EROC Curves and Ideal EROC Observers

We are proposing a general framework, the Estimation ROC curve (EROC), for the evaluation of observers on combined detection and estimation tasks. We define the EROC curve for the detection of a signal and the estimation of a set of signal parameters. We show how the area under the EROC curve (AEROC) is related to a 2AFC test. We also formulate the EROC ideal observer, whose EROC curve lies above those of all other observers for the given task, and study its properties.

An observer for a combined detection and estimation task is provided a data vector $\mathbf{g}$ that is drawn from either a signal-absent probability distribution $pr\left(\mathbf{g}|H_0\right)$ or a signal-present probability distribution $pr\left(\mathbf{g}|\boldsymbol{\theta}, H_1\right)$. The vector $\boldsymbol{\theta}$ is a parameter vector associated with the signal. The observer must decide whether the signal is present or not. If we assume that the observer is not subject to internal noise, then this decision can be reduced to the comparison of a test statistic $T\left(\mathbf{g}\right)$ with a threshold T_0. When the observer decides that the signal is present, then an estimate $\hat{\boldsymbol{\theta}}\left(\mathbf{g}\right)$ of the parameter $\boldsymbol{\theta}$ must be produced. The utility of the estimate $\hat{\boldsymbol{\theta}}\left(\mathbf{g}\right)$ is denoted by $u\left[\hat{\boldsymbol{\theta}}\left(\mathbf{g}\right), \boldsymbol{\theta}\right]$ when the signal is actually present and the true parameter vector is $\boldsymbol{\theta}$.

For the EROC curve, we define the false-positive fraction in the usual way as

$$P_{FP}\left(T_0\right) = \int p\left(\mathbf{g}|H_0\right) step\left[T\left(\mathbf{g}\right) - T_0\right] d\mathbf{g}.$$

This number is the abscissa of the point on the EROC curve corresponding to the threshold value T_0. For the ordinate, we use the expected utility for those data vectors where the utility function is defined, i.e., the true-positive fraction. Using the prior distribution $pr\left(\boldsymbol{\theta}\right)$ on the signal parameter, this expected utility is

$$U_{TP}\left(T_0\right) = \int \int pr\left(\boldsymbol{\theta}\right) pr\left(\mathbf{g}|\boldsymbol{\theta}, H_1\right) u\left[\hat{\boldsymbol{\theta}}\left(\mathbf{g}\right), \boldsymbol{\theta}\right] step\left[T\left(\mathbf{g}\right) - T_0\right] d\boldsymbol{\theta} d\mathbf{g}.$$

The plot of $U_{TP}\left(T_0\right)$ versus $P_{FP}\left(T_0\right)$ as the threshold is varied is the EROC curve.

The area under the EROC curve (AEROC) can be used as a figure of merit for the observer on the combined detection and estimation task. One useful property of the AEROC as a figure of merit is that it can be computed from a 2AFC test. For this test, the observer is shown many pairs of data vectors, with each pair consisting of a signal-absent case and a signal-present case. The observer must decide which of the pair of data vectors is from the signal-present ensemble, and estimate the parameter for that vector. The average utility of the estimates for the pairs where the correct data vector was chosen is then an estimate of this observer's AEROC.

Another useful property of the EROC curve is that there is an ideal EROC observer, one whose EROC curve lies above all others for the given probability distributions. To formulate this ideal observer, we first define a conditional likelihood ratio as

$$\Lambda\left(\mathbf{g}|\boldsymbol{\theta}\right) = \frac{pr\left(\mathbf{g}|\boldsymbol{\theta}, H_1\right)}{pr\left(\mathbf{g}|H_0\right)}.$$

The ideal EROC observer test statistic is given by the maximum of a likelihood-ratio-weighted average of the utility function:

$$T_I\left(\mathbf{g}\right) = \max_{\boldsymbol{\theta}'} \left\{ \int pr\left(\boldsymbol{\theta}\right) \Lambda\left(\mathbf{g}|\boldsymbol{\theta}\right) u\left(\boldsymbol{\theta}', \boldsymbol{\theta}\right) d\boldsymbol{\theta} \right\}.$$

The ideal EROC observer estimator is actually computed along with the test statistic:

$$\hat{\boldsymbol{\theta}}_I\left(\mathbf{g}\right) = \arg\max_{\boldsymbol{\theta}'} \left\{ \int pr\left(\boldsymbol{\theta}\right) \Lambda\left(\mathbf{g}|\boldsymbol{\theta}\right) u\left(\boldsymbol{\theta}', \boldsymbol{\theta}\right) d\boldsymbol{\theta} \right\}.$$

The AEROC is a figure of merit that can be used to measure the performance of an observer for any task which combines detection and estimation. We have shown how to estimate this figure of merit from 2AFC tests. The ideal EROC observer has a simple analytical form and can be related to many common methods of estimation. Finally, the ideal AEROC can be calculated with methods that have already been developed for the ideal AUC.

E. Clarkson, Estimation ROC curves and their corresponding ideal observers, *Proc. SPIE*, 6515, 2007.