Computational Methods

Short Course on Image Quality

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What we Will Cover

- Sources of randomness
- Computation of linear-observer performance
- Computation of ideal-observer performance
- Miscellaneous

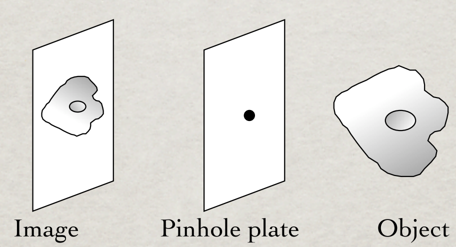
Sources of Randomness

$$g = \mathcal{H}f + n$$

- Every term in the imaging equation may have random variations*
- Neglecting a source of randomness can cause spurious results
- Addressing all sources of randomness can be computationally difficult

*Random variations in $\mathcal{H}$ will not be covered

Example



What size pinhole maximizes detectability?

Example

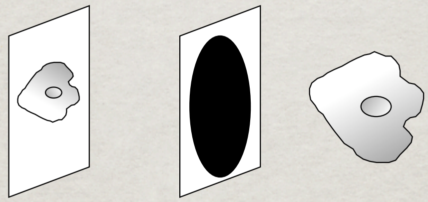
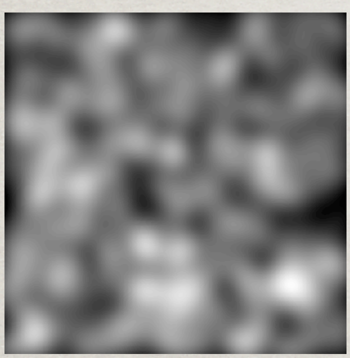
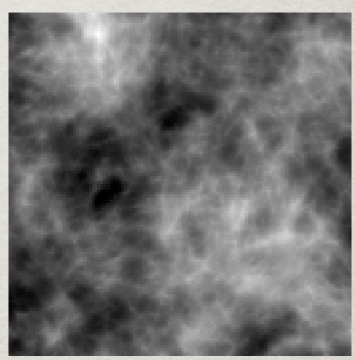


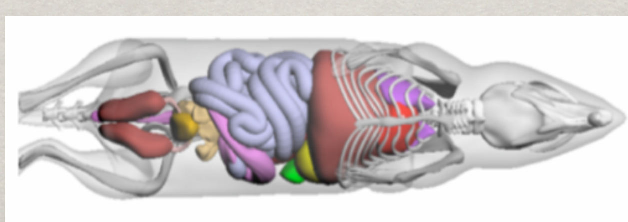
Image Pinhole plate Object

For a non-random background f , the ideal observer uses a wide-open aperture!

Object Models







Object Models

- Reduce the object model $\boldsymbol{f}$ down to a finite number of parameters $\boldsymbol{f}(\boldsymbol{\rho})$

Linear Observers

Hotelling observer

$$\mathrm{SNR}^2 = \Delta\overline{\boldsymbol{g}}^{\dagger}\boldsymbol{K}_{\boldsymbol{g}}^{-1}\Delta\overline{\boldsymbol{g}}$$

Wiener estimator

$$\mathrm{EMSE} = \mathrm{tr}\left[\boldsymbol{K}_{\boldsymbol{\theta}} - \boldsymbol{K}_{\boldsymbol{g},\boldsymbol{\theta}}^{\dagger}\boldsymbol{K}_{\boldsymbol{g}}^{-1}\boldsymbol{K}_{\boldsymbol{g},\boldsymbol{\theta}}\right]$$

Data Covariance

- If $\boldsymbol{g}$ is an $M \times 1$ vector where M is the number of pixels
- $\boldsymbol{K_g}$ is an $M \times M$ matrix
- We need to both **estimate** and **invert** this very large matrix!

Data Covariance

- Let us assume that we have a single 128 by 128 detector
- The matrix $\boldsymbol{K_g}$ has 270 million elements
- We need at least 128^2 or 16,000 images to produce an invertible estimate of $\boldsymbol{K_g}$

Data Covariance

- Without any assumption of independence, we can decompose the covariance matrix
  - In the case of the Hotelling observer

    $$\boldsymbol{K_g} = \overline{\boldsymbol{K}}_{\boldsymbol{n}} + \boldsymbol{K}_{\overline{\boldsymbol{g}}} \qquad (14.27)$$

- $\overline{\boldsymbol{K}}_{\boldsymbol{n}}$ is the noise covariance averaged over objects
- $\boldsymbol{K}_{\overline{\boldsymbol{g}}}$ is the covariance of the noise-free background images

Data Covariance

- For photon-counting detectors, the noise covariance $\overline{\boldsymbol{K}}_{\boldsymbol{n}}$ is a diagonal matrix with elements given by $\overline{\boldsymbol{g}}$
- An estimate of $\boldsymbol{K}_{\overline{\boldsymbol{g}}}$ can be produced using long-exposure images of random backgrounds
- The addition of a known diagonal matrix ensures that the overall covariance $\boldsymbol{K_g}$ is invertible even with few samples

Data Covariance

- We now have a full-rank estimate of the data covariance
- We still need to invert this estimate
  - Neumann series
  - Matrix-inversion lemma
  - Channels

Neumann Series

$$\boldsymbol{K_g} = \sigma^2 \boldsymbol{I} + \boldsymbol{A} = \sigma^2 \left[I + \frac{1}{\sigma^2} \boldsymbol{A} \right] \qquad (14.37)$$

$$\boldsymbol{K_g}^{-1} = \frac{1}{\sigma^2} \sum_{j=0}^{\infty} \left[-\frac{1}{\sigma^2} \boldsymbol{A} \right]^j \qquad (14.38)$$

Matrix-Inversion Lemma

$$\boldsymbol{K_g} = \overline{\boldsymbol{K}}_{\boldsymbol{n}} + \hat{\boldsymbol{K}}_{\overline{\boldsymbol{g}}} = \overline{\boldsymbol{K}}_{\boldsymbol{n}} + \boldsymbol{W}\boldsymbol{W}^{\dagger} \qquad (14.46)$$

- Where $\boldsymbol{W}$ is a $M \times N_s$ matrix

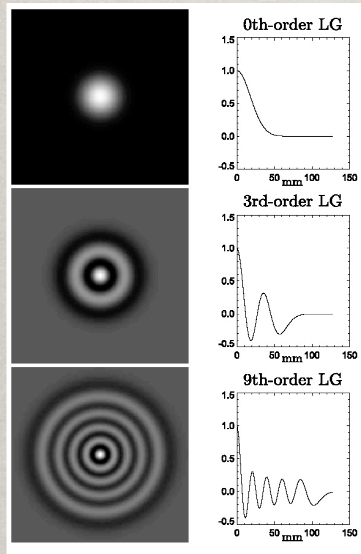
$$\left[\overline{\boldsymbol{K}}_{\boldsymbol{n}} + \boldsymbol{W}\boldsymbol{W}^{\dagger}\right]^{-1} = \left[\overline{\boldsymbol{K}}_{\boldsymbol{n}}\right]^{-1} - \left[\overline{\boldsymbol{K}}_{\boldsymbol{n}}\right]^{-1}\boldsymbol{W}\left[\boldsymbol{I} + \boldsymbol{W}^{\dagger}\left[\overline{\boldsymbol{K}}_{\boldsymbol{n}}\right]^{-1}\boldsymbol{W}\right]^{-1}\boldsymbol{W}^{\dagger}\left[\overline{\boldsymbol{K}}_{\boldsymbol{n}}\right]^{-1} \qquad (14.47)$$

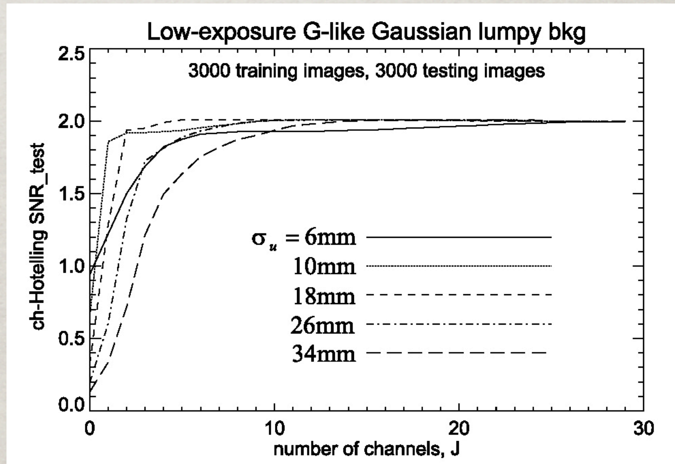
Channelized Hotelling Observer

$$\boldsymbol{v} = \boldsymbol{U}^{\dagger}\boldsymbol{g} \qquad (14.22)$$

$$\begin{aligned}\mathrm{SNR}_{\boldsymbol{v}}^2 &= \Delta\overline{\boldsymbol{v}}^{\dagger}\boldsymbol{K}_{\boldsymbol{v}}^{-1}\Delta\overline{\boldsymbol{v}} \\ &= \Delta\overline{\boldsymbol{g}}^{\dagger}\boldsymbol{U}\left[\boldsymbol{U}^{\dagger}\boldsymbol{K}_{\boldsymbol{g}}\boldsymbol{U}\right]^{-1}\boldsymbol{U}^{\dagger}\Delta\overline{\boldsymbol{g}} \qquad (14.23)\end{aligned}$$

Channelized Hotelling Observer





The Wiener Estimator

Hotelling observer

$$\boldsymbol{K_g} = \overline{\boldsymbol{K}}_{\boldsymbol{n}} + \boldsymbol{K}_{\overline{\boldsymbol{g}}}$$

Wiener estimator

$$\boldsymbol{K_g} = \overline{\boldsymbol{K}}_{\boldsymbol{g}}^{\text{noise}} + \overline{\boldsymbol{K}}_{\overline{\boldsymbol{g}}}^{\text{bg}} + \boldsymbol{K}_{\overline{\overline{\boldsymbol{g}}}}^{\text{param}}$$

Ideal Observers

- Bayesian ideal observer

$$\Lambda(\boldsymbol{g}) = \frac{pr(\boldsymbol{g}|H_2)}{pr(\boldsymbol{g}|H_1)} = \langle \Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}) \rangle_{\boldsymbol{b}|\boldsymbol{g},H_1} \qquad (14.80)$$

$$\Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}) = \frac{pr(\boldsymbol{g}|H_2,\boldsymbol{b})}{pr(\boldsymbol{g}|H_1,\boldsymbol{b})} \qquad (14.81)$$

Ideal Observers

$$\Lambda(\boldsymbol{g}) = \frac{pr(\boldsymbol{g}|H_2)}{pr(\boldsymbol{g}|H_1)} = \langle \Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}) \rangle_{\boldsymbol{b}|\boldsymbol{g},H_1} \qquad (14.80)$$

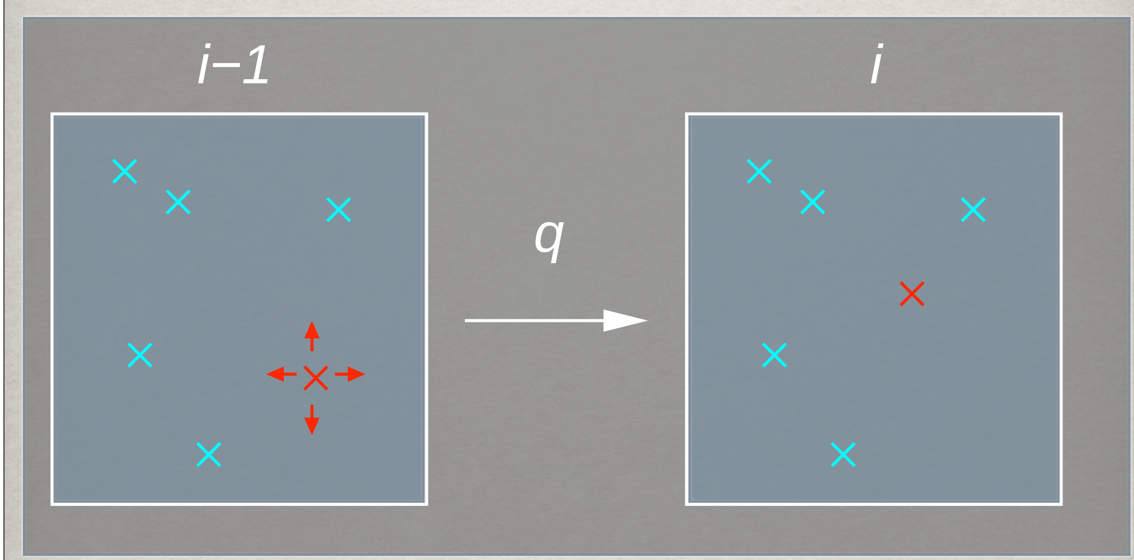
- If we could draw samples from $pr(\boldsymbol{b}|\boldsymbol{g},H_1)$, then we could approximate the ideal observer by,

$$\Lambda(\boldsymbol{g}) \approx \frac{1}{N_s} \sum_{n=1}^{N_s} \Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}_n) \qquad (14.87)$$

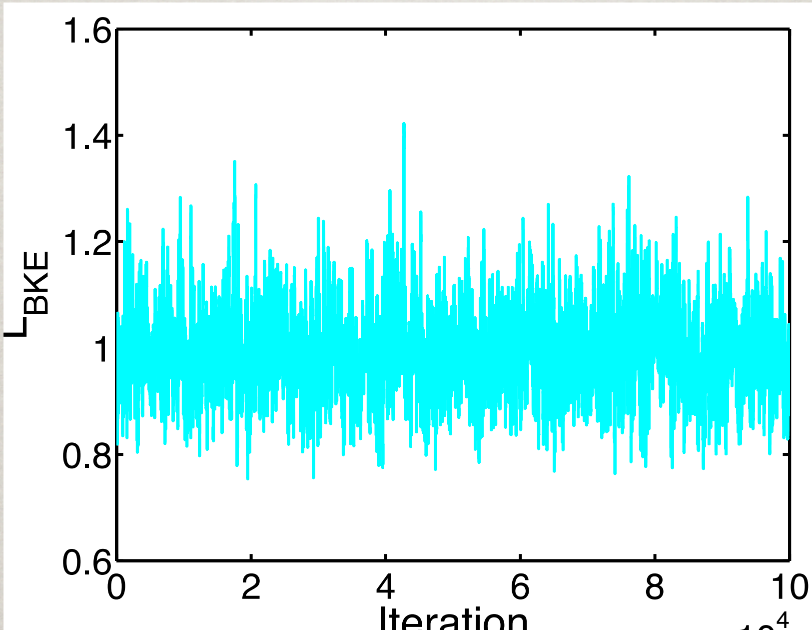
MCMC

- Markov chain Monte Carlo
  - Requires parameterized object model $\boldsymbol{f}(\boldsymbol{\rho})$
  - Directed search through parameter space
  - Each iteration only depends on the previous one
  - $pr(\boldsymbol{b}|\boldsymbol{g}, H_1)$ need only be known to a scale

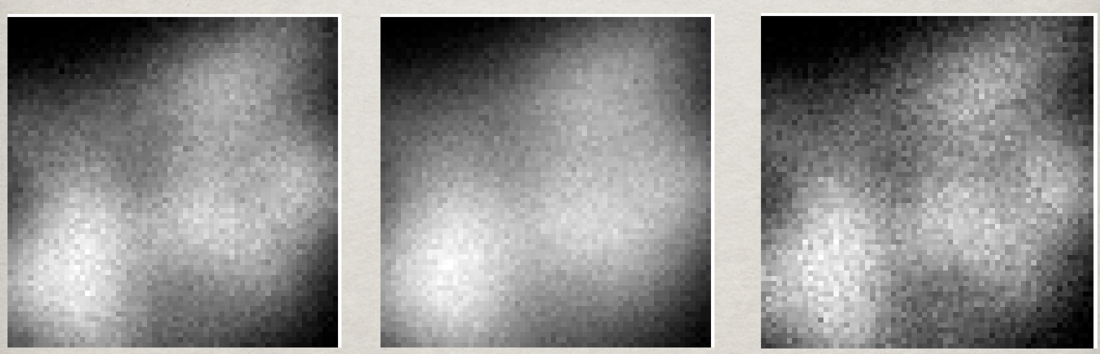
MCMC



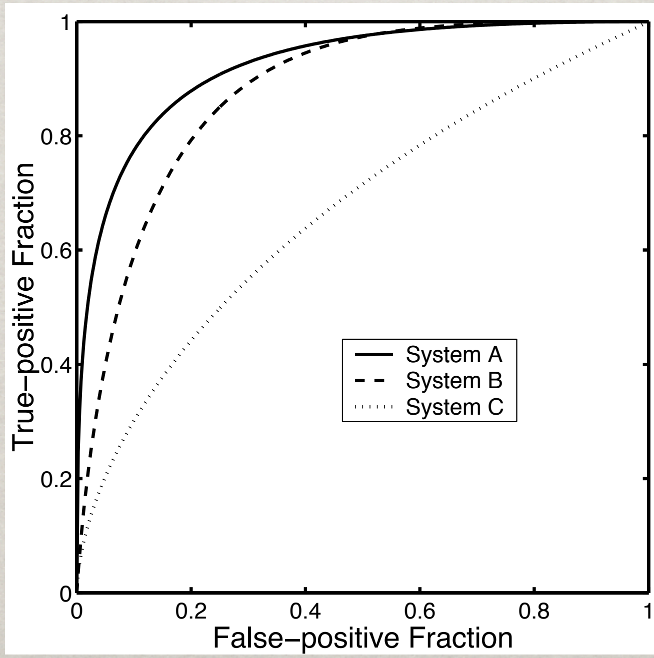
MCMC



MCMC



MCMC



What about Estimation?

Bayesian estimation

$$\hat{\theta}_{\mathrm{MMSE}} = \left\langle \theta \left\langle \Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}, \theta, \theta_0) \right\rangle_{\boldsymbol{b}|\boldsymbol{g},\theta_0} \right\rangle$$

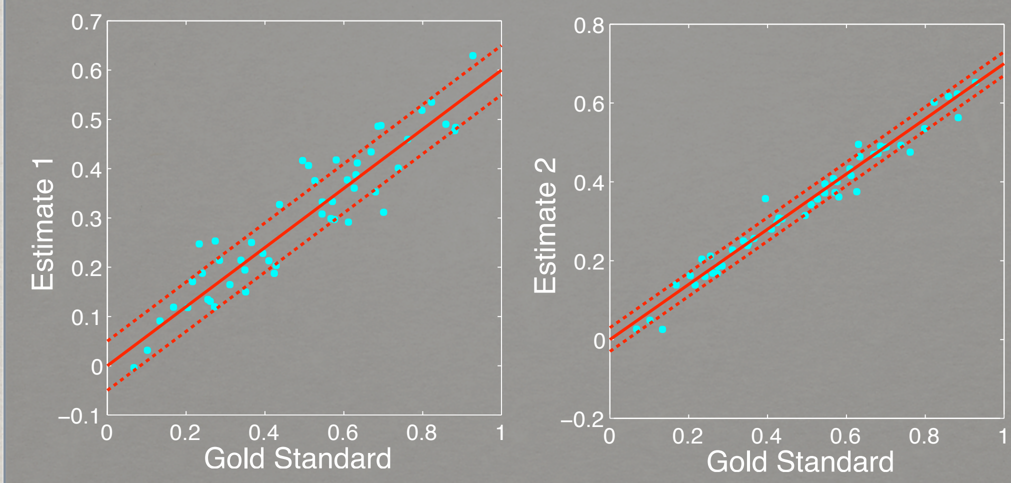
$$\Lambda_{\mathrm{BKE}}(\boldsymbol{g}|\boldsymbol{b}, \theta, \theta_0) = \frac{pr(\boldsymbol{g}|\boldsymbol{b}, \theta)}{pr(\boldsymbol{g}|\boldsymbol{b}, \theta_0)}$$

Maximum-likelihood estimation

$$\boldsymbol{s} = \int d\boldsymbol{b}\, pr(\boldsymbol{b}|\boldsymbol{g}, \boldsymbol{\theta}) \nabla_{\boldsymbol{\theta}} \log\left[pr(\boldsymbol{g}|\boldsymbol{b}, \boldsymbol{\theta})\right]$$

$$F = \left\langle \boldsymbol{s}\boldsymbol{s}^{\dagger} \right\rangle_{\boldsymbol{g}|\boldsymbol{\theta}}$$

No Gold



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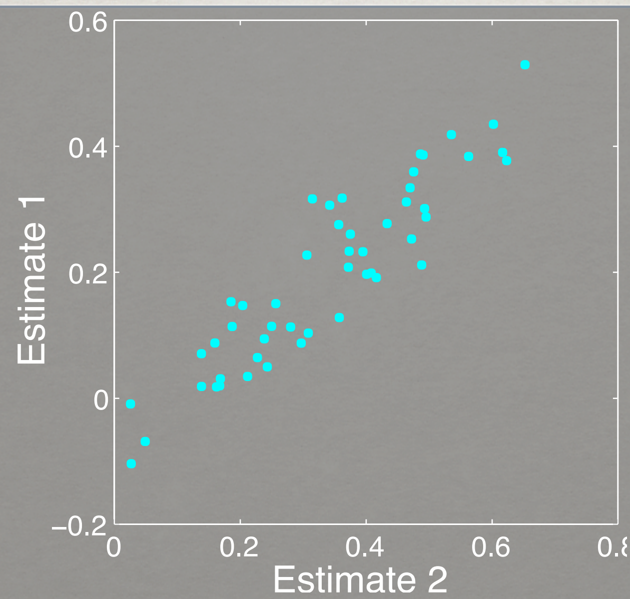

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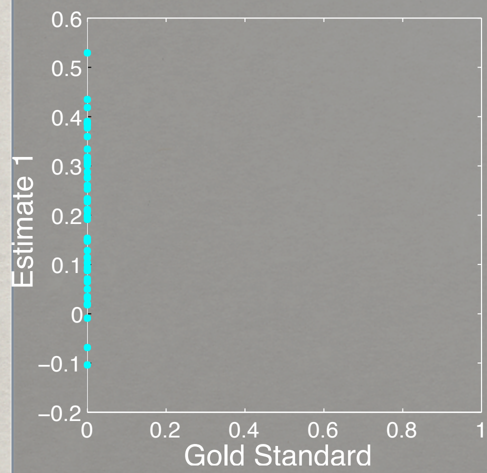

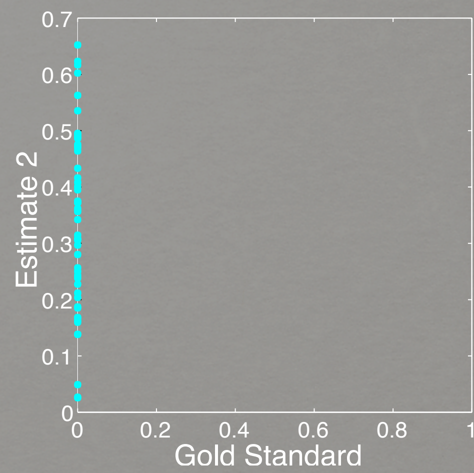

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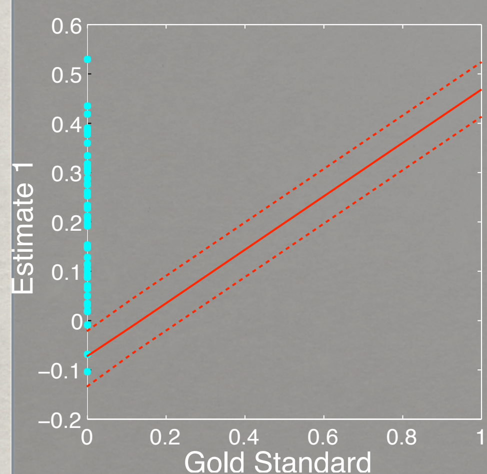

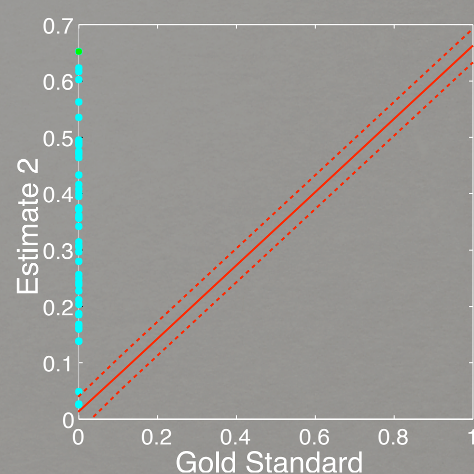

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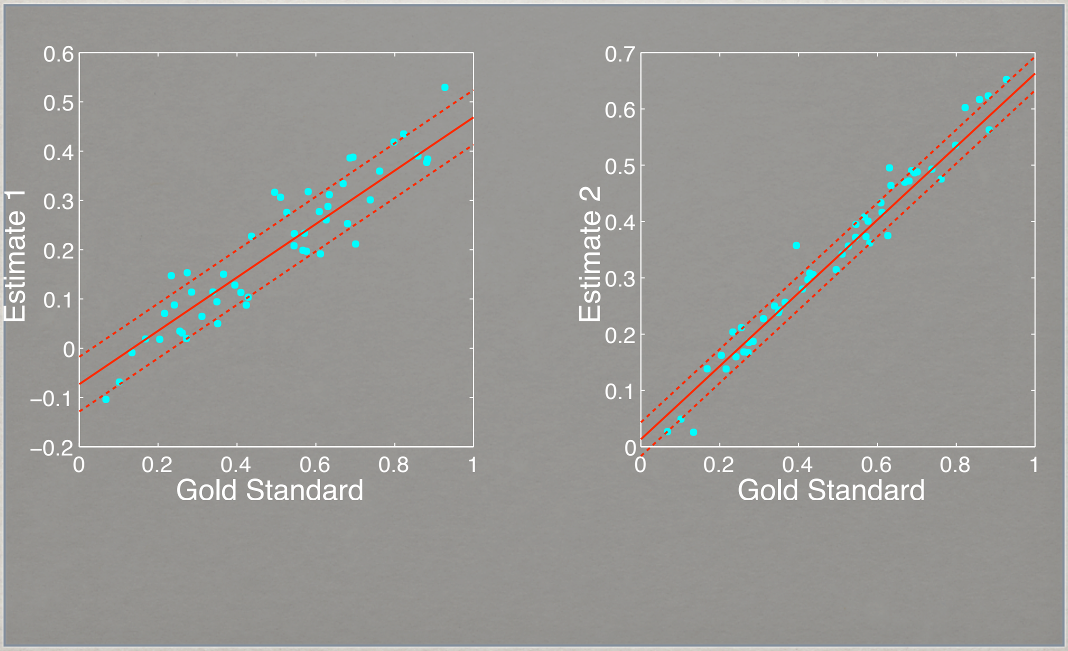


No Gold

Model $\hat{\theta}_{pm} = a_m\theta_p + b_m + n_{pm}$ (14.111)

- θ_p : Gold standard
- $\hat{\theta}_{pm}$: Estimate for modality m
- n_{pm} : Noise
- a_m, b_m, σ_m : Linear model parameters
- Note: The same population of patients are imaged on all modalities

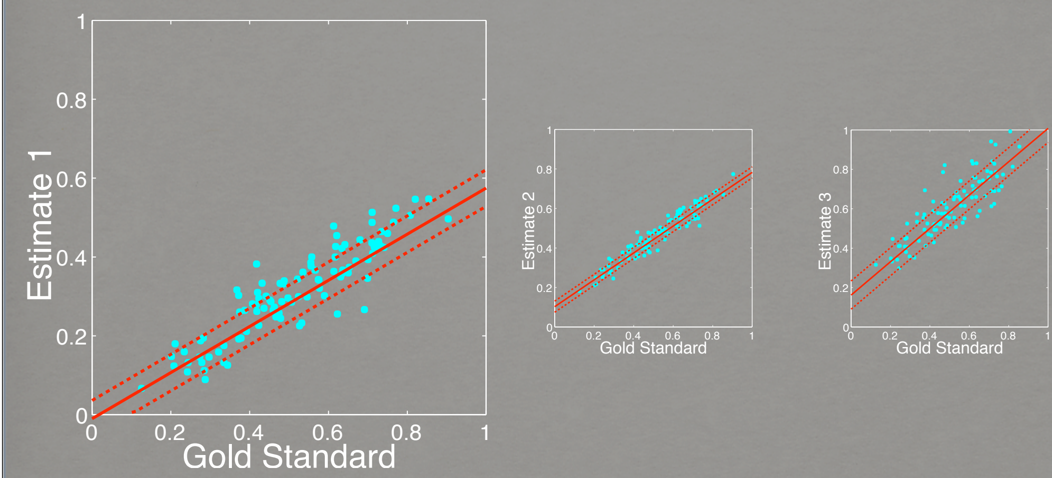
Assumptions

- Gold standard does not vary from one modality to the next
- There is an underlying distribution on the gold standard: $pr(\theta|\boldsymbol{\rho})$
- The noise n_{pm} is Gaussian distributed with variance σ_m
- Patients are independent

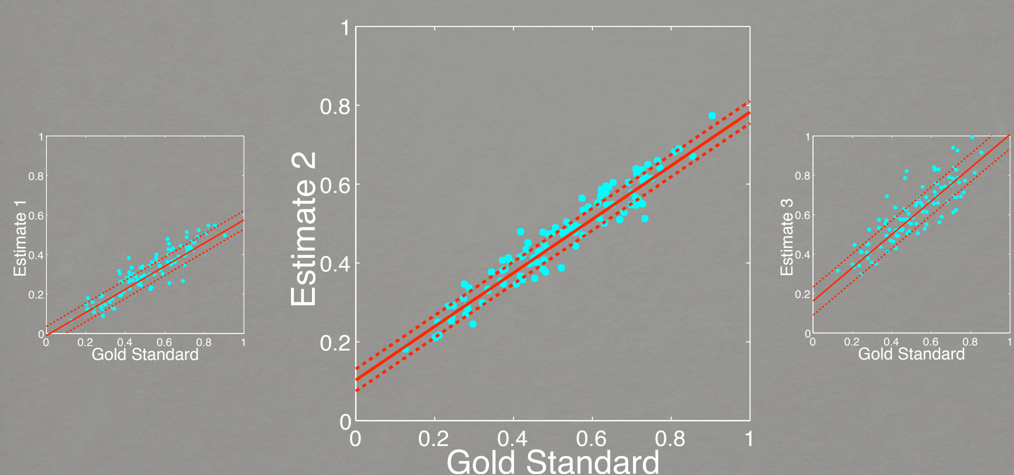
Method

- We can estimate $\{a_m, b_m, \sigma_m, \boldsymbol{\rho}\}$ using maximum-likelihood estimation

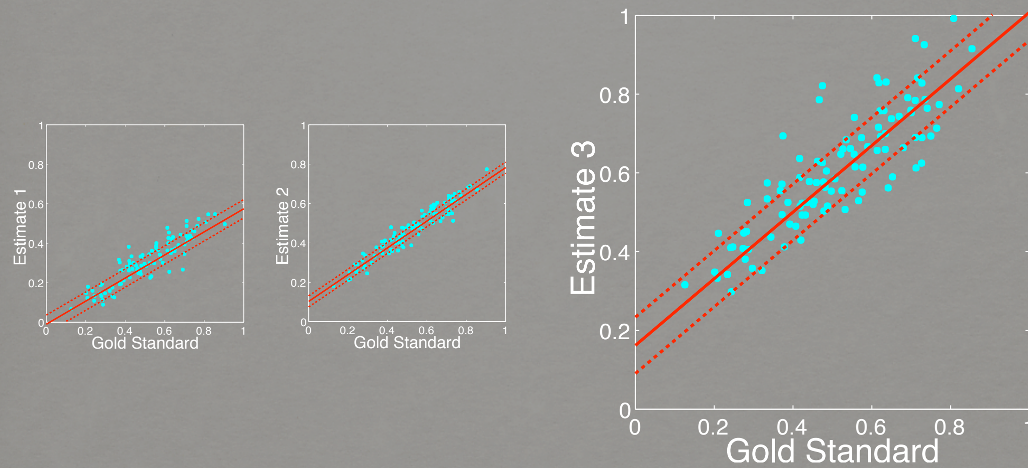
Simulation Study



Simulation Study



Simulation Study



Software

- No-gold comparisons
- Channelized Hotelling observer
- MCMC via the Image-Quality Toolkit (coming soon)

http://gamma.radiology.arizona.edu/IQ

Concluding Remarks

- Knowledge of forward model and statistics
- The more images, the better
- Many methods for computing task performance are computationally feasible