shifting interferometry is being used to test an optical system. Unfortunately, the detector has some linearity such that if the intensity incident upon the detector is I, the response of the detector goes as I + ϵ I^2. If the phase being measured is ϕ, how much error is introduced into the phase measurement?

Solution

Looking at the 4 buckets we get

$s_1 = i_1 + i_2 \text{Cos}[\phi] + \epsilon\,(i_1 + i_2 \text{Cos}[\phi])^2;$

$s_2 = i_1 - i_2 \text{Sin}[\phi] + \epsilon\,(i_1 - i_2 \text{Sin}[\phi])^2;$

$s_3 = i_1 - i_2 \text{Cos}[\phi] + \epsilon\,(i_1 - i_2 \text{Cos}[\phi])^2;$

$s_4 = i_1 + i_2 \text{Sin}[\phi] + \epsilon\,(i_1 + i_2 \text{Sin}[\phi])^2;$

$\text{Simplify}\left[\frac{s_4 - s_2}{s_1 - s_3}\right]$

$\text{Tan}[\phi]$

Therefore, the quadratic non-linearity introduces no error.

DP-14

In phase shifting interferometry the measured phase values at six consecutive data points are 0°, 150°, 300°, 90°, 240°, and 30°. We know that for the particular sample being measured the phase difference between consecutive data points must be less than 180°. What are the phase values after correcting for the 2 π ambiguities arising from using the arc tangent function?

Solution

Need to add or subtract multiple of 360° so difference between consecutive phase values less than 180°.

```
{0, 150, 300, 90 + 360, 240 + 360, 30 + 2 (360)} °
```

```
{0, 150 °, 300 °, 450 °, 600 °, 750 °}
```

DP-15

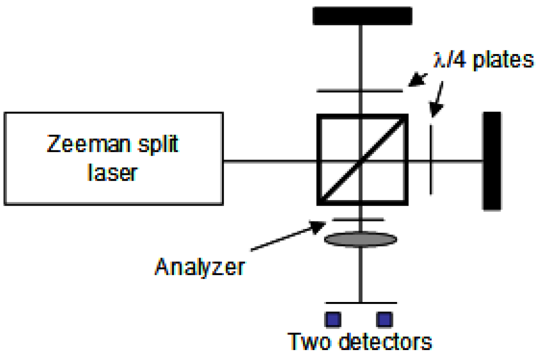
The zero crossing technique is used in a Twyman-Green interferometer to measure a mirror. The frequency difference between the two interfering beams is 1000 Hz.

a) Sketch the setup showing one possible method for producing the frequency difference.

b) If the time difference between the zero crossings of the output of two detectors is 0.0003 seconds, what is the corresponding surface height difference on the sample?

Solution

a)



b)

$$\frac{\text{timeDifference}}{\text{period}}\,\frac{\lambda}{2} = \frac{0.0003}{0.001}\,\frac{\lambda}{2} = 0.15\,\lambda$$

DP-16

Phase-shifting interferometry is used with a Twyman-Green interferometer to measure a nearly spherical mirror that has a small aspheric height variation that goes as $A\,\rho^4$, where $0 \le \rho \le 1$. Let the detector be a 1024 x 1024 element CCD having a pixel spacing of 9 microns. The wavelength is 633 nm.

a) How large can A, the coefficient of the ρ^4 term, be such that there are no 2π phase discontinuities in the measurements if conventional phase-shifting techniques are used? You should adjust the interferometer to maximize A. State any assumptions being made.

b) Repeat part b) for the case where the detector is a 1024 x 1024 element CCD having a pixel spacing of 12 microns. State any assumptions being made.

Solution

a)

Assume image of mirror fills the detector. Must have at least two detector elements per fringe. Therefore, the maximum wavefront slope is 256 waves/radius. If we introduce defocus to minimize the maximum slope,

$$\text{wavefrontSlope} = 2\,A\left(4\,\rho^3 - 3\,\rho\right) = 2\,A\,\text{waves}/\text{radius}$$

$$A = 128\,\text{waves}$$

b)

As long as the image of the mirror fills the detector the answer is independent of the size of the detector elements.

DP-17

A 20X Nomarski interference microscope contains a Wollaston prism of unknown material and prism angle. It is known that interference fringes having a spacing of 5 mm are obtained if the Wollaston prism is illuminated with a linearly polarized collimated HeNe laser beam and a properly oriented analyzer is placed after the prism. One method for obtaining the phase shift required for a phase-shifting interferometer is to translate the Wollaston prism. How far does the Wollaston prism need to be translated to obtain a 90^o phase shift between the two interfering beams?

Solution

The phase difference between the two beams changes 360^oas we move 5 mm, therefore it will change by 90^oas we move 1.25 mm. Since in the Nomarski we go through the Wollaston twice, the Wollaston will need to be translated only 625 μm to obtain the 90^ophase shift.

DP-18

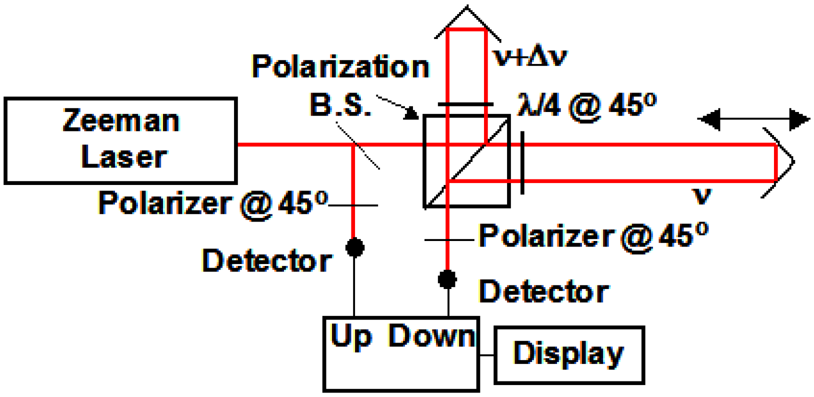
A system using a Zeeman split laser producing two orthogonally polarized beams having a difference frequency of 100 MHz as the light source and up-down counters is used to measure the motion of one mirror in Twyman-Green interferometer. The wavelength is approximately 633 nm.

a) Sketch the system showing all important components.

b) If the electronics works well for frequencies between 80 and 120 MHz, how fast can we move the mirror in the Twyman-Green interferometer?

Solution

a)



b)

$$\Delta\nu = \frac{2\,v}{\lambda} = \pm 20\ 10^6 \Big/ \text{sec}$$

$$v = \pm \frac{20 \times 10^6 / \text{sec}}{2}\ 633 \times 10^{-9}\ \text{m}\ //\ \text{N} = \pm \frac{6.33\ \text{m}}{\text{sec}}$$

DP-19

Briefly describe the spatial synchronous method of direct phase measurement. Be sure to give the major advantage and the major disadvantage compared to phase-shifting interferometry.

Solution

- The interference signal is compared to reference sinusoidal and cosinusoidal signals.

The two reference signals are

```
rcos[x_, y_] := Cos[2 π f x]
```

and

```
rsin[x_, y_] := Sin[2 π f x]
```

Multiplying the reference signal times the irradiance signal gives sum and difference signals.

```
TrigReduce[irradiance[x, y] rcos[x, y]]
```

$$\frac{1}{2}\left(2\ \text{iavg}\ \text{Cos}[2\ f\ \pi\ x] + \text{iavg}\ \gamma\ \text{Cos}[\phi[x, y]] + \text{iavg}\ \gamma\ \text{Cos}[4\ f\ \pi\ x + \phi[x, y]]\right)$$

```
TrigReduce[irradiance[x, y] rsin[x, y]]
```

$$\frac{1}{2}\left(2\,\text{iavg}\,\text{Sin}[2\,f\,\pi\,x]-\text{iavg}\,\gamma\,\text{Sin}[\phi[x,y]]+\text{iavg}\,\gamma\,\text{Sin}[4\,f\,\pi\,x+\phi[x,y]]\right)$$

The low frequency second term in the two signals can be written as

$$s1=\frac{\text{iavg}\,\gamma}{2}\,\text{Cos}[\phi[x,y]];\quad s2=-\frac{\text{iavg}\,\gamma}{2}\,\text{Sin}[\phi[x,y]]$$

$$\text{Tan}[\phi[x,y]]=\frac{-s2}{s1}$$

The only effect of having the frequency of the reference signals slightly different from the average frequency of the interference signal is to introduce tilt into the final calculated phase distribution.

A large amount of tilt must be introduced. Since a spatially limited system is not band limited, the orders are never completely separated and the resulting wavefront will always have some ringing at the edges. Also, the requirement for large tilt always limits the accuracy of the measurement. The major advantage of the technique is that only a single interferogram is required and vibration and turbulence cause less trouble than if multiple interferograms were required.

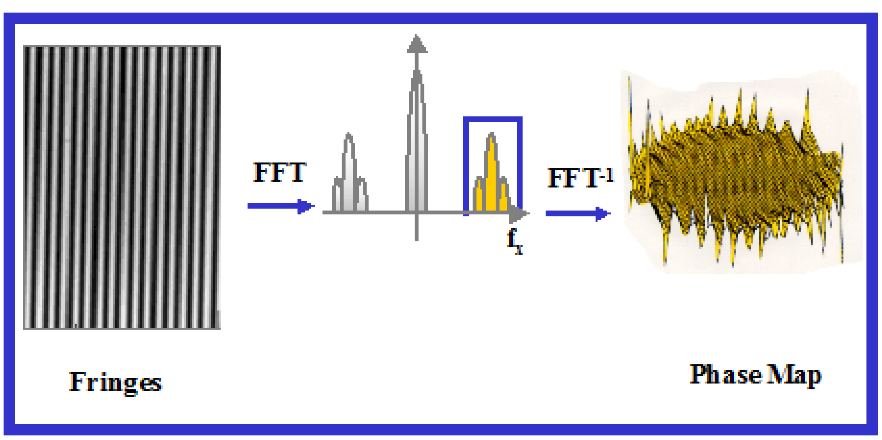
DP-20

The Fourier Method is used for direct phase measurement interferometry for testing a mirror whose surface departs from a spherical surface by an amount $8\lambda\,\rho^4$, where $0 \le \rho \le 1$. Remember that in the Fourier method a single interferogram having a large amount of tilt is used and the irradiance falling on the detector array is Fourier transformed and filtered to obtain the wavefront. For simplicity we will assume the detector array consists of an array of point detectors.

a) What is the minimum amount of tilt in units of fringes/radius required to perform the test in order to separate the orders if we are allowed to introduce defocus?

b) What is the minimum number of point detectors needed in the two-dimensional array used to detect the interferogram?

Solution



a)

The wavefront will be $16\,\lambda\,\rho^4$. To minimize tilt we must subtract $24\,\lambda\,\rho^2$. Thus, the wavefront we are testing is

$$16\,\lambda\left(\rho^4 - 1.5\,\rho^2\right)$$

The maximum slope is 16 fringes per radius. Therefore to separate orders we must have at least 16 tilt fringes across the radius. (In practice we would probably want a little more, perhaps 1.5 times as much.)

b)

The slope difference between the plane wave and the wavefront being tested would range from 0 to a maximum of 32 fringes per radius. Since we need at least two detectors per fringe (2.5 or 3 would be better) we need at least 64 detectors across a radius. The detector array must be at least 128 x 128.

DP-21

A concave spherical mirror is measured in a phase-shifting laser-based Fizeau interferometer made for the testing of flats. The diameter of the spherical mirror is 10 cm. A 1024 x 1024 element CCD array having a 5 micron pixel separation is used in the interferometer. The wavelength is 633 nm.

a) Assuming we want to test the entire mirror surface, what is the shortest radius of curvature mirror that we can measure? State any assumptions you are making.

b) Repeat part a) for an 8 micron pixel separation.

Solution

a)

Assume the image of the mirror fills the detector. We must have at least two detector elements per fringe so the maximum wavefront slope is 256 waves/radius.

$$\text{opd} = 2\,\frac{x^2}{2\,r};\quad \text{slope} = 2\,\frac{x}{r};\quad r = 2\,\frac{x}{\text{slope}}$$

$$r = 2\,\frac{5\times 10^{-2}\,\text{m}}{256\,\frac{0.6328\times 10^{-6}\,\text{m}}{5\times 10^{-2}\,\text{m}}} = 30.8648\,\text{m}$$

b)

As long as the image of the mirror fills the detector the answer is independent of the pixel separation.