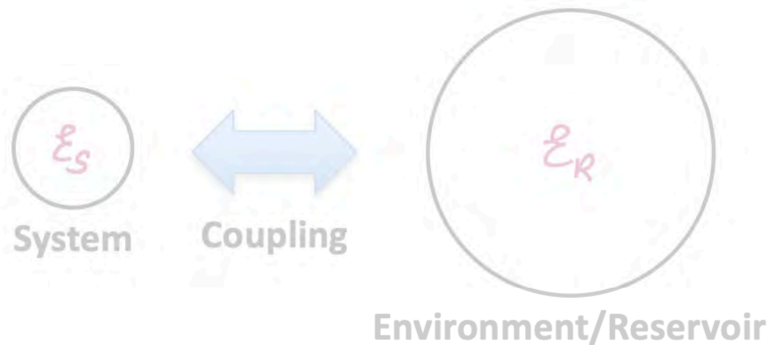
Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #1: Coupling to an Environment

(Lecture 10-04-23)



* System + Environment evolves unitarily, become entangled ⇨ the system on its own evolves non-unitarily

* Reasonable assumptions about the environment

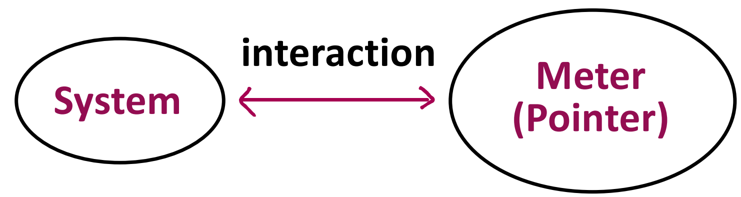
"Master Equation" for ϱ_S

$$\boxed{\dot{\varrho}_S = \frac{1}{i\hbar}[H_S, \varrho_S] + \mathcal{L}(\varrho_S)}$$

* The Liouvillian $\mathcal{L}$ accounts for relaxation and decoherence

Example #2: Coupling to a Meter

(Lecture 10-04-2023)



⇨ Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

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Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\varrho = \varrho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

$$\varrho \Rightarrow U_{AB}\left(\varrho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}$$

Reduced density operator for system A in basis $\{|\mu\rangle_B\}$

$$\varrho_A' = \mathrm{Tr}_B\left[U_{AB}\left(\varrho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}\right]$$

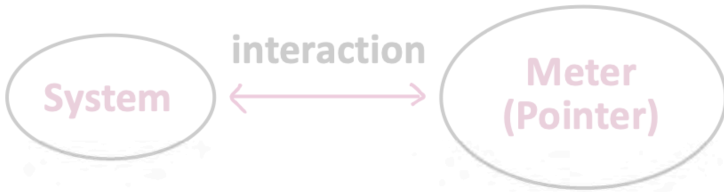
$$= \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0\rangle_B}\, \varrho_A \; {}_B\langle 0 | U_{AB}^{+} | \mu\rangle_B$$

operator M_μ acting on ϱ_A

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #2: Coupling to a Meter

(Lecture 10-18-2022)



Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\rho = \rho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

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$$\rho_A' = Tr_B\left[U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{\dagger}\right]$$

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operator M_μ acting on ρ_A

We can now write

$$\boxed{\rho_A' = \$(\rho_A) = \sum_{\mu} M_\mu \rho_A M_\mu^{\dagger}}$$

Furthermore, since U_{AB} is unitary, the M_μ's have the property

$$\sum_{\mu} M_\mu^{\dagger} M_\mu = \sum_{\mu} {}_B\langle 0 | U_{AB}^{\dagger} | \mu\rangle_{BB}\langle \mu | U_{AB} | 0\rangle_B$$

$$= {}_B\langle 0 | U_{AB}^{\dagger} U_{AB} | 0\rangle_B = \mathbb{1}_A$$

We conclude:

$\$$ defines a Linear Map

$\$$: Linear Operator → Linear Operator

If $\sum_{\mu} M_\mu^{\dagger} M_\mu = \mathbb{1}_A$ then $\$$ is a SuperOperator

and $\$(\rho_A) = \sum_{\mu} M_\mu \rho_A M_\mu^{\dagger}$ is the Operator-Sum or Krauss representation of $\$$

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Note: $\$$ maps density operators to density operators because

* ρ_A' is Hermitian: $\rho_A'^\dagger = \sum_\mu M_\mu \rho_A^\dagger M_\mu^\dagger = \rho_A'$
* ρ_A' has unit trace: $\mathrm{Tr}\,\rho_A' = \sum_\mu \mathrm{Tr}[\rho_A M_\mu^\dagger M_\mu]$
* ρ_A' is positive:

$${}_A\langle\psi|\rho_A'|\psi\rangle_A = \sum_\mu ({}_A\langle\psi|M_\mu)\rho_A(M_\mu^\dagger|\psi\rangle_A) \geq 0$$

Used $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ and
Trace invariance under cyclic permutation

Theorem: Given some $\$$ with an operator-sum representation, we can choose $\mathcal{H}_B$ and find the corresponding unitary U_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$

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> **Theorem**: Given some $\$$ with an operator-sum representation, we can choose $\mathcal{H}_B$ and find the corresponding unitary U_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$

Note:

* Superoperators provide a formalism to describe decoherence, i. e., maps from pure to mixed states

* Unitary evolution is a special case with only one term in the operator-sum expansion

* Two or more terms → initial pure states $\in \mathcal{H}_A$ become entangled w/states $\in \mathcal{H}_B$ due to U_{AB}
  → mixed final state ρ_A'

* Superoperators can be concatenated to form new ones, $\$ = \$_1 \$_2$

> **Theorem**: If $(\$)^{-1}(\$) = \mathbb{1}$ then $\$$ must necessarily be unitary

Non-unitary evolution cannot be reversed
→ "arrow of time"

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We summarize:

A mapping $\$: \rho \to \rho'$ where ρ, ρ' are density operators, is a mapping of operators to operators that satisfy

(0) $\$$ is Linear

(1) $\$$ preserves Hermiticity

(2) $\$$ is Trace preserving

(3) $\$$ is completely positive,

$\$_A \otimes \mathbb{1}_B$ positive in $\mathcal{H}_A \otimes \mathcal{H}_B$ for all $\mathcal{H}_B$

Krauss Representation Theorem

Any $\$$ satisfying (0) – (3) has an Operator-Sum Representation

See Preskill, Ch. 3.2 for more on Superoperator formalism

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Measurement as a Superoperator:

Von Neumann: Entangle System *A* with Meter *B*

$$U_{AB}: |\varphi\rangle_A |0\rangle_B \to \sum_\mu M_\mu |\varphi\rangle_A |\mu\rangle_B \quad (1)$$

Orthogonal measurement on *B* in "pointer basis" $|\mu\rangle_B$ yields outcome μ and tells us the meter is in $|\mu\rangle_B$.

This projects out a state $|\psi_\mu\rangle_{AA}\langle\psi_\mu| = \dfrac{M_\mu|\varphi\rangle_{AA}\langle\varphi|M_\mu^\dagger}{{}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A}$

with probability $P(\mu) = {}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A$

Generally ρ_A is mixed

Meas. on *B* projects out $\rho_A^\mu = \dfrac{M_\mu \rho_A M_\mu^\dagger}{Tr[M_\mu \rho_A M_\mu^\dagger]}$

with probability $P(\mu) = Tr_A[M_\mu^\dagger M_\mu \rho_A] = Tr_A[F_\mu \rho_A]$

This is a POVM with elements

$$F_\mu = M_\mu^\dagger M_\mu, \quad \sum_\mu F_\mu = \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}_A \quad (2)$$

If no access to the measurement outcome then

$$\rho_A \to \rho_A' = \sum_\mu P(\mu)\rho_A^\mu = \sum_\mu M_\mu \rho_A M_\mu^\dagger = \underbrace{\$}_{\text{Superoperator}}(\rho_A)$$

Most general measurement: POVM $\{F_\mu\}$ on ρ_A

In this case we have $\begin{cases} P(\mu) = Tr_A[F_\mu \rho_A] \\ \rho_A' = \sum_\mu \sqrt{F_\mu}\,\rho_A\sqrt{F_\mu} \end{cases}$

Note that F_μ Hermitian $\to$ $\sqrt{F_\mu}$ Hermitian

$\sum_\mu F_\mu = \mathbb{1}_A$ Follows from the operator sum Normalization condition (2) above

Compare w/ (1) above to see this POVM has the unitary representation

$$U_{AB}: |\varphi\rangle_A|0\rangle_B \to \sum_\mu \sqrt{F_\mu}\,|\varphi\rangle_A|\mu\rangle_B$$

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<u>Summary</u>:

* The discussion so far highlights the relationship between measurement and decoherence. We can always view the latter as the environment Doing a measurement and extracting information that we cannot retrieve. The loss of information causes an initial pure state to evolve into a statistical mixture, which is the definition of decoherence

* Sometimes we can “guess” what kind of “measurements” the environment implements. This is useful in the modeling of decohering “Quantum Channels”

* The example that follows is based on the first of four examples of decohering quantum channels given in Preskills notes. These will be particularly Relevant for those of you working in the area of Quantum communication over quantum photonic Networks.

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Decohering Quantum Channels
– a simple example –

Communication Scenario:

imperfect quantum channel

Alice ⟶ $ ⟶ Bob

what happens to a qubit
when transmitted ?

Alternatively: Transmission in time

imperfect evolution

Alice ⟶ $ ⟶ Bob

evolution of qubit in
presence of environment?

These are generic input-output maps !

Example: Depolarizing Channel

Probability of error $= p$, 3 types, equal probability

(1) Bit flip $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$ ⇨ $|\psi\rangle \to \sigma_1 |\psi\rangle$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2) Phase flip $|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$ ⇨ $|\psi\rangle \to \sigma_3 |\psi\rangle$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(3) Both $|0\rangle \to i|1\rangle$, $|1\rangle \to -i|0\rangle$ ⇨ $|\psi\rangle \to \sigma_2 |\psi\rangle$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Unitary Representation: Channel is a unitary map on $\mathcal{H}_A \otimes \mathcal{H}_E$

One choice (not unique, can always find one)

$$\boxed{U_{AE}: |\psi\rangle_A |0\rangle_E \to \sqrt{1-p}\, |\psi\rangle_A |0\rangle_E + \sqrt{\tfrac{p}{3}} \left[\sigma_1^A |\psi\rangle_A |1\rangle_E + \sigma_2^A |\psi\rangle_A |2\rangle_E + \sigma_3^A |\psi_A\rangle |3\rangle_E \right]}$$

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Note:

The 4 orthogonal states in $\mathcal{H}_E$ keep records of what happened. If available through measurement in $\mathcal{H}_E$ the errors would in principle be reversible. We must have $\mathrm{Dim}\,\mathcal{H}_E \geq 4$ to allow 4 distinct evolutions

On Operator Form: we have

$$U_{AE} = \underbrace{\sqrt{1-p}\, \mathbb{1}_{AE}} + \underbrace{\sqrt{\tfrac{p}{3}}\, \sigma_1^A |1\rangle_{EE}\langle 0|} + \underbrace{\sigma_2^A |2\rangle_{EE}\langle 0|} + \underbrace{\sigma_3^A |3\rangle_{EE}\langle 0|}$$

⇨ $\rho_A' = \mathrm{Tr}_E \left[U_{AE} \left(\rho_A |0\rangle_{EE}\langle 0| \right) U_{AE}^\dagger \right]$

$$= \sum_\mu \underbrace{{}_E\langle \mu | U_{AE} |0\rangle_E}_{M_\mu} \rho_A\, {}_E\langle 0 | U_{AE}^\dagger |\mu\rangle_E = \sum_\mu M_\mu \rho_A M_\mu^\dagger$$

From this we find Kraus operators

$$\boxed{M_0 = \sqrt{1-p}\, \mathbb{1}_A ,\ M_1 = \sqrt{\tfrac{p}{3}}\, \sigma_1^A ,\ M_2 = \sqrt{\tfrac{p}{3}}\, \sigma_2^A ,\ M_3 = \sqrt{\tfrac{p}{3}}\, \sigma_3^A}$$

Check ($\sigma_i^2 = 1$) : $\sum_\mu M_\mu^\dagger M_\mu = \left(1 - p + 3\tfrac{p}{3}\right) \mathbb{1} = \mathbb{1}$

From earlier:

$$\rho_A' = \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} |0\rangle_B}\, \rho_A\, {}_B\langle 0 | U_{AB}^\dagger |\mu\rangle_B$$

operator M_μ acting on ρ_A

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$$\Rightarrow \quad \rho_A' = \mathrm{Tr}_E\left[U_{AE}\left(\rho_A |0\rangle_E {}_E\langle 0|\right) U_{AE}^\dagger\right]$$

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Evolution of the Qubit:

$$\boxed{\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1^A \rho_A \sigma_1^A + \sigma_2^A \rho_A \sigma_2^A + \sigma_3^A \rho_A \sigma_3^A\right)}$$

Bloch Sphere representation:

Bloch vector

Let: $\rho_A = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + P_3\sigma_3\right)$ (1)

Choose $\vec{e}_3$ along $\vec{P} = (0, 0, P_3)$

Sub in expression for ρ_A' above and use

$$\sigma_1\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_2 = -\sigma_3, \quad \sigma_3\sigma_3\sigma_3 = \sigma_3$$

Can show that (Math details)

$$\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1\rho_A\sigma_1\right) + \sigma_2\rho_A\sigma_2 + \sigma_3\rho_A\sigma_3)$$

$$= (1-p)\frac{1}{2}\left(\mathbb{1} + P_3\sigma_3 + \frac{p}{3}\left[\frac{1}{2}\left(\mathbb{1} - P_3\sigma_3\right) + \frac{1}{2}\left(\mathbb{1} - P_3\sigma_3\right) + \frac{1}{2}\left(\mathbb{1} + P_3\sigma_3\right)\right]\right.$$

$$= \frac{1}{2}\left[\mathbb{1} + \left(1 - \frac{4p}{3}\right)P_3\sigma_3\right] = \frac{1}{2}\left(\mathbb{1} + P_3'\sigma_3\right) \Rightarrow P_3' = \left(1 - \frac{4p}{3}\right)P_3$$

By symmetry of (1) we have $\boxed{\vec{P}' = \left(1 - \frac{4}{3}p\right)\vec{P}}$

Uniform shrinking Bloch Sphere

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Choose $\hat{z}$ along $\vec{P} = (0,0,P_3)$

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$$\sigma_1\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_2 = -\sigma_3, \quad \sigma_3\sigma_3\sigma_3 = \sigma_3$$

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$$= (1-p)\frac{1}{2}(\mathbb{1} + P_3\sigma_3) + \frac{p}{3}\left[\frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} + P_3\sigma_3)\right]$$

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Uniform shrinking Bloch Sphere

Continuous limit:

$$p = \Gamma dt \Rightarrow P_3(t+dt) = \left(1 - \frac{4}{3}\Gamma dt\right)P(t)$$

$$\Rightarrow \frac{dP_3}{dt} = -\frac{4}{3}\Gamma P_3 \Rightarrow P_3(t) = P_3(0)\, e^{-4/3\,\Gamma t}$$

Bloch Sphere shrinking at constant rate

$$\boxed{\vec{P}(t) = \vec{P}(0)\, e^{-4/3\,\Gamma t}}$$

This turns out to be identical to the Master Equation result

Other Examples:

- Phase Damping (Bloch sphere shrinks along *x* , *y*)
- Amplitude Damping (Bloch sphere shrinks along *z*)

Classical Information Theory (Preskill ch. 5)

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Main Topics of QIT:

(1) Transmission of classical info over quantum channels

(2) Information/disturbance tradeoff in QM

(3) Quantifying entanglement

(4) Transmission of quantum info over quantum channels

Our Program: (1) & (4) ~ 2 Lectures

Key Concept – Incompressible information content

Classical Measure:	Shannon Entropy
Quantum Measure:	von Neumann Entropy

Review of Classical Information Theory

(Shannon for Dummies, Preskill 5.1)

Shannon, 1948: Core findings of classical info theory

(1) How much data can be compressed (Redundancy)

(2) Reliable communication rate over noisy channel (Redundancy needed to protect against errors)

Shannon Entropy and Data Compression

(Shannon's noiseless coding theorem)

Message = String of letters chosen from $\{a_1, a_2, \ldots, a_k\}$

A priori probability of occurrence: $p(a_x)$, $\sum p(a_x) = 1$

Basic Question: given message w/ $n \gg 1$ letters

Can we compress to length $< n$?

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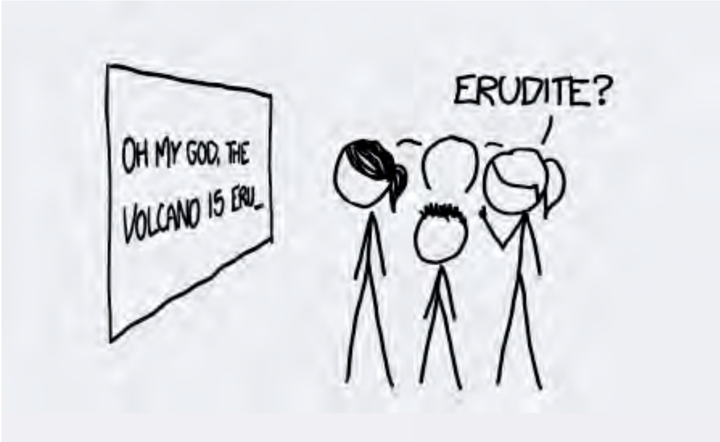

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(Shannon's noiseless coding theorem)

Message = String of letters chosen from $\{a_1, a_2, \ldots, a_k\}$

A priori probability of occurrence: $p(a_x)$, $\sum p(a_x) = 1$

Basic Question: given message w/ $n \gg 1$ letters

Can we compress to length $< n$?

Classical Information Theory (Preskill ch. 5)

Look at Binary case:		**Typical Occurrence**
0	$p(0) = 1-p$	$n(1-p)$
1	$p(1) = p$	np

binomial coeff.

Number of distinct typical strings $\sim \binom{n}{np}$

$$\log \binom{n}{np} = \log \frac{n!}{(np)!\,[n(1-p)]!}$$

base 2

Stirlings formula

$$\log n! = n\log n - n + \mathcal{O}(\log n)$$

$$\simeq n\log n - n - [np\log(np) - (np) + n(1-p)\log[n(1-p)] - n(1-p)]$$

$$\equiv nH(p)$$

\# of bits needed to specify all typical strings, for a given n

$$\boxed{H(p) = -p\log p - (1-p)\log(1-p) = \sum_{x=0,1} p(x)\log p(x)}$$

Entropy function

Classical Information Theory (Preskill ch. 5)

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Entropy function

Basic idea of Data Compression:

* Assign integer code letter to each typical string
* This block code has $2^{nH(p)}$ letters
* Each code letter specified by $n\,H(p)$ bits

$0 \leq p \leq 1 \rightarrow 0 \leq H(p) \leq 1$

$H(p) = 1$ only for $p = 1/2$

Block code compresses message for $p \neq 1/2$

Generalization:

k letters, prob. $p(x)$
Ensemble $X = \{x, p(x)\}$ of letters

n - letter string → x occurs $\sim np(x)$ times

# of typical strings $\sim \dfrac{n!}{\prod_x [np(x)]!} \sim 2^{-nH(X)}$

$$\boxed{H(X) = -\sum_x p(x)\,\text{Log}\, p(x)}$$ ← **Shannon entropy !**

Classical Information Theory (Preskill ch. 5)

Look at Binary case: | **Typical Occurrence**

$0 \quad p(0) = 1-p \qquad n(1-p)$

$1 \quad p(1) = p \qquad np$

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Classical Information Theory (Preskill ch. 5)

Shannons Noiseless Coding Theorem

Consider a specific message $x_1 x_2 \dots x_n$

with $x_j \in X$ in j' th place

statistically independent, x_j occurs n times on average

$$\begin{cases} p(x_1 \dots x_n) = p(x_1)\, p(x_2) \dots p(x_n) \\ p(x_j) = \text{a priori prob. of } x \text{ in place } j \end{cases}$$

Then $$\log P(x_1, \dots, x_j) = \sum_{j=1}^{n} \log p(x_j)$$

Applying the central limit theorem to this sum, we conclude that for "most sequences"

$$-\frac{1}{n} \log P(x_1, \dots, x_j) \sim \langle -\log p(x) \rangle \equiv H(x)$$

where brackets denote the mean with respect to the probability distribution governing the random variable x

Now: For any $\varepsilon, \delta > 0$ there exist an n large enough s. t.

$$H(X) - \delta \leq -\frac{1}{n} \log p(x_1 \dots x_n) \leq H(X) + \delta \quad (1)$$

$$\Rightarrow 2^{-n(H-\delta)} \geq p(x_1 \dots x_n) \geq 2^{-n(H+\delta)}$$

But: $p(x_1 \dots x_n)$ is just one of many typical strings with the same number of occurrences of each letter and thus identical a priori probabilities $p(\text{typical})$.
Then for n large enough, we also have

$$1 - \varepsilon \leq \underbrace{\sum p(\text{typical})}_{= N(\varepsilon, \delta) \times p(\text{typical})} \leq 1 \quad (2)$$

# of typical strings

Taking the ratio $\frac{1}{(1)} \times (2)$ gives us the final result

$$\boxed{(1-\varepsilon)\, 2^{n(H-\delta)} \leq N(\varepsilon, \delta) \leq 2^{n(H+\delta)}}$$

Classical Information Theory (Preskill ch. 5)

Joint and Conditional Entropy, Mutual Information

Consider the following scenario:

noisy channel

Alice $\longrightarrow$ Bob

$X = \{x, p(x)\}$ errors specified by $p(y|x)$ $Y = \{y, p(y)\}$

known about channel

known about Alices alphabet

Bayes Rule: $p(x|y) = \dfrac{p(y|x)\,p(x)}{p(y)}$

$p(y) = \sum_x p(y|x)\,p(x)$

Bob uses this to estimate the prob. that Alice sent *x* given he received *y*. The "width" of the distribution $p(x|y)$ is thus a measure of Bob's information gain per letter.

Think about this in terms of joint events

$$\{X, Y\} = \{(x,y),\ p(x,y)\}$$

Joint entropy

$$\boxed{H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y)}$$

This is a measure of information content per letter in the combined strings

* Assume Bob measures the value of a letter y in the message
* He gets $H(Y)$ bits of info about the letter pair x, y
* Bob's remaining uncertainty about the letter x is then tied to his lack of knowledge about x given that he knows y.

Classical Information Theory (Preskill ch. 5)

The entropy of X conditioned on Y is therefore

$$H(X,Y) = H(Y) + H(X|Y)$$

$$\Rightarrow H(X|Y) = H(X,Y) - H(Y)$$

The **Conditional Entropy** $H(X|Y)$

is the number of bits of info per letter in Alice's message that Bob is missing due to channel errors

– measure of information loss due to errors –

Equivalently, it is the # of extra bits Alice must send to ensure Bob gets the complete message in the presence of channel errors.

Note: From the above,

$$H(X|Y) = H(X,Y) - H(Y)$$

$$= -\sum_{(x,y)} p(x,y) \, Log \, p(x|y) = \sum_{(x,y)} -p(x,y) \, Log \frac{p(x,y)}{p(y)}$$

$$= -\sum_{(x,y)} p(x,y) \, Log(x,y) + \sum_{y} p(y) \, Log \, p(y)$$

We can similarly quantify the # of bits of info about X that Bob has gained by measuring Y .

This is the **Mutual Information**:

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

$$= H(X) - H(X|Y) = H(Y) - H(Y|X)$$

Note: When we added the info content of **X** to the info content of **Y** we overcounted the total info because some info is common to **X** and **Y**, and must be subtracted to get the proper measure for the Mutual Information

Classical Information Theory (Preskill ch. 5)

Quantum Information Theory (Preskill ch. 5)