Aberrations

A-1

I have an optical system with +4 waves (λ = 0.5 μm) of third order spherical aberration. If the pupil diameter is 4 cm and the distance between the exit pupil and the paraxial image plane is 10 cm, find

a) Geometrical spot size at paraxial focus.

b) Longitudinal aberration, LA.

c) Show that the focal shift required to make the OPD zero at the edge of the pupil is 0.5 LA.

d) What focal shift is required to make the transverse aberration equal to zero at the edge of the pupil?

e) Show that the minimum geometrical spot size due to third-order spherical aberration occurs at a focus three quarters the distance from the paraxial focus to the marginal focus. What is the minimum geometrical spot size? If $\Delta w = W_{040}$ $(\rho^4 - a\,\rho^2)$, what is a?

f) What is the maximum wavefront slope in units of waves/radius and arc-seconds?

Solution

$\Delta w = w_{040}\,\rho^4;\qquad w_{040} = 2\,\mu m;$

a)

$$\epsilon_\rho = -\frac{R}{h}\frac{\partial \Delta w}{\partial \rho};\ R = 10\text{ cm};\ h = 2\text{ cm};$$

$$\epsilon_{\rho max} = \text{Abs}\left[-\frac{10\text{ cm}}{2\text{ cm}}\,4\,(2\,\mu m)\right] = 40\,\mu m$$

$$\boxed{\text{geometricalSpotDiameter} = 80\,\mu m}$$

b)

$$\epsilon_z = -\frac{R^2}{\rho\,h^2}\frac{\partial \Delta w}{\partial \rho};\qquad LA_{\rho=1} = -\frac{4\,R^2}{h^2}\,w_{040};$$

$$\boxed{LA_{\rho=1} = -4\,\frac{(10\text{ cm})^2}{(2\text{ cm})^2}\,(2\,\mu m) = -200\,\mu m}$$

c)

$$\Delta w = w_{040}\,\rho^4 + \frac{\epsilon_z\,h^2}{2\,R^2}\,\rho^2 = 0 \text{ when } \rho = 1;$$

$$\epsilon_z = -\frac{2\,R^2\,w_{040}}{h^2}; \quad \text{but } LA_{\rho=1} = -\frac{4\,R^2}{h^2}\,w_{040};$$

Therefore,

$$\boxed{\epsilon_z = \frac{1}{2}\,LA_{\rho=1}}$$

d)

$$\epsilon_\rho = -\frac{R\,4\,w_{040}\,\rho^3}{h} - \frac{\epsilon_z\,h\,\rho}{R} = 0 \quad \text{when } \rho = 1;$$

$$\epsilon_z = -R^2\,4\,\frac{w_{040}}{h^2} = -\left(\frac{10\text{ cm}}{2\text{ cm}}\right)^2\,4\,(2\,\mu\text{m});$$

$$\boxed{\boldsymbol{\epsilon_z = LA_{\rho=1} = -200\,\mu\text{m}}}$$

e)

$$\epsilon_\rho = -\frac{R\,4\,w_{040}\,\rho^3}{h} - \frac{\epsilon_z\,h\,\rho}{R}$$

$$\text{For edge ray } \epsilon_\rho = -\frac{4\,R\,w_{040}}{h} - \frac{\epsilon_z\,h}{R}$$

Local extreme transverse ray error given by

$$\frac{\partial\epsilon_\rho}{\partial\rho} = -\frac{12\,R\,w_{040}\,\rho^2}{h} - \frac{\epsilon_z\,h}{R} = 0$$

$$\epsilon_z = -\frac{12\,R^2\,w_{040}\,\rho^2}{h^2}, \text{ therefore}$$

$$\epsilon_\rho = -\frac{4\,R\,w_{040}\,\rho^3}{h} + \frac{12\,R\,w_{040}\,\rho^3}{h} = \frac{8\,R\,w_{040}\,\rho^3}{h}$$

Minimum spot size occurs where edge ray meets the caustic. That is

$$-\frac{4\,R\,w_{040}}{h} - \frac{\epsilon_z\,h}{R} = \frac{8\,R\,w_{040}\,\rho^3}{h}$$

Setting the shift at which the edge ray meets the caustic equal to the shift for the local extreme ray yields

$$\epsilon_z = -\frac{R^2}{h^2}\,w_{040}\,\left(8\,\rho^3 + 4\right) = -\frac{12\,R^2\,w_{040}\,\rho^2}{h^2}$$

$$8\,\rho^3 + 4 = 12\,\rho^2; \quad \rho = -\frac{1}{2} \quad \text{or } \rho = 1$$

at the zone where $\rho = -\frac{1}{2}$

$$\boxed{\epsilon_z = -\frac{12\,R^2\,w_{040}\left(\frac{1}{4}\right)}{h^2} = -\frac{3\,R^2\,w_{040}}{h^2} = \frac{3}{4}\,\text{LA}}$$

$$\epsilon_\rho = -\frac{4\,R\,w_{040}}{h} + \frac{3\,R^2\,w_{040}\,h}{R\,h^2} = -\frac{R\,w_{040}}{h} = 10\,\mu\text{m}$$

$$\boxed{\text{Minimum spot size} = 20\,\mu\text{m} = \frac{1}{4}\,\text{spot size at paraxial focus}}$$

$$\Delta w = w_{040}\,\rho^4 + \frac{\epsilon_z\,h^2}{2\,R^2}\,\rho^2$$

$$\Delta w = w_{040}\,\rho^4 - \frac{3\,R^2\,w_{040}}{h^2}\,\frac{h^2}{2\,R^2}\,\rho^2$$

$$\boxed{\Delta w = w_{040}\left(\rho^4 - 1.5\,\rho^2\right)}$$

f)

$$\text{slope} = \frac{-1}{h}\,\frac{\partial \Delta w}{\rho} = \frac{-4}{h}\,w_{040}\,\rho^3$$

$$\boxed{\mathbf{slope}_{\rho\,\mathbf{max}} = \frac{-4}{2\,\text{cm}}\left(2\times 10^{-4}\,\text{cm}\right) = -0.0004\,\text{radian} = -82.5\,\text{sec}}$$

$$\boxed{\mathbf{slope}_{\rho\,\mathbf{max}} = -\frac{4\,(2\,\mu\text{m})}{\text{radius}}\,\frac{\text{wave}}{0.5\,\mu\text{m}} = -16\,\frac{\text{wave}}{\text{radius}}}$$

A-2

How does the minimum blur diameter due to third-order spherical aberration compare to

a) the blur diameter due to astigmatism at the circle of least confusion?

b) the width of blur due to third-order coma which is twice the sagittal coma?

Assume equal amounts of third-order spherical, coma, and astigmatism as the maximum wavefront aberration.

Solution

Transverse aberration given by $\epsilon_\rho = -\frac{R}{h}\frac{\partial \Delta w}{\partial \rho}$

$\Delta W_{040} = \Delta W_{131}\, y_o = \Delta W_{222}\, y_o{}^2$

The blur diameter is given by

$$\text{diameter} = 2\,\frac{R}{h}\,\frac{\partial \Delta w}{\partial \rho}$$

For minimum blur diameter due to third-order spherical aberration $\Delta W = \Delta W_{040}\left(\rho^4 - 1.5\,\rho^2\right)$.

$$\text{diameter} = 2\,\frac{R}{h}\,\Delta W_{040}\,(4-3) = 2\,\frac{R}{h}\,\Delta W_{040}$$

a)

Minimum spot size for astigmatism.

$\Delta W = \Delta W_{222}\, y_o{}^2\left(y^2 - \frac{1}{2}\rho^2\right) = \Delta W_{040}\left(y^2 - \frac{1}{2}\rho^2\right)$

$$\text{diameter} = 2\,\frac{R}{h}\,\Delta W_{040}\,(2-1) = 2\,\frac{R}{h}\,\Delta W_{040}$$

b)

Width of blur due to third-order coma (twice the sagittal coma).

$\Delta W = \Delta W_{131}\, y_o\, y\left(x^2 + y^2\right) = \Delta W_{040}\;\; y\left(x^2 + y^2\right)$.

$$\epsilon_x = -\frac{R}{h}\frac{\partial \Delta w}{\partial x} = -\frac{R}{h}\,\Delta W_{040}\,(2\,x\,y) = -\frac{R}{h}\,\Delta W_{040}\left(\rho^2\,\text{Sin}[2\,\theta]\right)$$

$$\epsilon_y = -\frac{R}{h}\frac{\partial \Delta w}{\partial y} = -\frac{R}{h}\,\Delta W_{040}\left(3\,y^2 + x^2\right) = -\frac{R}{h}\,\Delta W_{040}\,\rho^2\,(2 + \text{Cos}[2\,\theta])$$

$$\text{width} = 2\,\frac{R}{h}\,\Delta W_{040}$$

Thus, the minimum blur diameter due to third-order spherical aberration, the blur diameter due to astigmatism at the circle of least confusion (minimum spot size), and the width of the blur due to third-order coma (twice the sagittal coma) are all equal for the same maximum amount of wavefront aberration.

A-3

A 20 cm diameter, 80 cm radius of curvature spherical mirror is illuminated with a collimated beam of light with wavelength 633 nm. Describe the on-axis wavefront aberrations in terms of Zernike polynomials. Do not worry about terms for which the aberration corresponds to less than 1/20 wave.

Solution

Using the first three terms of the expansion the spherical aberration for a sphere used at infinite conjugate can be written as

```
Δw[r_, s_, λ_] := 2/λ (-(s^4/(2^2 2! r^3)) ρ^4 - (3 s^6/(2^3 3! r^5)) ρ^6 - (3 (5) s^8/(2^4 4! r^7)) ρ^8)
```

$$\Delta w[r\_, s\_, \lambda\_] := \frac{2}{\lambda}\left(-\frac{s^4}{2^2\, 2!\, r^3}\rho^4 - \frac{3\, s^6}{2^3\, 3!\, r^5}\rho^6 - \frac{3\,(5)\, s^8}{2^4\, 4!\, r^7}\rho^8\right)$$

$\Delta w\left[80\times 10^4\ \mu m,\ 10\times 10^4\ \mu m,\ 0.633\ \mu m\right]$ // Expand

$-77.1376\,\rho^4 - 0.602638\,\rho^6 - 0.00588513\,\rho^8$

Thus we need to use only the first two terms.

$\Delta wWaves = -77.14\,\rho^4 - 0.60\,\rho^6;$

The four Zernikes of interest are

$z15 = 20\,\rho^6 - 30\,\rho^4 + 12\,\rho^2 - 1;$

$z8 = 6\,\rho^4 - 6\,\rho^2 + 1;$

$z3 = 2\,\rho^2 - 1;$

$z0 = 1;$

We solve for the Zernike coefficients A0, A3, A8, and A15 knowing that

$$\Delta wWaves == A0\, z0 + A3\, z3 + A8\, z8 + A15\, z15.$$

SolveAlways[ΔwWaves == A0 z0 + A3 z3 + A8 z8 + A15 z15, {ρ}]

$\{\{A15 \to -0.03,\ A3 \to -38.84,\ A8 \to -13.0067,\ A0 \to -25.8633\}\}$

Therefore,

$\Delta wWaves = -0.03\, z15 - 13.0067\, z8 - 38.84\, z3 - 25.8633$

A-4

I want to maximize the Strehl ratio by balancing third-order and fifth-order spherical aberrations with defocus. What are the optimum amounts of third-order spherical and defocus if I have 8 λ of fifth-order spherical?

Solution

Zernike #15 gives the correct balance to minimize the rms and maximize the Strehl ratio.

$z15 = 20\,\rho^6 - 30\,\rho^4 + 12\,\rho^2 - 1$

$8\,\lambda$ fifth-order spherical

$-\frac{8}{20}\, 30\,\lambda = -12\,\lambda$ third-order spherical

$\frac{8}{20}\, 12\,\lambda = \frac{24}{5}\,\lambda$ defocus

A-5

Derive the relationship between Zernike polynomials and the Seidel aberrations shown in Table V (page 37) in Volume XI of Applied Optics and Optical Engineering.

Solution

Let z_n be the coefficient of the n^{th} Zernike.

$$\triangle W = z_0 + z_1 \rho \,\mathrm{Cos}[\theta] + z_2 \rho \,\mathrm{Sin}[\theta] + z_3 \left(2\rho^2 - 1\right) + z_4 \rho^2 \mathrm{Cos}[2\theta] + z_5 \rho^2 \mathrm{Sin}[2\theta] + z_6 \left(3\rho^2 - 2\right) \rho \,\mathrm{Cos}[\theta] + z_7 \left(3\rho^2 - 2\right) \rho \,\mathrm{Sin}[\theta] + z_8 \left(6\rho^4 - 6\rho^2 + 1\right)$$

$$\triangle W = (z_0 - z_3 + z_8) + (z_1 - 2 z_6) \rho \,\mathrm{Cos}[\theta] + (z_2 - 2 z_7) \rho \,\mathrm{Sin}[\theta] + (2 z_3 - 6 z_8) \rho^2 + (z_4 \mathrm{Cos}[2\theta] + z_5 \mathrm{Sin}[2\theta]) \rho^2 + 3 (z_6 \mathrm{Cos}[\theta] + z_7 \mathrm{Sin}[\theta]) \rho^3 + 6 z_8 \rho^4$$

But

$$a \,\mathrm{Cos}[\phi] + b \,\mathrm{Sin}[\phi] = \sqrt{a^2 + b^2} \,\mathrm{Cos}\left[\phi - \mathrm{ArcTan}\left[\frac{b}{a}\right]\right]$$

We can see this as follows

$$\mathbf{FullSimplify}\left[\mathbf{TrigToExp}\left[\mathbf{TrigExpand}\left[\sqrt{a^2 + b^2} \,\mathbf{Cos}[\phi - \mathbf{ArcTan}[a, b]]\right]\right]\right]$$

$$a \,\mathrm{Cos}[\phi] + b \,\mathrm{Sin}[\phi]$$

Therefore

$$(z_4 \mathrm{Cos}[2\theta] + z_5 \mathrm{Sin}[2\theta]) \rho^2 = \sqrt{z_4{}^2 + z_5{}^2} \,\mathrm{Cos}[2\theta - \mathrm{ArcTan}[z_4, z_5]] \rho^2$$

$$\mathrm{But} \quad \mathrm{Cos}[2\phi] = 2 \,\mathrm{Cos}[\phi]^2 - 1$$

$$\sqrt{z_4{}^2 + z_5{}^2} \,\mathrm{Cos}[2\theta - \mathrm{ArcTan}[z_4, z_5]] \rho^2 = \sqrt{z_4{}^2 + z_5{}^2} \,\rho^2 \left(2 \,\mathrm{Cos}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2 - 1\right)$$

$$(z_4 \mathrm{Cos}[2\theta] + z_5 \mathrm{Sin}[2\theta]) \rho^2 = -\sqrt{z_4{}^2 + z_5{}^2} \,\rho^2 + \sqrt{z_4{}^2 + z_5{}^2} \,\rho^2 \left(2 \,\mathrm{Cos}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2\right)$$

Likewise

$$3 (z_6 \mathrm{Cos}[\theta] + z_7 \mathrm{Sin}[\theta]) \rho^3 = 3 \sqrt{z_6{}^2 + z_7{}^2} \,\mathrm{Cos}[\theta - \mathrm{ArcTan}[z_6, z_7]] \rho^3$$

Piston

$$z_0 - z_3 + z_8$$

$$\mathbf{piston} = (\mathbf{z_0} - \mathbf{z_3} + \mathbf{z_8});$$

Tilt

$$\mathrm{x\,component:}\ z_1 - 2 z_6$$
$$\mathrm{y\,component:}\ z_2 - 2 z_7$$

$$\mathrm{magnitude:} \sqrt{(z_1 - 2 z_6)^2 + (z_2 - 2 z_7)^2}$$

angle : $\mathrm{ArcTan}[(z_1 - 2 z_6), (z_2 - 2 z_7)]$

tilt $= \sqrt{(z_1 - 2 z_6)^2 + (z_2 - 2 z_7)^2}\ \mathrm{Cos}[\theta - \mathrm{ArcTan}[(z_1 - 2 z_6), (z_2 - 2 z_7)]]\ \rho$

Focus

$2 z_3 - 6 z_8 - \sqrt{z_4{}^2 + z_5{}^2}$;

focus $= \left(2 z_3 - 6 z_8 - \sqrt{z_4{}^2 + z_5{}^2}\right) \rho^2$

Astigmatism

$2 \sqrt{z_4{}^2 + z_5{}^2}$;

angle is $\frac{1}{2}\ \mathrm{ArcTan}[z_4, z_5]$

astigmatism $= 2 \sqrt{z_4{}^2 + z_5{}^2}\ \mathrm{Cos}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2 \rho^2$

Sometimes $2(\sqrt{z_4{}^2 + z_5{}^2})\ \rho^2$ is added to the focus term to make its absolute value smaller and then $2(\sqrt{z_4{}^2 + z_5{}^2})\ \rho^2$ must be subtracted from the astigmatism term. This gives a focus term equal to

focusModified $= \left(2 z_3 - 6 z_8 + \sqrt{z_4{}^2 + z_5{}^2}\right) \rho^2$

and astigmatism given by

astigmatismModified =
Simplify$\left[\mathbf{Together}\left[2 \sqrt{z_4{}^2 + z_5{}^2}\ \mathrm{Cos}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2 \rho^2 - 2 \left(\sqrt{z_4{}^2 + z_5{}^2}\right) \rho^2\right]\right]$

$-2 \rho^2\ \mathrm{Sin}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2 \sqrt{z_4^2 + z_5^2}$

but

$\mathrm{Sin}\left[\theta - \frac{1}{2} \mathrm{ArcTan}[z_4, z_5]\right]^2 == \mathrm{Cos}\left[\theta - \left(\frac{1}{2} \mathrm{ArcTan}[z_4, z_5] + \frac{\pi}{2}\right)\right]^2$

True

therefore

astigmatismModified $= -2 \sqrt{z_4^2 + z_5^2}\ \mathrm{Cos}\left[\theta - \left(\frac{1}{2} \mathrm{ArcTan}[z_4, z_5] + \frac{\pi}{2}\right)\right]^2 \rho^2$

Note that not only are we changing the sign of the astigmatism term, but we are also rotating it 90°.

Coma

$3 \sqrt{z_6{}^2 + z_7{}^2}$; angle is $\mathrm{ArcTan}[z_6, z_7]$;

$$\text{coma} = 3\sqrt{z_6^2 + z_7^2}\ \text{Cos}[\theta - \text{ArcTan}[z_6, z_7]]\ \rho^3$$

Spherical

$6\ z_8$

$$\text{spherical} = 6\ z_8\ \rho^4;$$

A-6

If the coefficients for the first 8 Zernike polynomials are 5.9568, -0.5239, 0.0753, -0.1537, 0.1290, 0.0275, 0.0226, and -1.0185, determine the tilt, focus, astigmatism, coma, and fourth order spherical. Give both magnitude (including sign) and angle.

Solution

```
z1 = 5.9568; z2 = -.5239; z3 = .0753; z4 = -.1537;
z5 = .1290; z6 = .0275; z7 = .0226; z8 = -1.0185;
```

$$\text{tiltMagnitude} = \sqrt{(z1 - 2\ z6)^2 + (z2 - 2\ z7)^2} = 5.93$$

$$\text{tiltAngle} = \frac{180}{\pi}\ \text{ArcTan}[z1 - 2\ z6, z2 - 2\ z7] = -5.5\ °$$

There are two possible values of the focus. We will select the one that minimizes the absolute value of magnitude.

$$\text{focusPossible} = \left\{2\ z3 - 6\ z8 + \sqrt{z4^2 + z5^2},\ 2\ z3 - 6\ z8 - \sqrt{z4^2 + z5^2}\right\} = \{6.46226, 6.06094\};$$

$$\text{focus} = 6.06$$

$$\text{astPossible} = \left\{-2\sqrt{z4^2 + z5^2},\ 2\sqrt{z4^2 + z5^2}\right\} = \{-0.40132, 0.40132\}$$

We select the one having the sign opposite to the sign chosen for focus. In focus we selected the term having the minus sign so here we pick the plus sign.

$$\text{ast} = 2\sqrt{z4^2 + z5^2} = 0.40$$

$$\text{astAngle} = \frac{180}{\pi}\ \frac{1}{2}\ \text{ArcTan}[z4, z5] = 70\ °$$

$$\text{coma} = 3\sqrt{z6^2 + z7^2} = 0.11$$

$$\text{comaAngle} = \frac{180}{\pi}\,\text{ArcTan}[z6, z7] = 39.4\,°$$

$$\text{spherical} = 6\,z8 = -6.11$$

A-7

Starting with Eq. (3), derive Eq. (4) (page 5) in Volume XI of Applied Optics and Optical Engineering.

Solution

$$E_r = E_o\left(\frac{J_1[x]}{x}\right)^2, \text{ where } x = \frac{\pi\,r}{\lambda\,f\,\#}$$

$$\text{fractionOfEncircledEnergy} = \frac{\int_0^r \left(\frac{J_1[x]}{x}\right)^2 r\,dr}{\int_0^\infty \left(\frac{J_1[x]}{x}\right)^2 r\,dr} = \frac{\int_0^{r'} \left(\frac{J_1[x]}{x}\right)^2 x\,dx}{\int_0^\infty \left(\frac{J_1[x]}{x}\right)^2 x\,dx}$$

$$\text{fractionOfEncircledEnergy} = \frac{\int_0^{r'} \left(\frac{\text{BesselJ}[1,x]}{x}\right)^2 x\,dx}{\int_0^\infty \left(\frac{\text{BesselJ}[1,x]}{x}\right)^2 x\,dx} = 2\int_0^{r'} \left(\frac{\text{BesselJ}[1,x]}{x}\right)^2 x\,dx$$

From Born & Wolf page 440

$$\frac{d}{dx}\left(x^{n+1} J_{n+1}[x]\right) = x^{n+1} J_n[x]$$

From *Mathematica*

$$\partial_x\left(x^{n+1}\,\text{BesselJ}[n+1, x]\right) = x^{1+n}\,\text{BesselJ}[n, x]$$

If n = 0 and multiplying by BesselJ[1,x] yields

$$\text{BesselJ}[1, x]\,\partial_x\,(x\,\text{BesselJ}[1, x]) = x\,\text{BesselJ}[0, x]\,\text{BesselJ}[1, x]$$

But

$$\partial_x\,(x\,\text{BesselJ}[1, x]) = \text{BesselJ}[1, x] + x\,\partial_x\,(\text{BesselJ}[1, x])$$

Therefore

$$(\text{BesselJ}[1, x])^2 + x\,\partial_x\,(\text{BesselJ}[1, x])\,(\text{BesselJ}[1, x]) = x\,\text{BesselJ}[0, x]\,\text{BesselJ}[1, x]$$

or

$$\frac{(\text{BesselJ}[1, x])^2}{x} = \text{BesselJ}[0, x]\,\text{BesselJ}[1, x] - \partial_x\,(\text{BesselJ}[1, x])\,\text{BesselJ}[1, x]$$

Likewise

$$\partial_x\,(x^{-n}\,\text{BesselJ}[n, x]) = -x^{-n}\,\text{BesselJ}[1+n, x]$$

If n = 0

∂_x (BesselJ[0, x]) = -BesselJ[1, x]

Therefore

$$\frac{(\text{BesselJ}[1,x])^2}{x} = -\partial_x(\text{BesselJ}[0,x])\,\text{BesselJ}[0,x] - \partial_x(\text{BesselJ}[1,x])\,\text{BesselJ}[1,x]$$

or

$$\frac{(\text{BesselJ}[1,x])^2}{x} = -\frac{1}{2}\partial_x\left(\text{BesselJ}[0,x]^2 + \text{BesselJ}[1,x]^2\right)$$

But

BesselJ[0, 0] = 1; BesselJ[1, 0] = 0;

Hence

fractionOfEncircledEnergy =

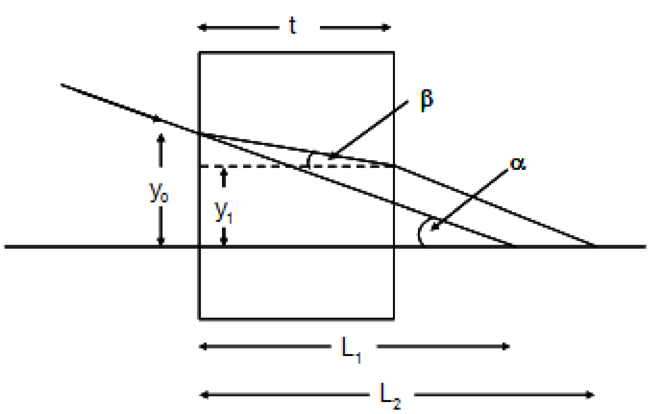
$$2\int_0^{r'}\left(\frac{\text{BesselJ}[1,x]}{x}\right)^2 x\,dx = 1 - \text{BesselJ}\left[0, \frac{\pi r}{\lambda f\#}\right]^2 - \text{BesselJ}\left[1, \frac{\pi r}{\lambda f\#}\right]^2$$

A-8

Derive Eqs. (68) (page 41), (70), and the exact version of Eq. (72) in Volume XI of Applied Optics and Optical Engineering.

Solution

Eq. 68



$$L_1 = \frac{y_o}{\text{Tan}[\alpha]};\quad L_2 = t + \frac{y_1}{\text{Tan}[\alpha]};$$

$$\text{Tan}[\beta] = \frac{y_o - y_1}{t};\quad y_1 = y_o - t\,\text{Tan}[\beta];$$

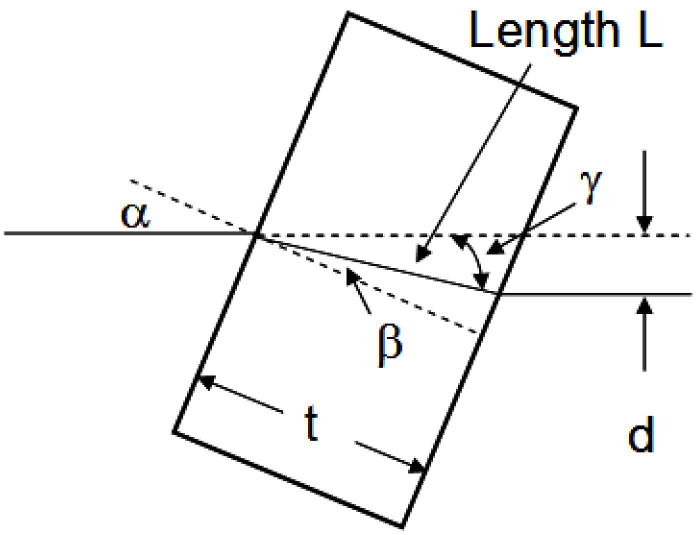
$$L_2 - L_1 = t + \frac{y_o - t\,\text{Tan}[\beta]}{\text{Tan}[\alpha]} - \frac{y_o}{\text{Tan}[\alpha]} = t - t\,\frac{\text{Tan}[\beta]}{\text{Tan}[\alpha]}$$

For small angles

$$\frac{\mathrm{Tan}[\beta]}{\mathrm{Tan}[\alpha]} \sim \frac{1}{n}$$

$$L_2 - L_1 = t \left(1 - \frac{1}{n}\right) = t \left(\frac{n-1}{n}\right);$$

Eq. 70



$$\mathrm{Cos}[\beta] = \frac{t}{L}; \quad \mathrm{Sin}[\gamma] = \frac{d}{L};$$

$$d = \frac{t\,\mathrm{Sin}[\gamma]}{\mathrm{Cos}[\beta]} = \frac{t\,\mathrm{Sin}[\alpha - \beta]}{\mathrm{Cos}[\beta]};$$

$$d = \frac{t}{\mathrm{Cos}[\beta]} (\mathrm{Sin}[\alpha]\,\mathrm{Cos}[\beta] - \mathrm{Cos}[\alpha]\,\mathrm{Sin}[\beta])$$

$$d = t\,\mathrm{Sin}[\alpha] - t\,\mathrm{Cos}[\alpha] \frac{\mathrm{Sin}[\beta]}{\mathrm{Cos}[\beta]}$$

$$d = t\,\mathrm{Sin}[\alpha] \left(1 - \frac{\mathrm{Cos}[\alpha]}{\mathrm{Sin}[\alpha]} \frac{\mathrm{Sin}[\beta]}{\mathrm{Cos}[\beta]}\right)$$

$$n\,\mathrm{Sin}[\beta] = \mathrm{Sin}[\alpha]$$

$$d = t\,\mathrm{Sin}[\alpha] \left(1 - \frac{\sqrt{1 - \mathrm{Sin}[\alpha]^2}}{n\sqrt{1 - \mathrm{Sin}[\beta]^2}}\right) =$$

$$t\,\mathrm{Sin}[\alpha] \left(1 - \frac{\sqrt{1 - \mathrm{Sin}[\alpha]^2}}{n\sqrt{1 - \frac{\mathrm{Sin}[\alpha]^2}{n^2}}}\right) = t\,\mathrm{Sin}[\alpha] \left(1 - \frac{\sqrt{1 - \mathrm{Sin}[\alpha]^2}}{\sqrt{n^2 - \mathrm{Sin}[\alpha]^2}}\right)$$

Eq. 72

From the solution for Eq. 68

$$L_2 - L_1 = t - t\,\frac{\mathrm{Tan}[\beta]}{\mathrm{Tan}[\alpha]}$$

For paraxial rays

$$L_2 - L_1 = t\left(1 - \frac{1}{n}\right);$$

Therefore,

$$L' - l' = t - t\,\frac{\mathrm{Tan}[\beta]}{\mathrm{Tan}[\alpha]} - t\left(1 - \frac{1}{n}\right);$$

$$L' - l' = \frac{t}{n}\left(1 - n\,\frac{\mathrm{Tan}[\beta]}{\mathrm{Tan}[\alpha]}\right)$$

$$n\,\mathrm{Sin}[\beta] = \mathrm{Sin}[\alpha]$$

$$\mathrm{Tan}[\beta] = \frac{\mathrm{Sin}[\alpha]}{\sqrt{n^2 - \mathrm{Sin}[\alpha]^2}}$$

$$L' - l' = \frac{t}{n}\left(1 - \frac{n\,\mathrm{Cos}[\alpha]}{\sqrt{n^2 - \mathrm{Sin}[\alpha]^2}}\right)$$

A-9

The coefficients for Zernike #8, Zernike #6, and Zernike #3 are 2, 3, and 4, respectively. All other Zernike polynomials have zero coefficients. How many waves of tilt, defocus, astigmatism, coma, and spherical aberration are present?

Solution

$$\mathbf{z8 = 6\,\rho^4 - 6\,\rho^2 + 1;}$$

$$\mathbf{z6 = \left(3\,\rho^2 - 2\right)\rho\,\mathrm{Cos}[\theta];}$$

$$\mathbf{z3 = 2\,\rho^2 - 1;}$$

$$\mathbf{\Delta w = 2\,z8 + 3\,z6 + 4\,z3\ //\ Expand}$$

$$-2 - 4\,\rho^2 + 12\,\rho^4 - 6\,\rho\,\mathrm{Cos}[\theta] + 9\,\rho^3\,\mathrm{Cos}[\theta]$$

-6 waves tilt
-4 waves defocus
no astigmatism
9 waves coma
12 waves third-order spherical

A-10

Compare the relative maximum dimension in the geometrical spot image at

a) paraxial focus for a system having 2 waves of spherical aberration and

b) sagittal focus for a system having 4 waves of astigmatism.

Solution

a) Spherical Aberration

$$\Delta w = w_{040}\,\rho^4;\ \ w_{040} = 2\,\lambda;$$

$$\epsilon_x = \frac{-R}{h}\,\frac{\partial \Delta w}{\partial x} = -\frac{R\,4\,w_{040}}{h}\,x\,\left(x^2 + y^2\right);$$

$$TA = -\frac{R\,4\,w_{040}}{h}\,\rho^3;\ \ BlurDiameter = 16\,\lambda\,\frac{R}{h};$$

b) Astigmatism

$$\Delta w = w_{222}\,y_o{}^2\,\rho^2\,Cos[\phi]^2;\ \ w_{222}\,y_o{}^2 = 4\,\lambda;$$

$$\epsilon_y = \frac{-2\,R}{h}\,\left(w_{222}\,y_o{}^2\right)\,y;$$

$$LengthOfFocalLine = 16\,\lambda\,\frac{R}{h};$$

Length of sagittal focal line = length of tangential focal line. Therefore, if $\frac{R}{h}$ is the same for both systems we have the same maximum dimension in the geometrical spot image for 2λ of spherical as for 4λ of astigmatism.

A-11

A 50 mm diameter, f/8 lens has 2 waves of third-order spherical when it is illuminated with a collimated beam of HeNe light (λ = 633 nm).

a) What is the diameter of the geometrical image of a collimated beam at paraxial focus?

b) How many waves of defocus should be added to give the minimum geometrical spot image?

c) What is the minimum diameter of the geometrical spot image?

Solution

a)

$\Delta w = w_{040}\,\rho^4;\quad w_{040} = 2\,\lambda;\quad \lambda = 0.633\text{ microns};$

$\epsilon_\rho = -\frac{R}{h}\frac{\partial \Delta w}{\partial \rho};\; R = 400\text{ mm};\; h = 25\text{ mm};$

$$\epsilon_{\rho max} = \text{Abs}\left[-\frac{400\text{ mm}}{25\text{ mm}}\,4\,(2)\,(0.633\,\mu\text{m})\right] = 81.024\,\mu\text{m}$$

geometricalSpotDiameter = 162.05 μm

b)

For every wave of third-order spherical we should add -1.5 waves of defocus. Therefore, we need -3 waves of defocus to get minimum spot size.

c)

$$\Delta w = w_{040}\left(\rho^4 - 1.5\,\rho^2\right)$$

Therefore the spot size will be $\frac{1}{4}$the spot size at paraxial focus.

geometricalSpotDiameter = 40.51 μm

A-12

I have a special optical system where the only aberration is third-order coma. The Strehl ratio for the system must be greater than 0.7. Which one or two Zernike polynomials are we interested in for this system? How large can the coefficient be before the optical system is unacceptable?

Solution

z[6] and z[7] are the two Zernikes of interest, n = 2 and m = 1.

$\text{Strehl} = e^{-(2\pi\sigma)^2} = 0.7$

FindRoot$\left[e^{-(2\pi\sigma)^2} == 0.7, \{\sigma, 0.01\}\right]$

$\{\sigma \to 0.095051\}$

$$\sigma^2 = \sum_{n=1}^{\infty}\left(\frac{A_n^{\,2}}{2\,n+1} + \frac{1}{2}\sum_{m=1}^{n}\frac{B_{nm}^{\,2} + C_{nm}^{\,2}}{2\,n+1-m}\right)$$

$$\sigma^2 = \frac{1}{2}\,\frac{B_{21}^{\,2}}{2\,(2)+1-1}$$

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B21 = √8 σ = 0.27;
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A-13

Use the table of Zernike polynomials provided in class to help answer the following.

a) What is the smallest number of waves of defocus that will introduce zero irradiance on axis for an aberration free image of a point source?

b) As the obscuration ratio of a circular aperture increases, does the size of the central core of the diffraction pattern increase or decrease?

c) We have 4 waves of third-order spherical aberration. How many waves of defocus should be added to maximize the Strehl ratio?

d) We have 4 waves of third-order spherical aberration. How many waves of defocus should be added to minimize the geometrical spot size?

Solution

a)

1 wave.

b)

Decrease.

c)

-4 waves.

d)

-6 waves.

A-14

I have a conic having a conic constant k = -0.5, a radius of curvature of 40 cm, and a semi-diameter of 5 cm. If I place a point source of wavelength 0.5 microns on axis a distance of 30 cm away from the conic, how many waves of aberration (peak-valley) are present in the reflected beam?

Solution

The ideal conic would have a distance to the foci of 30 cm which is 0.75 the radius of curvature. Thus the conic constant we want is

$$k = \frac{2\,n - 1}{n^2} - 1 \,/.\, n \to 0.75$$

-0.111111

The conic we have is

$k_c = -0.5;$

The spherical aberration we have, in units of waves, is

$$\text{thirdOrder} = \frac{2}{\lambda}\left(\frac{(k_c + 1) - (k + 1)}{2^2\, 2!\, r^3}\right) s^4;$$

$$\text{fifthOrder} = \frac{2}{\lambda}\left(\frac{3\left((k_c + 1)^2 - (k + 1)^2\right)}{2^3\, 3!\, r^5}\right) s^6;$$

$$\text{seventhOrder} = \frac{2}{\lambda}\left(\frac{3\,(5)\left((k_c + 1)^3 - (k + 1)^3\right)}{2^4\, 4!\, r^7}\right) s^8;$$

spherical = thirdOrder + fifthOrder + seventhOrder;

variables = $\{\lambda \to 0.5 \times 10^{-4}\ \text{cm},\ s \to 5\ \text{cm},\ r \to 40\ \text{cm}\}$;

thirdOrder /. variables

-18.9887

fifthOrder /. variables

-0.206041

seventhOrder /. variables

-0.00215073

The spherical aberration in waves is given by

spherical /. variables

-19.1969

A-15

I have a system having an aberration of the form $(7\lambda)\ \rho^3\ \text{Cos}[\phi] - (7\ \lambda)\ \rho^3 \text{Sin}[\phi]$ where $0 \le \rho \le 1$. Linear tilt of the form A x + B y is introduced to minimize the rms wavefront variation over the circular aperture of unit radius. What should A and B be equal to?

Solution

aberration = $(7\ \lambda)\ \rho^3\ \text{Cos}[\phi] - (7\ \lambda)\ \rho^3\ \text{Sin}[\phi]$

$z[6] = \left(3\,\rho^2 - 2\right)\rho\,\text{Cos}[\phi]$; and $z[7] = \left(3\,\rho^2 - 2\right)\rho\,\text{Sin}[\phi]$;

Therefore,

$A = \frac{-2}{3}(7\lambda) = -4.67\lambda;$ $\quad B = \frac{-2}{3}(-7\lambda) = 4.67\lambda;$

A-16

The coefficient of the Zernike polynomial for which n = 2 and m = 0 is equal to 5λ. How many waves of third-order spherical aberration are present?

Solution

n = 2 and m = 0 for z8.

$z8 = 6\rho^4 - 6\rho^2 + 1;$

If the coefficient is 5λ we will have 30 waves of third-order spherical aberration present.

A-17

I am given the job of measuring the position of the first zero of the image of the point source formed using an optical system having a central obscuration. As part of the job, I need to calculate the theoretical position of the first zero. Let ϵ = diameter of central obscuration divided by diameter of optical system and let β = diameter of central maximum of diffraction pattern for system having the central obscuration divided by the diameter of central maximum of diffraction pattern for system if it did not have a central obscuration. Derive an expression in terms of J_1, the first order Bessel function, relating β and ϵ.

Solution

Use Babinet's principle.

$$i = \frac{E_A(\pi/4)^2}{\lambda^2}\left(d_o^2\frac{2J_1\left[\frac{2\pi}{\lambda}\frac{d_o}{2f}r\right]}{\left(\frac{2\pi}{\lambda}\frac{d_o}{2f}r\right)} - d_i^2\frac{2J_1\left[\frac{2\pi}{\lambda}\frac{d_i}{2f}r\right]}{\left(\frac{2\pi}{\lambda}\frac{d_i}{2f}r\right)}\right)^2$$

Let $\epsilon = \frac{d_i}{d_o}$.

For the first zero of the unobstructed aperture $\frac{2\pi}{\lambda}\frac{d_o}{2f}r = 1.22\pi$. If β is the distance to the first zero in units of the Airy disk radius for the unobstructed aperture,

$$i = \frac{E_A\left(\pi d_o^2/4\right)^2}{\lambda^2}\left(\frac{2J_1[1.22\pi\beta]}{(1.22\pi R)} - \epsilon^2\frac{2J_1[1.22\pi\epsilon\beta]}{(1.22\pi\epsilon R)}\right)^2$$

For the first zero

$$\epsilon J_1[1.22\pi\epsilon\beta] = J_1[1.22\pi\beta]$$

A-18

The first nine Zernike polynomials can be written as

#0 1

#1 ρ Cos[ϕ]

#2 ρ Sin[ϕ]

#3 $2\rho^2 - 1$

#4 ρ^2 Cos[2ϕ]

#5 ρ^2 Sin[2ϕ]

#6 $\left(3\rho^2 - 2\right)\rho$ Cos[ϕ]

#7 $\left(3\rho^2 - 2\right)\rho$ Sin[ϕ]

#8 $6\rho^4 - 6\rho^2 + 1$

a) What are the names of the aberrations we associate with each term?

b) Give the Zernike coefficients for an aberration of the form $6\rho^3\cos[\phi] + 4\rho^4 + 6\rho^2$.

c) I have a system having an aberration of the form $(7\lambda)\,\rho^3\cos[\phi] - (5\lambda)\,\rho^3\sin[\phi]$ where $0 \le \rho \le 1$. Linear tilt of the form Ax + By is introduced to minimize the rms wavefront variation over the circular aperture of unit radius. What should A and B be equal to?

Solution

a)

#0 Piston

#1 and #2 Tilt

#3 Defocus and Piston

#4 and #5 Astigmatism and Defocus

#6 and #7 Coma and Tilt

#8 Spherical, Defocus, and Piston

b)

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aberration = 6 ρ^3 Cos[θ] + 4 ρ^4 + 6 ρ^2;

z8 = 6 ρ^4 - 6 ρ^2 + 1;

z6 = (3 ρ^2 - 2) ρ Cos[θ];

z3 = 2 ρ^2 - 1;

z1 = ρ Cos[θ];

z0 = 1;
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Expand[2/3 z8 + 2 z6 + 5 z3 + 4 z1 + 13/3 z0] == aberration
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True

c)

$\text{aberration} = (7\,\lambda)\,\rho^3 \cos[\phi] - (5\,\lambda)\,\rho^3\ \sin[\phi];$

$z6 = \left(3\,\rho^2 - 2\right)\,\rho\,\text{Cos}[\phi];$

$z7 = \left(3\,\rho^2 - 2\right)\,\rho\,\text{Sin}[\phi];$

$A = (7\,\lambda)\,\left(\frac{-2}{3}\right) = -\frac{14}{3}\,\lambda$

$B = (-5\,\lambda)\,\left(\frac{-2}{3}\right) = \frac{10}{3}\,\lambda$