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Quantitative Analysis of Qualitative Images

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ABSTRACT

We show optical evidence that demonstrates artists as early as Jan van Eyck and Robert Campin (c1425) used optical projections as aids for producing their paintings. We also have found optical evidence within works by later artists, including Bermejo (c1475), Lotto (c1525), Caravaggio (c1600), de la Tour (c1650), Chardin (c1750) and Ingres (c1825), demonstrating a continuum in the use of optical projections by artists, along with an evolution in the sophistication of that use. However, even for paintings where we have been able to extract unambiguous, quantitative evidence of the direct use of optical projections for producing certain of the features, this does not mean that paintings are effectively photographs. Because the hand and mind of the artist are intimately involved in the creation process, understanding these complex images requires more than can be obtained from only applying the equations of geometrical optics.

Keywords: image analysis, human vision, Renaissance, paintings, art, van Eyck, Holbein

1. INTRODUCTION

We have found optical evidence that demonstrates some artists as early as c1425 used lenses or concave mirrors as direct aids for producing some features of some of their paintings. Here we address only evidence of the <u>direct</u> use of optical projections by artists. Such use can significantly aid the artist in making measurements and noting key points on the subject, resulting in an identifiable optical base to certain features within paintings that one of us (DH) observed to first appear early in the 15th century.[1] Today everyone is completely familiar with the distinct appearance of "photographic" imagery, but this was not the case in the early 15th century, so the identification of an optical base is an important piece of evidence. However, the use of lenses does not mean artists were compelled to slavishly trace all of the details of the projected images. Also, no less important is the <u>indirect</u> use of optics. Viewing an optical projection, rather than the actual scene, allows the artist to move their head without the reflections and highlights from shiny materials like satin or steel armor moving. On an optical projection the relationship of three-dimensional objects and the volatile form and position of reflections and highlights remains fixed even when the artist moves. Even without making marks from the projected image, these and other indirect uses of optics offer representational possibilities to the artist hitherto unrealized, and perhaps inconceivable without an optical model.

We selected the examples in this manuscript to show how quantitative evidence can be extracted from images even though they were produced by hand, and thus contain both optical and non-optical ("eyeballed") features. As a consequence of the way they were produced, these images are much more complex than photographs. However, even in the painting we refer to below as our "Rosetta Stone," Lorenzo Lotto's *Husband and Wife* of c1523–24, only approximately 5% of the surface area exhibits evidence of having been produced with the direct aid of projected images. All the details of these paintings—the composition, the colors, the textures, the tonal range—were decided by the artist, not dictated by a lens. As we show, the amazing perspective and certain of the features of the chandelier in Jan van Eyck's *Arnolfini Marriage* contains evidence that it was produced with the direct aid of a projected image. However, the small dog elsewhere in this painting would not have remained stationary long enough for a lens to be of any possible help in painting its image, again demonstrating that these works of art are collections of lens-based and "eyeballed" elements.[2] Because the hand and mind of the artist are intimately involved in the creation process, understanding these images requires more than can be obtained from only applying the equations of geometrical optics as if they were photographs whose latent images were fixed by paint rather than by chemicals.

As the many independent pieces of visual and optical evidence we have assembled demonstrate,[1,3,4] some artists—Robert Campin, Jan van Eyck, and Rogier van der Weyden, to name three of the earliest practitioners—as early as c1425 used lenses as aids for producing some of the features in some of their paintings. Whether or not any of these artists had the skill to have precisely reproduced at the same level of detail the same features in those particular works without having used lenses certainly is an interesting question to contemplate. Irrespective of this question, the optical evidence presented in this manuscript shows that they did indeed use lenses, rather than alternatives such as sighting devices or Alberti's grid,[5,6] and hence these discoveries provide insights that were previously unavailable into the practices of many important artists.

2. HISTORICAL BACKGROUND

Briefly, both the theoretical knowledge of optics as well as the suitable optic elements—both refracting and reflecting—were available by the early Renaissance. Sixty-one texts on optics written between the years 1000, the time of al-Haytham, and 1425, that of van Eyck, have survived,[7] showing this 425-year period was one of remarkable scientific activity. Although only a few of these texts have been translated from Latin, those that have provide detailed descriptions for fabricating suitable concave mirrors from metal. For example, *"Make a spherical mirror as before [from clear iron]; smooth and polish its interior along the concave portion of its curvature…"*.[8]

Tomaso da Modena's 1352 paintings of *Hugh of Provence* and *Cardinal Nicholas of Rouen* show, respectively, spectacles and a magnifying glass, and *Isnardo of Vicenza* and *St. Jerome* both show concave mirrors. About these mirrors, Robert Gibbs writes[9] that *"Isnardo da Vicenza is preparing his office; there is a reading glass (an enlarging-concave-mirror) on the shelf behind him."* In a footnote to that sentence, Gibbs explains *"Mirrors, despite their inconvenient habit of reversing the text, were used alongside lenses to enlarge small and faded handwriting"*.[10] Gibbs continues in his footnote *"The use of mirrors for reading continued into the sixteenth century, and the second (not the first) representation, of a variant type set in a leather horn rather than on a fixed metal stand, appears on St. Jerome's shelf..."*. Not only were the necessary optics available by the early Renaissance, as shown by the da Modena paintings, they also were inexpensive.[11]

3. QUALITATIVE IMAGES; QUANTITATIVE EVIDENCE (FIVE EXAMPLES)

3.1 Lorenzo Lotto, *Husband and Wife*, c1523–24

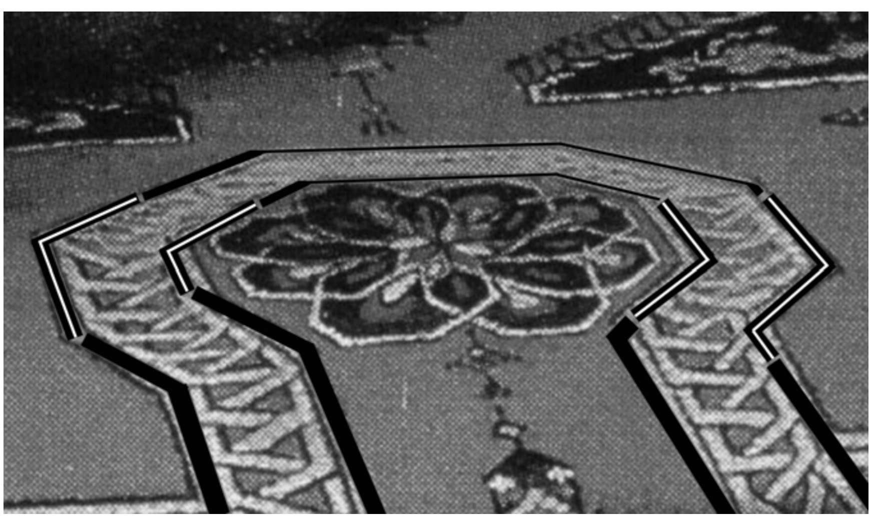

Figure 1. *Husband and Wife* (detail). The overlays are perspective-corrected sections of an octagonal pattern that we fit to the painting. As described in the text, the details of this portion of the painting are in excellent qualitative and quantitative agreement with the three-segment, perspective-corrected octagon that is predicted by the laws of geometrical optics for such a projected image.

Figure 1 is a detail from Lorenzo Lotto's *Husband and Wife* of c1523–4. This painting contains so much information that we have referred to it as our "Rosetta Stone." Figure 1 shows an octagonal pattern on an oriental carpet that appears to go out of focus at some depth into the painting. Overlaid on this painting are three segments of a perspective-corrected octagon whose overall fit to the pattern is excellent, and whose quantitative details we calculate below.[12]

As we show below, the magnification of this painting is $M \approx 0.56$. Any optical projection at such a high magnification intrinsically has a relatively shallow depth of field (DOF), the value of which depends on the focal length and diameter of the lens as well as the magnification. To change the distance of sharp focus requires physically moving the lens with respect to the subject and the image plane. To refocus a lens further into a scene from its original plane of focus requires moving the lens further away from the scene, resulting in a small decrease in the magnification of the projected scene, as well as in a slight change in the vanishing points. Although such effects are fundamental characteristics of images projected by lenses, they are extremely unlikely to occur in a painting if an artist had instead laid out patterns using sighting devices or following geometrical rules first articulated in the fifteenth century.[5,6]

Since we already have discussed several aspects of this painting elsewhere,[1,3,4] here we provide a variation of our previous analysis. The distance across the wife's shoulders in the painting provides an internal length scale that lets us determine the magnification to be $M \approx 0.56$.[13] This in turn allows us to determine the repeat distance of the triangular pattern on the actual carpet to be 3.63 cm. The first place where the image of the carpet changes character is approximately 4–5 triangular-repeats into the scene, from which we calculate the depth of field to be DOF=16±1.5 cm.

The focal length (FL) and magnification (M) are given by the following equations from geometrical optics:

$$1/FL = 1/(d_{lens-subject}) + 1/(d_{lens-image}) \qquad (1)$$

and

$$M = (d_{lens-image})/(d_{lens-subject}) \qquad (2)$$

As indicated by the overlays on Figure 1, there are three regions of this octagonal pattern, which result from Lotto having refocused twice as he exceeded the DOF of his lens. We label these Regions 1, 2, and 3, with Region 1 the closest to the front of the painting. Thus, for the first two Regions,

$$1/FL = 1/(d_{lens-subject1}) + 1/(d_{lens-image1}) \qquad (3)$$

and

$$1/FL = 1/(d_{lens-subject2}) + 1/(d_{lens-image2}) \qquad (4)$$

However, the measured DOF is 16±1.5 cm, so for Region 2

$$d_{lens-subject2} \approx d_{lens-subject1} + 16\text{ cm} \qquad (5)$$

and thus

$$1/FL = 1/(d_{lens-subject1} + 16\text{ cm}) + 1/(d_{lens-image2}) \qquad (6)$$

Because Region 2 is further into the scene, and therefore at slightly lower magnification, than is Region 1, its DOF will be somewhat larger than 16 cm. We can calculate DOF_2 from[14]

$$DOF_2 = 2\,C \times f\# \times (1+ M_2)/ M_2^2 \qquad (7)$$

where C is the circle of confusion, f# is the lens diameter / focal length, and M_2 is the magnification of Region 2. Hence,

$$DOF_2 = DOF_1 \times (1+M_2) / (1+M_1) \times (M_1/M_2)^2 \qquad (8)$$

Region 3 of the pattern thus starts at a depth of 16 cm + DOF_2 into the scene, so

$$d_{lens-subject3} = d_{lens-subject1} + 16\text{ cm} + DOF_2 \qquad (9)$$

and

$$1/FL = 1/(d_{lens-subject1} + 16\text{ cm} + DOF_2) + 1/(d_{lens-image3}) \quad (10)$$

The magnifications M of the three regions are given by:

$$0.56 = d_{lens-image1} / d_{lens-subject1} \quad (11)$$
$$M_2 = d_{lens-image2} / (d_{lens-subject1} + 16\text{ cm}) \quad (12)$$
$$M_3 = d_{lens-image3} / (d_{lens-subject1} + 16\text{ cm} + DOF_2) \quad (13)$$

This analysis give us seven equations (3, 6, 8, 10, 11, 12, and 13) and eight unknowns: FL, $d_{lens-subject1}$, $d_{lens-image1,2,3}$, DOF_2, $M_{1,2}$. If we make a single assumption about any one of these unknowns, we can solve these equations uniquely for the other seven unknowns. Assuming that the distance from the lens to the carpet was at least 1.5 meters, but no greater than 2.0 meters (i.e. $d_{lens-subject1} = 175\pm25$ cm) we find

$$\text{focal length} = 62.8\pm9.0\text{ cm}$$
$$M_2 = 0.489\pm0.9$$
$$M_3 = 0.423\pm1.5$$

The magnification when moving from Region 1 to Region 2, as measured from our fit of a perspective-corrected octagon, decreases by 13.1% from the original 0.56 of the painting, in excellent agreement with the –12.6±1.5% calculated from the above equations. Similarly, the measured magnification decreases by a further 13.3% when going to Region 3, again in excellent agreement with the calculated value of –13.5±1.6%.

From Eq. (7),

$$f\# = [DOF_1 \times M_1^2] / [2\,C\,(1+ M_1)]$$

If we assume the simple lens available to Lotto resulted in a circle of confusion on the painting of 2 mm, we find f# ≈ 22, and hence a diameter of 2.9±0.4 cm. As we have confirmed with our own experiments, a lens or concave mirrors with these properties projects a quite useful image of a subject that is illuminated by daylight.

To summarize, using only the measured magnification of this painting (0.56, i.e. roughly half life size, as determined from the size of the wife[13]), and making a reasonable assumption about the distance Lotto would have positioned his lens from the carpet (175±25 cm), the laws of geometrical optics uniquely determine both changes in magnification (–13.1% and –13.3%) of the central octagonal pattern, as well as the focal length and diameter of the lens (62.8±9.0 cm and ~3 cm, respectively) used to project this image. The three sets of vanishing points exhibited by the octagonal pattern, as well as the depths into the painting where they occur, are a direct and inevitable consequence of the use of a lens to project this portion of the painting.[6] Other quantitative information extracted from this painting is discussed elsewhere.[4]

3.2 Robert Campin, *The Annunciation Triptych (Merode Altarpiece)*, c1425–30

The center and right panels of Robert Campin's *Merode Triptych* of c1425–28 contain the earliest evidence we have found to date of the use of direct optical projections. A detail of the right panel is shown at the lower left of Figure 2. As we previously showed,[4] this portion of the painting exhibits the same complex changes in perspective seen in Lorenzo Lotto's *Husband and Wife,* resulting from Campin also having refocused his lens twice.

The upper right of Figure 2 shows one of the two sets of slats (the set that is numbered on the lower inset), with each slat individually rotated to be vertical and expanded horizontally by a factor of 3.5× to accentuate any deviations from being straight. Marked on the slats are the locations of "kinks" exhibited by each of them, with those kinks connected by lines. The positions of the lines connecting the kinks are shown on the inset at the lower left. Comparing with Fig. 2 of Reference 4 it can be seen that the slats are kinked at the same two depths into the painting where we previously showed, with a different type of analysis using different data, that Campin had to refocus due to the DOF of his lens. Geometrical constructions can be devised which exhibit kinks, but not in the overall configuration of this painting. The complex perspective exhibited by the latticework in this portion of the painting is a direct and inevitable outcome from the DOF of a lens, but would be extremely unlikely to have resulted from any geometrical construction, or from the use of a straightedge.[6]

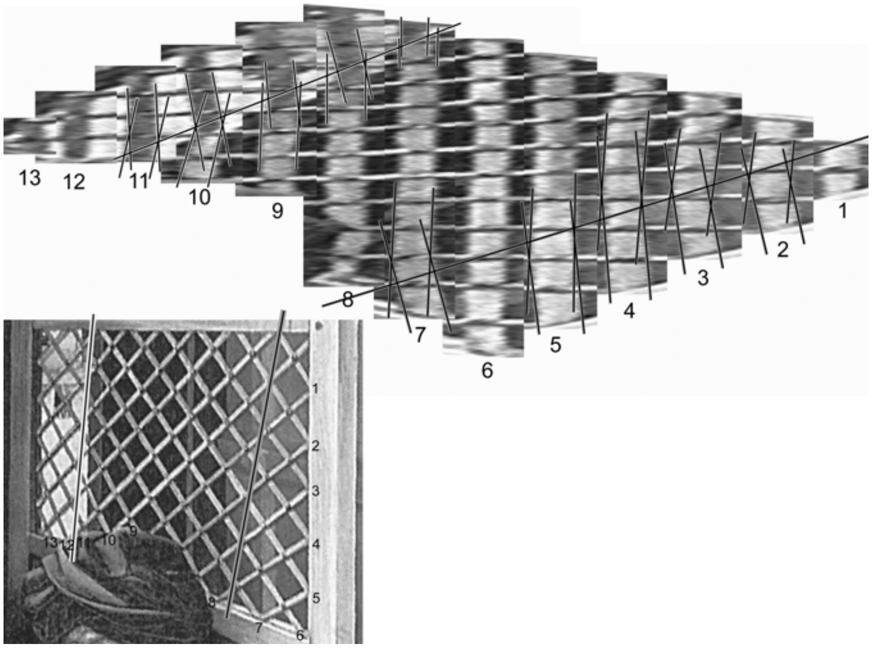

Figure 2. (Lower Left) detail of the *Merode Altarpiece* with one set of slats numbered. **(Upper Right)** slats rotated to be vertical and expanded horizontally by 3.5×. We have connected the "kinks" that are apparent in the slats by lines, the locations of which are shown in the detail at the lower left.

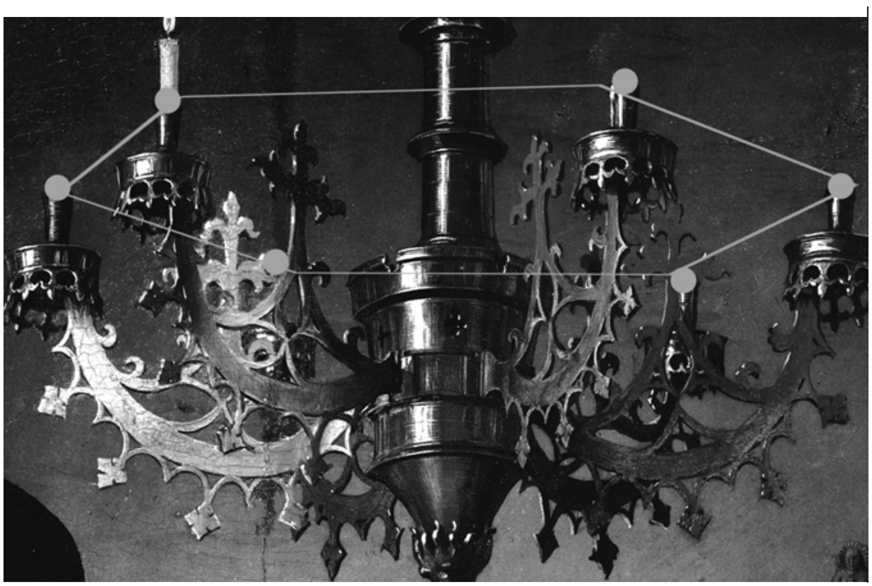

Figure 3. *The Arnolfini Marriage* (detail). As can be seen, a perspective-corrected hexagon fits the positions of the tops of the candle holders to a remarkable accuracy, with small deviations from ideal symmetry consistent with a large, handmade 15th century object.

Using the height of the head in the full painting as a scale, the magnification of this portion of the painting is M≈0.27. If we assume a circle of confusion of 1 mm,[15] Eq. 7 yields f#=25.2. We can obtain an estimate for the focal length with the assumption the lens or concave mirror had a diameter of 3 cm, in which case the focal length FL = f# × 3 cm = 76 cm, which is quite reasonable.

3.3 Jan van Eyck, *The Arnolfini Marriage*, 1434

The remarkable perspective, texture, and detail of the complex chandelier shown in Figure 3 caused us to examine it for optical evidence that certain critical aspects of it might be based on a direct optical projection. Such a projection would have allowed van Eyck to make measurements and note key points that enabled him to obtain a level of accuracy for the overall perspective of this complex object that never had been previously achieved in any painting. The use of a lens resulted in the identifiable optical base to certain of the features within the chandelier, even though a skilled artist would not have needed to trace every detail in order to produce a work of art even as convincing as this one is.

As we noted previously,[4] it is important to analyze appropriate aspects of the chandelier to determine whether or not it is based on an optical projection, since it would have been easier for van Eyck to "eyeball" many of the features of the chandelier after having established its optical base. Paintings such as the *Arnolfini Marriage* are collages, made up of optical and non-optical elements, with even the optical elements containing eyeballed features as well.[2] Another important point is that any painting of a three-dimensional object, such as this chandelier, has reduced that object to a two-dimensional space, so some spatial information inevitably has been lost. This differs from the Lotto and Campin paintings discussed earlier, where the carpet and seat back both are two-dimensional.

Previously we estimated that the magnification of the chandelier element of the *Arnolfini Marriage* is 0.16, and hence that the outer diameter of the original chandelier was approximately 1 m.[4] This magnification is small enough that the DOF for a lens falling within any reasonable range of focal lengths and diameters would be over 1 m, and hence van Eyck could have seen the entire depth of the image without needing to refocus. If van Eyck had used measurements from an optical projection of a real chandelier to establish the overall symmetry of it on the painting, it is reasonable to expect that, after correcting for perspective, the positions of the tops of each of the six candle holders should exhibit something close to perfect hexagonal symmetry, although with deviations from ideal symmetry due to the imperfections of a large, handmade 15th century object. If, however, he had painted this very complex three-dimensional object only by eyeball, and lacking any knowledge of analytical perspective, larger deviations in the positions of these candle holders would be expected.

The dots in Figure 3 are the positions of the tops of each of the candle holders, and the six-sided shape is an ideal perspective-corrected hexagon. As can be seen, the agreement of the six candle holders with the points of a perfect hexagon is remarkable. The hexagon has been rotated 6^{o} counterclockwise, due to the chandelier being viewed from that angle. The maximum deviation of any of the candle holders from a perfect hexagon is only 7^{o}, which in turn corresponds to the end of that half-meter-long arm being bent only 6.6 cm away from its "ideal" hexagonal position. The root-mean-square deviation of all six candle holders from perfect hexagonal symmetry is only 4.2^{o}, or 4.1 cm. While there is no reason to expect a hand-made 15th century chandelier to exhibit accuracy any greater than this in the first place, there is also the possibility some or all of the individual deviations could have resulted from slight bends during transportation, hanging, or cleaning.

Figure 4. The outlines of all six arms on the Arnolfini chandelier, corrected for perspective and with the arms to the viewer's right flopped horizontally to overlay on the arms to the left. Our analysis scheme shows the main arc of all six arms are the same to within 1.5% in length and 5% in width. Variations in the decorative elements are consistent with these features having been eyeballed as well as having been hand attached to the original chandelier.

As we previously showed,[4] only in the oversimplified version of perspective described in most textbooks must perspective lines converge to foci, and the foci all lie on a single horizon line. Consequently, drawing lines to determine vanishing points can produce deceptive results when applied to a large, imperfect, complex object like this chandelier. Because of this, we devised a new scheme for analyzing it.[4]

Although chandeliers are three dimensional objects, note that the individual arms of the one in the *Arnolfini Marriage* are two dimensional. Taking advantage of this feature is fundamental to our analysis scheme, shown in Figure 4. In this Figure we have individually corrected each of the six arms of the chandelier for perspective and overlaid them to reveal similarities and differences. Where Figure 4 does not show a complete arm it is because it is partially hidden by the arms in front of it. The loss of information when transforming a three-dimensional object into two dimensions introduces ambiguities. The scheme we used to analyze this chandelier avoids this difficulty.

The main arcs of all six arms are identical to within 5% in width and 1.5% in length. The fact all of the arms are identical in length to within 1.5% is consistent with our independent analysis of the radial positions of the candle holders.[4] However, since it would have been easier for van Eyck to eyeball many aspects of this chandelier, rather than to trace the entire projected image, it is not surprising that there are variations in the positions of the decorative features attached to those arcs.

From this evidence, along with that we published previously,[4] we conclude with a high degree of confidence that van Eyck's chandelier is based on an optical projection of a real chandelier. Although there are some small differences (e.g. the height of one arm) that provide insights into artistic choices van Eyck made to deviate from tracing the projection, possibly to accentuate some features, overall the unprecedented realistic perspective of this complex object is a result of the optical projection technique that our analysis shows he used.

3.4 Jan van Eyck, *Portrait of Cardinal Niccolò Albergati*, c1432

Jan van Eyck produced a drawing of Cardinal Albergati that is approximately 40% smaller than his painting of the same subject. However, as Figure 5 shows, when we enlarge the drawing and overlay it on the painting, the correspondence between the major features (eyes, nose, mouth, etc.) as well as the minor details (wrinkles, lines, creases, etc.) within each of the three large regions shown in this Figure, is to a precision of better than 1 mm. This level of precision, in three separate image segments, provides strong evidence that van Eyck used a lens to enlarge the drawing to the size of the painting, accidentally (or, possibly, deliberately) moving it twice when doing so.

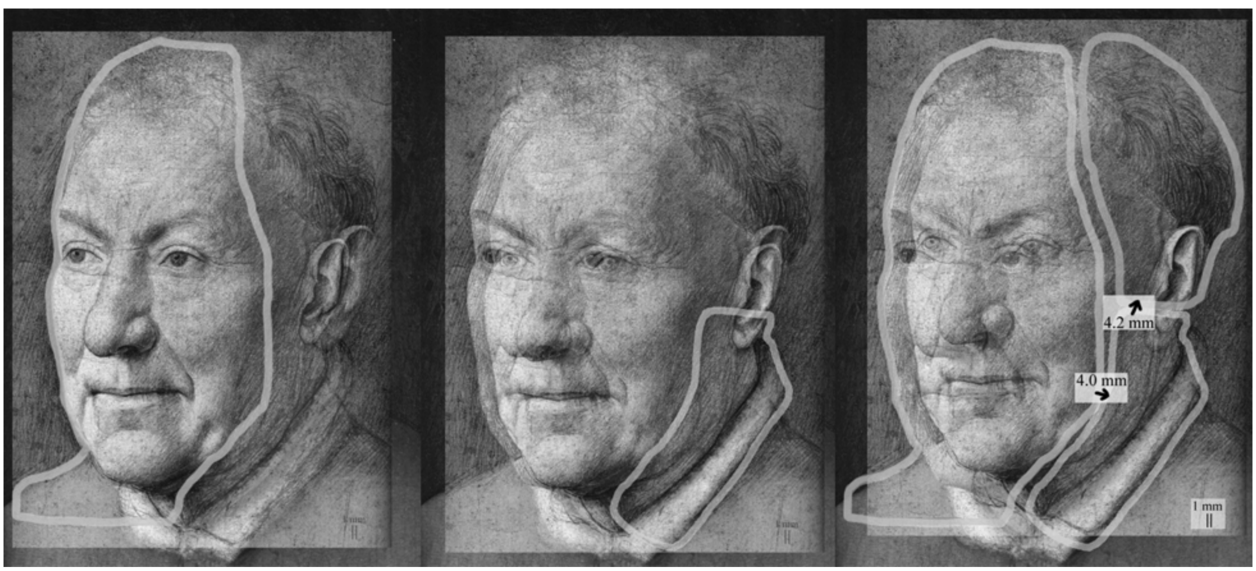

Figure 5. *Portrait of Cardinal Niccolò Albergati* (detail). Van Eyck’s drawing of the Cardinal has been enlarged by approximately 40% and overlaid on his painting. **(Left)** position of enlarged drawing adjusted for optimum registration of the features on the Cardinal’s face. **(Center)** position of drawing moved 4.0 mm to the right and slightly down, giving optimum registration of the features in the collar, neck, and lower ear. **(Right)** position moved 4.2 mm up and to the right to register the upper ear and head.

There are various possible explanations for why van Eyck might have produced his drawing of the Cardinal at a smaller scale than his painting. For example, van Eyck simply might not have had paper of the requisite size available, or something about the protocol in dealing with a Cardinal may have been involved. However, another possibility is suggested by the fact Cardinal Albergati visited Bruges for four days, 8–11 December 1431. At that time of year the daylight in northern Europe is at its lowest intensity, and the probability of having overcast skies is relatively high. Since the brightness of a projected image scales as the square of the magnification, this ~40% reduction in size of the drawing would have resulted in an image ~2× as bright for van Eyck to work with if he had used a lens to aid him in making his drawing of the Cardinal. Later, the same lens and optical setup could be used to enlarge the drawing to the size that the Cardinal presumably had commissioned for the painting. To do this, the drawing would have been placed in the sunlight, rather than the Cardinal, and its enlarged image projected onto the canvas in the identical fashion van Eyck could have used to produce the drawing in the first place. The three major regions on the canvas are accounted for by two very slight "bumps" of the optical setup during this process of producing the enlarged painting. The lengths and directions of the arrows are drawn to scale: the region in Figure 5(Center) is 4.0 mm to the right and slightly down from that in Figure 5(Left); the region in the Figure 5(Right) is 4.2 mm up and to the right of that shown in Figure 5(Center).

3.5 Hans Holbein the Younger, *The French Ambassadors to the English Court,* 1532

An anamorphic skull is a prominent feature of *The French Ambassadors to the English Court*, Hans Holbein, 1532. This feature is shown in Figure 6. By linearly compressing it by 6×, the way this anamorphic skull appears to someone viewing the painting at a grazing angle is shown in Figure 7(Right),[16] with a real skull for comparison in Figure 7(Left). Very obvious differences include that the jaw of Holbein's skull is much longer than the real skull, the slope of the top of the skull is steeper, and the eye sockets and nose are much more pronounced as well as aimed more in the direction of the viewer.

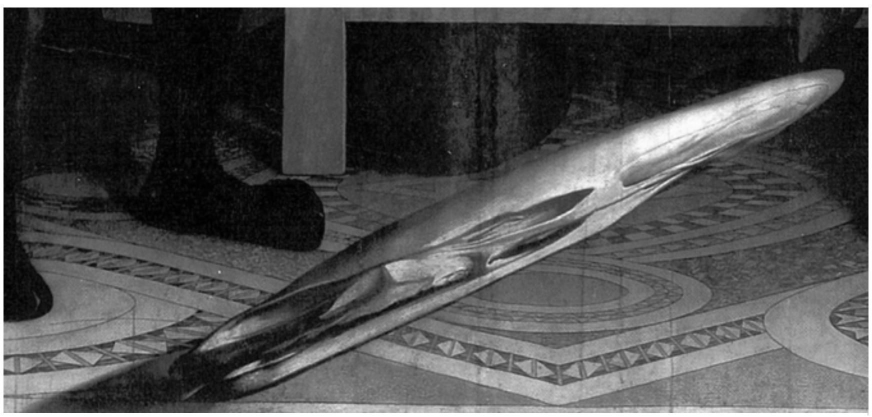

Figure 6 *The French Ambassadors to the English Court* (detail). This detail shows the unusual feature at the bottom of Holbein's painting. Viewed from a grazing angle to visually compress it, this feature appears as shown in Figure 7(Right). Possibly not apparent in this small B&W reproduction is that this anamorphic skull does not occupy the same visual space as the rest of the painting.

To see if optical projections may account for the appearance of this skull in the painting, we used a concave mirror of focal length 41 cm to project the image of a real skull onto a screen at a grazing angle[17] in order to produce an anamorphic image. Figure 7(Left) is a photograph of the real skull taken from precisely the location of that concave mirror after it had been removed from its holder. However, because of the limited depth of focus[18] of the projected image on the tilted screen, it was necessary to refocus the concave mirror a number of times in order to generate the composite anamorphic image that we have compressed linearly to produce

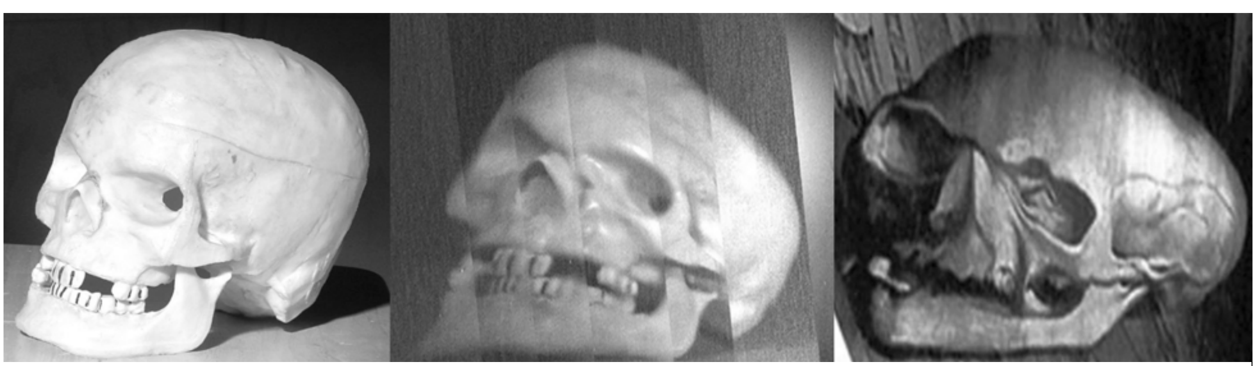

Figure 7. (Left) photograph of a skull taken from the position of the concave mirror used to project its image onto a tilted screen to form an anamorphic image. **(Center)** composite of the individual in-focus segments of the projected anamorphic image of the skull after linearly compressing it horizontally. **(Right)** anamorphic skull in *The French Ambassadors* after linearly compressing it horizontally.

Figure 7(Center).

The segments of each of the in-focus images are visible in this composite. What is striking about Figure 7(Center) is how well it reproduces the very unusual visual appearance of the linearly-compressed skull from Holbein's painting.[19] Although mathematical and graphical methods can be used to construct anamorphic images, the optics-produced composite of Figure 7(Center) is far more complex than is obtained from any such construction. The magnification of each segment in the anamorphic photographic composite is linear in the vertical direction, but is proportional to 1/sin of the grazing angle in the horizontal. The overall composite of Figure 7(Center) is thus the result of a highly nonlinear, piecewise-segmented transformation. Although this complex transformation was naturally produced by the optical projection, it would be quite implausible to have resulted from any sort of a graphical or mathematical construction.[6] We conclude that the probability is extremely small that Holbein could have accidentally reproduced these complex features without having projected them with a lens.

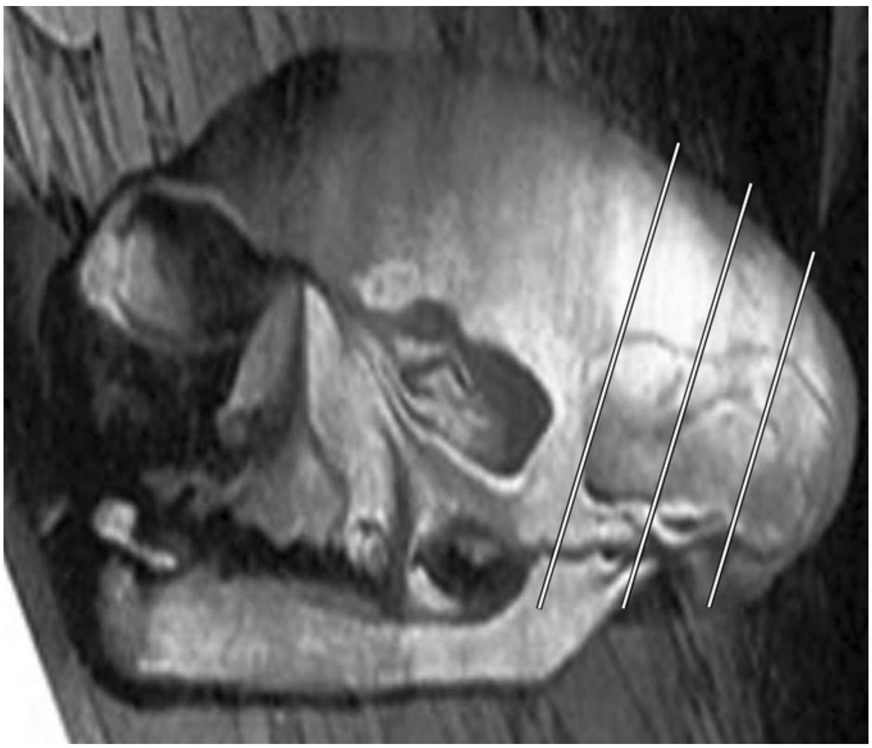

Figure 8. Anamorphic skull in *The French Ambassadors to the English Court*. For this Figure we have rotated the feature in Figure 6 clockwise by 25^o and then linearly compressed it by 6×. The height of the skull in this image compared to a real one gives a magnification M=0.71±0.5. The lines indicate two regions where it can be seen Holbein duplicated features (notably, the two dark depressions just above the jaw, and the double-humped line midway up the skull). A discontinuity in the slope of the top of the skull is also visible at the left edge of the leftmost marked region.

Figure 8 shows Figure 7(Right) at a larger scale. Marked on this Figure are two regions where we observed that Holbein has duplicated features of the skull. Because the lens and canvas (or, less likely, the skull) has to be moved a number of times when piecing together an anamorphic image from segments projected at such a high magnification, it is very easy to accidentally duplicate a region, so its presence provides additional evidence that Holbein had to refocus a lens. The duplicated segment corresponds to a region 3.0±0.5 cm wide on a real skull. That same region corresponds to a width of 8.2 cm on the actual painting, Figure 6, which gives us an approximate lower limit measure for the depth of focus. From the results of our experiments shown in Figures 7(Left) and (Center), that region of the skull is at an angle of $25^o \pm 5^o$ with respect to the perpendicular to the axis of the lens, so its depth into the scene is 1.3±0.5 cm. Although a more accurate value for the depth of focus can be obtained by convoluting this measured depth of field into the calculation, for our purposes here the approximate value 8.2 cm will suffice.[20] Using this value, along with a circle of confusion of 2 mm and the measured M=0.71, we calculate as a lower limit[14]

$$\begin{aligned} f\# &\geq \text{Depth of Focus} / [2C \times (M+1)] \\ &= 12.0 \end{aligned}$$

Because we have neglected the depth of field in the calculation shown here, this value for the f# of Holbein's lens is somewhat smaller than the actual value, as well as represents a lower limit.[20] However, this calculation is sufficient to show that the f# of Holbein's lens is consistent with the values we obtained for Lotto's and Campin's lenses (22 and 25.2, respectively).

4. SUMMARY AND CONCLUSIONS

As a consequence of the way they were constructed, the optics-based paintings discussed here are much more complex than photographs. However, in spite of this complexity, the examples in this manuscript show that we are able to extract quantitative evidence from these images, even though they were produced by hand and contain both optics-based and non-optics-based ("eyeballed") features. These discoveries demonstrate that highly influential artists used optical projections as aids for producing some of their paintings early in the 15th century, at the dawn of the Renaissance, at least 150 years earlier than previously thought possible. In addition to Holbein, we also have found optical evidence within works by later artists, including Bermejo (c1475), Caravaggio (c1600), de la Tour (c1650), Chardin (c1750) and Ingres (c1825), demonstrating a continuum in the use of optics by artists, along with an evolution in the sophistication of that use.[21] However, even for paintings where we have been able to extract unambiguous, quantitative evidence of the direct use of optical projections for producing certain of the features, it does not mean that these paintings are effectively photographs. Because the hand and mind of the artist are intimately involved in the creation process, understanding these images requires more than can be obtained from only applying the equations of geometrical optics. Also, although we have only briefly addressed it in this manuscript, no less important for understanding the evolution of post-c1425 painting is the indirect use of optics.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

1. David Hockney, ***Secret Knowledge: Rediscovering the Lost Techniques of the Old Masters*** (Viking Studio, 2001).

2. "*Each element in the painting* [Bouts' *Last Supper*]*—each face, each object—was drawn separately… and then pieced together on the panel. In effect, it's a collage.*" Reference 1, p. 87.

3. David Hockney and Charles M. Falco, "Optical Insights into Renaissance Art," Optics & Photonics News **11**, 52 (2000).

4. David Hockney and Charles M. Falco, "Optics at the Dawn of the Renaissance," Technical Digest of the Optical Society of America, 87th Annual Meeting (OSA, 2003).

5. Martin Kemp, ***The Science of Art*** (Yale University Press, New Haven, 1990).

6. Ellen Winner, Professor of Psychology at Boston College, expressed this important point at the 2003 'Conference on Measuring Art: A Scientific Revolution in Art History' as *"[Hockney and Falco] aren't saying lenses are the only way to get the perspective right. They're saying lenses are the only way to get the perspective wrong in precisely the ways these artists erred."*

7. D.C. Lindberg, ***A Catalog of Medieval and Renaissance Optical Manuscripts*** (Pontifical Institute of Medieval Studies, 1975).

8. A.M. Smith, *translator,* ***Book V of Witelo's Perspectiva*** [c1274] (Polish Academy of Sciences, 1983), p. 98.

9. Robert Gibbs, ***Tomaso da Modena*** (Cambridge University Press, 1989).

10. Note: when properly oriented, concave mirrors magnify the text without reversing it.

11. Vincent Ilardi, ***Renaissance Vision, from Spectacles to Telescopes*** (American Philosophical Society, 2005).

12. For the purposes of the present manuscript, we traced lines over the actual perspective-corrected octagonal image segments we had previously fit to the pattern. For the original color version of this image, see Reference 4, which can be downloaded from:

http://www.optics.arizona.edu/ssd/FAQ.html

13. Although we used the size of the woman as the length scale to determine the magnification of the carpet, we have confirmed the reasonableness of this result with other measurements. For example, two similar keyhole pattern carpets are shown in Rosamond E. Mack's chapter "Lotto: A Carpet Connoisseur" in ***Lorenzo Lotto, Rediscovered Master of the Renaissance*** (National Gallery, Washington, 1997). The width of the octagonal pattern in the carpet in her Figure 1, from the Museum fűr Islamische Kunst, Berlin, is 25 cm, and in her Figure 3, from the Metropolitan Museum, New York, is 37 cm (although this latter carpet is significantly warped and stretched, making it unsuitable for detailed comparisons with Lotto's painting). Using the length scale we determined from the woman's shoulders, the width of the octagonal pattern in the carpet in Lotto's painting, Figure 1 of this manuscript, is 33 cm; intermediate in size between the 25 cm and 37 cm of the octagonal patterns on the two similar carpets in the collections of the abovementioned museums.

14. See, for example, Charles E. Engles, ***Photography for the Scientist*** (Academic Press, New York, 1986).

15. This circle of confusion is half the 2 mm we assumed for our calculations of Lorenzo Lotto's *Husband and Wife*, but is consistent since the magnification of Campin's painting is half that of Lotto's.

16. The anamorphic feature in Figure 6 is 106 cm long and 14.4 cm high. To visually compress its length to be the same as its height so that it appears approximately like Figure 7(Right) requires viewing the painting at a grazing angle of $\sin^{-1}(14.4/106) \approx 8^{o}$. At this angle the far end of the anamorphic feature is over 100 cm further away from the viewer than is the near end, so that for reasonable viewing distances the magnification of the far end is significantly less than that of the near end. Also, since for any reasonable viewing distance the depth of the feature is greater than the depth of field of the eye, it requires the viewer to scan back and forth through the feature, with their eyes constantly refocusing when doing so, in order to "construct" a composite image in their mind that does indeed strongly resemble Figure 7(Right). Although the feature in Figure 6 was constructed optically, the multiple positions of the lens need to generate it, coupled with the multiple movements and refocusing of the eye needed to view it, along with the mental compositing need to construct the final image of it in the brain, results in a fascinating underlying complexity to Figure 7(Right) that is beyond the scope of the present manuscript to explore in detail. However, for these reasons, Figure 7(Right) only approximately reproduces what the feature looks like to the viewer when examining the painting from a grazing angle.

17. The grazing angle of the screen on which we projected the image segments for Figure 7(Center) was approximately 80^{o}, i.e. nearly parallel to a line connecting the subject and the lens.

18. For the Lotto and Campin paintings we were concerned with the *depth of field*, which is the range of distances along the optical axis for which the subject appears in focus when projected onto an image plane that is perpendicular to the optical axis. Although this is the geometry for the cameras most people are familiar with, "view cameras" allow swings, tilts and shifts of the lens and film planes. To produce the anamorphic skull in the Holbein painting required the image plane to be tilted, so the *depth of focus* is quite important as well. The *depth of focus* is the distance on either side of the ideal image plane that is perpendicular to the optical axis for which the projected image will appear in focus.

19. In Figures 7(Center) and (Right) we show the compressed versions of our photographic composite and Holbein's anamorphic feature because it is much easier to compare the two images in this form. They are directly comparable in this Figure, just as they would be in their original uncompressed form, since both images shown here result from simply linear transformations from their anamorphic forms.

20. Taking into account the depth of field of 1.3 cm gives f# = 14.7 for Holbein's lens. Further taking into account the likely way we believe an artist would have used a lens when producing an anamorphic image like Holbein's, we estimate f# $\approx$ 20±5.

21. Although Vermeer (c1660) is of interest for his possible use of optics, he is not included in this list because we have not examined his paintings ourselves for optical evidence. For a detailed analysis of Vermeer's works in the context of the use of projected images, see Philip Steadman, ***Vermeer's Camera*** (Oxford, 2001).