General Theory of Quantum Measurement (Preskill ch. 3)

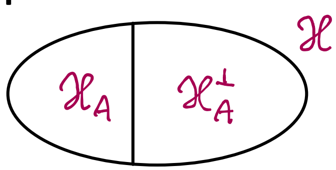
How to do it?

We can effectively do non-OM's in part of Hilbert space if we can add extra dimensions to $\mathcal{H}$:

$$\boxed{\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^{\perp} \quad \text{or} \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$$

Direct Sum Implementation

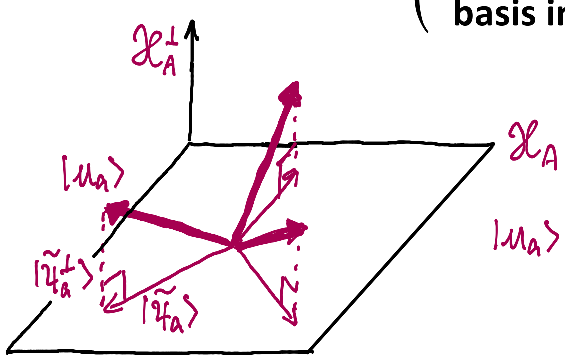
Let $\mathcal{H}_A \oplus \mathcal{H}_A^{\perp}$



Alice prepares states $\rho_A \in \mathcal{H}_A$

Bob (and/or Alice) makes OM $\{E_a\}$ in $\mathcal{H}$, $E_a = |u_a\rangle\langle u_a|$

Geometric visualization: (like an over complete basis in 2D subspace)



$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^{\perp}\rangle$

We can now define effective measurement operators

$$F_a = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_a \rho_A]$$

Properties:

* Each F_a is Hermitian & non-negative $\Rightarrow P(m_a) \geq 0$
* Individual F_a are not projectors unless $\lambda_a = 1$
* $\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A \leftarrow$ identity on $\mathcal{H}_A$

POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a's are non-negative & $\sum_a F_a = \mathbb{1}$

General Theory of Quantum Measurement (Preskill ch. 3)

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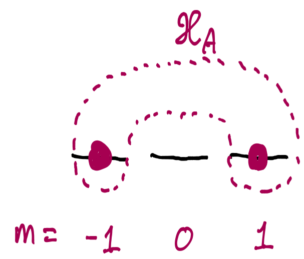
POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a 's are non-negative & $\sum_a F_a = \mathbb{1}$

Example : POVM on Qubit encoded in Qutrit

$^{87}Rb\,(F=1)$ atomic HF state

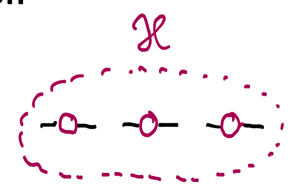
Unitary Transformation



$U: |\tilde{\psi}_1\rangle + |\tilde{\psi}_1^{\perp}\rangle \to |-1\rangle$

$U: |\tilde{\psi}_2\rangle + |\tilde{\psi}_2^{\perp}\rangle \to |0\rangle$

$U: |\tilde{\psi}_3\rangle + |\tilde{\psi}_3^{\perp}\rangle \to |1\rangle$



Alice prepares in $\mathcal{H}_A$

Bob measures

$E_m = |m\rangle\langle m| \in \mathcal{H}$

Choose the map U

$\Rightarrow$ any 1 qubit, 3 outcome POVM we want

Theorem: Any POVM can be realized by adding to $\mathcal{H}_A$ an orthogonal complement $\mathcal{H}_A^{\perp}$

If N F_a's are desired, where $N > \mathrm{Dim}\,\mathcal{H}_A$

then we need $\quad \mathrm{Dim}\,(\mathcal{H}_A + \mathcal{H}_A^{\perp}) \geq N$

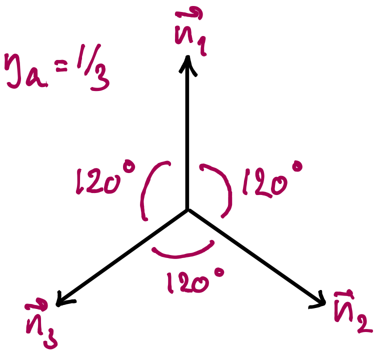
(Preskill 3.1.4)

General Theory of Quantum Measurement (Preskill ch. 3)

Toy Example: One Qubit POVM, illustrates different capabilities of OM & non-OM POVM's

Pick **3** unit vectors s. t. $\sum_a y_a \vec{n}_a = 0, \ \sum_a y_a = 1$

$y_a = 1/3$



Measurement operators

$$F_a = 2 y_a |\uparrow_{\vec{n}_a}\rangle\langle\uparrow_{\vec{n}_a}| \Rightarrow \sum_a F_a = \mathbb{1}$$

For the above & following, note that

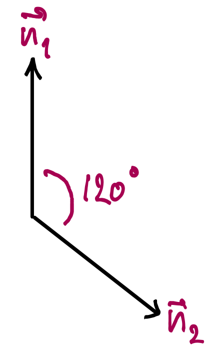
$$|\uparrow_{\vec{n}_2}\rangle = \cos(60°)\,|\uparrow_{\vec{n}_1}\rangle + \sin(60°)\,|\downarrow_{\vec{n}_1}\rangle = \frac{1}{2}|\uparrow_{\vec{n}_1}\rangle + \frac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

$$|\uparrow_{\vec{n}_3}\rangle = \cos(60°)\,|\uparrow_{\vec{n}_1}\rangle + \sin(-60°)\,|\downarrow_{\vec{n}_1}\rangle = \frac{1}{2}|\uparrow_{\vec{n}_1}\rangle - \frac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability

How can Bob best tell the difference ?



OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Bob's guess

Alice sends
- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $P = 1$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $P = 1/4$ — Bob's guess: $|\uparrow_{\vec{n}_1}\rangle$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $P = 3/4$ — Bob's guess: $|\uparrow_{\vec{n}_2}\rangle$

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$

General Theory of Quantum Measurement (Preskill ch. 3)

Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability

How can Bob best tell the difference ?

OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Alice sends	Bob gets	Bob's guess
$|\uparrow_{\vec{n}_1}\rangle \rightarrow$	Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=1$	$|\uparrow_{\vec{n}_1}\rangle$
$|\uparrow_{\vec{n}_2}\rangle \rightarrow$	Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=1/4$	$|\uparrow_{\vec{n}_1}\rangle$
	$|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=3/4$	$|\uparrow_{\vec{n}_2}\rangle$

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$

Fidelity of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \underset{(a)}{\tfrac{1}{2}\times 1} + \tfrac{1}{2}\left(\underset{(c)}{3/4 \times 1} + \underset{(d)}{1/4 \times 1/4}\right) = \frac{29}{32} \simeq \underline{0.9063}$$

(a) (b) (c) (d) (Quite good)

(a) A sent $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=1/2$, B guesses $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=1$ $(\mathcal{F}=1)$

(b) A sent $|\uparrow_{\vec{n}_2}\rangle$ w/ $\mathcal{P}=1/2$

(c) Given $|\uparrow_{\vec{n}_2}\rangle$ B gets $|\downarrow_{\vec{n}_1}\rangle$ & guesses $|\uparrow_{\vec{n}_2}\rangle$ w/ $\mathcal{P}=3/4$ $(\mathcal{F}=1)$

(d) Given $|\uparrow_{\vec{n}_2}\rangle$ B gets $|\uparrow_{\vec{n}_1}\rangle$ & guesses $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P}=1/4$ $(\mathcal{F}=1/4)$

Note: The content on pages 3, 4 and 5 needs updating, which I hope to do in the background while we continue "regular" programming

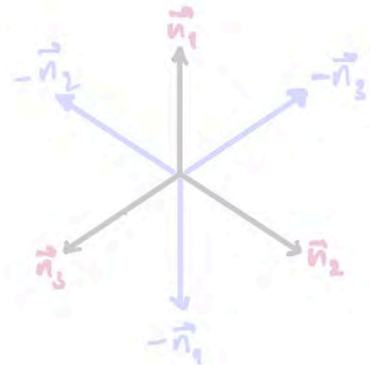
Preskill gives us the POVM elements that go with each measurement outcome. These are of the form $F_a = 2\eta_a |\uparrow_{\vec{n}_a}\rangle\langle\uparrow_{\vec{n}_a}|$, after which we can calculate the probabilities for Bob's measurement outcomes given what states Alice sent. To do so, we can follow the standard rule for POVM's, $P(m_a) = \langle\uparrow_{\vec{n}_a}|F_a|\uparrow_{\vec{n}_a}\rangle$. When we know the conditional probabilities for Bob's outcomes given what Alice sent, we can try to design a set of optimal decisions that Bob can use at each step.

General Theory of Quantum Measurement (Preskill ch. 3)

Instead

Bob does the POVM

$$\boxed{F_a = \frac{2}{3}\,|\downarrow_{\vec{n}_a}\rangle\langle\downarrow_{\vec{n}_a}|}$$



Alice sends:

- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $P = 0$
  - $|\downarrow_{\vec{n}_2}\rangle$ w/ $P = 1/2$
  - $|\downarrow_{\vec{n}_3}\rangle$ w/ $P = 1/2$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $P = 1/2$
  - $|\downarrow_{\vec{n}_2}\rangle$ w/ $P = 0$
  - $|\downarrow_{\vec{n}_3}\rangle$ w/ $P = 1/2$

Bob gets:

- $|\downarrow_{\vec{n}_1}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec{n}_2}\rangle$
- $|\downarrow_{\vec{n}_2}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec{n}_1}\rangle$
- $|\downarrow_{\vec{n}_3}\rangle \rightarrow$ Bob doesn't know (DK)

<u>Fidelity</u> of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \underbrace{\tfrac{1}{2}}_{P(\text{know})}\times 1 + \underbrace{\tfrac{1}{2}}_{P(\text{don't know})}\left(\tfrac{1}{2}\times 1 + \tfrac{1}{2}\times\tfrac{1}{4}\right) = \frac{13}{16} = \underline{0.8125}$$

(a) (b) (c)

(a) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B knows which one w/ $P = 1/2$ $(\mathcal{F} = 1)$

(b) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B DK, correct guess w/ $P = 1/2$ $(\mathcal{F} = 1)$

(c) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B DK, wrong guess w/ $P = 1/2$ $(\mathcal{F} = 1/4)$

<u>Note</u>: If in (c) Bob guesses $|\downarrow_{\vec{n}_3}\rangle$ w/ $\mathcal{F} = 3/4$ he gets a slightly better fidelity of

$$\mathcal{F}_{POVM} = \frac{14}{16} = \underline{0.8750}$$

<u>However</u>: if Bob sticks with <u>Heralded Success</u> he will have a subensemble w/ $\mathcal{F}_{POVM} = 1$!

General Theory of Quantum Measurement (Preskill ch. 3)

Rewind: how to implement a non-OM ?

* The Postulates of QM tells us we can do OM's in a given Hilbert space

* We can effectively do non-OM's in part of $\mathcal{H}$ if

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^{\perp} \quad \text{or} \quad \underbrace{\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$$

Look at this option !

Motivation:

* We cannot count on our system to be embedded in a larger Hilbert space

* A more realistic implementation is to juxtapose system A with a second system B and doing OM's in the resulting tensor product space

Systems A & B, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_{AB} = \rho_A \otimes \rho_B$

Set of orthogonal E_a acting in $\mathcal{H}$, $\sum_a E_a = \mathbb{1}$

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

$\mathcal{H}_B$, ρ_B & an OM $\{E_a\}$ in $\mathcal{H}_n = \mathcal{H}_A \otimes \mathcal{H}_B$ s. t.

$$\mathcal{P}(m_a) = \mathrm{Tr}_{AB}[E_a(\rho_A \otimes \rho_B)] = \mathrm{Tr}_A[F_a \rho_A]$$

where $F_a = \mathrm{Tr}_B[E_a \rho_B]$

and $\rho_{AB} \to \rho'_{AB}(m_a) = \dfrac{E_a(\rho_A \otimes \rho_B)E_a}{\mathcal{P}(m_a)}$

Math details

$$\mathrm{Tr}_{AB}[E_a(\rho_A \otimes \rho_B)] = \mathrm{Tr}_A\left[\mathrm{Tr}_B[(\rho_A \otimes \rho_B)E_a]\right]$$
$$= \mathrm{Tr}_A\left[\rho_A \mathrm{Tr}_B[\rho_B E_a]\right] = \mathrm{Tr}_A[F_a \rho_A]$$

where $F_a = \mathrm{Tr}_B[E_a \rho_B]$

General Theory of Quantum Measurement (Preskill ch. 3)

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

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$$\mathcal{P}(m_a) = Tr_{AB}[E_a(\rho_A \otimes \rho_B)] = Tr_A[F_a \rho_A]$$

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and $\rho_{AB} \to \rho'_{AB}(m_a) = \dfrac{E_a(\rho_A \otimes \rho_B)E_a}{\mathcal{P}(m_a)}$

The F_a have the properties of POVM elements:

∗ **Hermiticity**:

$$F_a^+ = Tr_B[E_a \rho_B]^+ = Tr_B[\rho_B^+ E_a^+] = Tr_B[E_a \rho_B] = F_a$$

∗ **Positivity**: E_a, ρ_B positive (eigenvalues ≥ 0) (i)

⇨ $E_a \rho_B$ positive, marginal $Tr_B[E_a \rho_B] = F_a$

∗ **Completeness**: $\sum_a F_a = \mathbb{1}$ (ii)

∗ **Non-orthogonality**: $\# F_a\text{'s} \begin{cases} > \mathcal{H}_A \\ \leq Dim(\mathcal{H}_A \otimes \mathcal{H}_B) \end{cases}$

Math Details

(i) Let $\rho_B = \sum_\mu p_\mu |\mu\rangle_{BB}\langle\mu|$ ⇨ $F_a = \sum_\mu p_\mu\, {}_B\langle\mu|E_a|\mu\rangle_B$

⇨ ${}_A\langle\psi|F_a|\psi\rangle_A = \sum_\mu p_\mu ({}_A\langle\psi| \otimes {}_B\langle\mu|) E_a (|\mu\rangle_B \otimes |\psi\rangle_A) \geq 0$

(ii) $\sum_a F_a = \sum_\mu p_\mu\, {}_B\langle\mu| \sum_a E_a |\mu\rangle_B = \mathbb{1}_A$

General Theory of Quantum Measurement (Preskill ch. 3)

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

$\mathcal{H}_B$, ρ_B & an OM $\{E_a\}$ in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ s. t.

$$\mathcal{P}(m_a) = \mathrm{Tr}_{AB}\left[E_a(\rho_A \otimes \rho_B)\right] = \mathrm{Tr}_A\left[F_a \rho_A\right]$$

where $F_a = \mathrm{Tr}_B\left[E_a \rho_B\right]$

and $\rho_{AB} \to \rho'_{AB}(m_a) = \dfrac{E_a(\rho_A \otimes \rho_B) E_a}{\mathcal{P}(m_a)}$

The F_a have the properties of **POVM elements**:

* **Hermiticity**:

$$F_a^{+} = \mathrm{Tr}_B\left[E_a \rho_B\right]^{+} = \mathrm{Tr}_B\left[\rho_B^{+} E_a^{+}\right] = \mathrm{Tr}_B\left[E_a \rho_B\right] = F_a$$

* **Positivity**: E_a, ρ_B positive (eigenvalues ≥ 0) (i)

  ➡ $E_a \rho_B$ positive, marginal $\mathrm{Tr}_B\left[E_a \rho_B\right] = F_a$

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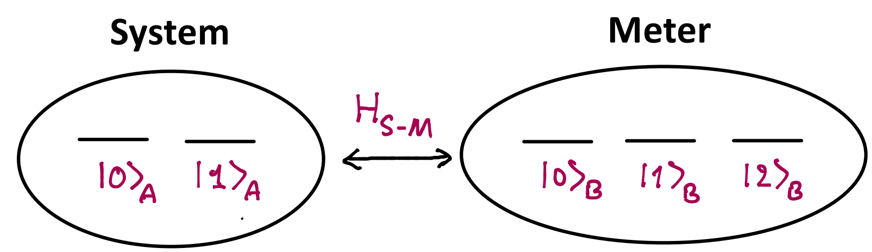
How to do it?



$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ has TP basis

$|00\rangle_{AB}$ $|01\rangle_{AB}$ $|02\rangle_{AB}$

$|10\rangle_{AB}$ $|11\rangle_{AB}$ $|12\rangle_{AB}$

↳ start meter in $|0\rangle_B$

The interaction drives a unitary map, for example

$$|00\rangle_{AB} \to \sum_{j=0}^{2} a_j |0\rangle_A |j\rangle_B$$

$$|10\rangle_{AB} \to \sum_{j=0}^{2} b_j |1\rangle_A |j\rangle_B$$

where the c-numbers a_j, b_j are chosen to ensure orthogonality

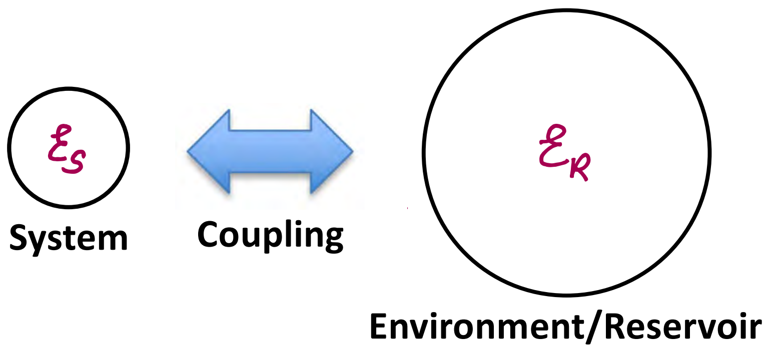
Measuring the meter ➡ 3 possible outcomes $j = 0, 1, 2$

General Theory of Quantum Measurement (Preskill ch. 3)

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #1: Coupling to an Environment

(Lecture 10-04-23)



- ∗ System + Environment evolves unitarily, become entangled ⇨ the system on its own evolves non-unitarily

- ∗ Reasonable assumptions about the environment

"Master Equation" for ϱ_S

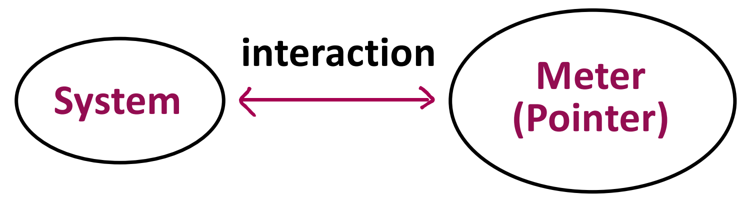
⇩

$$\boxed{\dot{\varrho}_S = \frac{1}{i\hbar}[H_S, \varrho_S] + \mathcal{L}(\varrho_S)}$$

- ∗ The Liouvillian $\mathcal{L}$ accounts for relaxation and decoherence

Example #2: Coupling to a Meter

(Lecture 10-04-2023)



⇨ Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

---

Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\varrho = \varrho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

$$\varrho \Rightarrow U_{AB}\left(\varrho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{\dagger}$$

Reduced density operator for system A in basis $\{|\mu\rangle_B\}$

$$\varrho_A' = \mathrm{Tr}_B\left[U_{AB}\left(\varrho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{\dagger}\right]$$

$$= \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0 \rangle_B}\, \varrho_A\, {}_B\langle 0 | U_{AB}^{\dagger} | \mu \rangle_B$$

operator $\mathcal{M}_\mu$ acting on ϱ_A

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #2: Coupling to a Meter

(Lecture 10-18-2022)

System $\xleftrightarrow{\text{interaction}}$ Meter (Pointer)

⇒ Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

---

Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\rho = \rho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

$$\rho \Rightarrow U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}$$

Reduced density operator for system A in basis $\{|\mu\rangle_B\}$

$$\rho_A' = \mathrm{Tr}_B\left[U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}\right]$$

$$= \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0\rangle_B}\, \rho_A\, {}_B\langle 0 | U_{AB}^{+} | \mu\rangle_B$$

operator M_μ acting on ρ_A

We can now write

$$\boxed{\rho_A' = \$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^{+}}$$

Furthermore, since U_{AB} is unitary, the M_μ's have the property

$$\sum_\mu M_\mu^{+} M_\mu = \sum_\mu {}_B\langle 0 | U_{AB}^{+} | \mu\rangle_{BB}\langle \mu | U_{AB} | 0\rangle_B$$

$$= {}_B\langle 0 | U_{AB}^{+} U_{AB} | 0\rangle_B = \mathbb{1}_A$$

We conclude:

$\$$ defines a Linear Map

$\$$: Linear Operator → Linear Operator

If $\sum_\mu M_\mu^{+} M_\mu = \mathbb{1}_A$ then $\$$ is a <u>SuperOperator</u>

and $\$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^{+}$ is the Operator-Sum or Krauss representation of $\$$

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

We can now write

$$\boxed{\rho_A' = \$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger}$$

Furthermore, since U_{AB} is unitary, the M_μ's have the property

$$\sum_\mu M_\mu^\dagger M_\mu = \sum_\mu {}_B\langle 0| U_{AB}^\dagger |\mu\rangle_{B\,B}\langle \mu| U_{AB} |0\rangle_B$$

$$= {}_B\langle 0| U_{AB}^\dagger U_{AB} |0\rangle_B = \mathbb{1}_A$$

We conclude:

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$\$$: Linear Operator → Linear Operator

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and $\$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger$ is the Operator-Sum or Krauss representation of $\$$

Note: $\$$ maps density operators to density operators because

* ρ_A' is Hermitian: $\rho_A'^\dagger = \sum_\mu M_\mu \rho_A^\dagger M_\mu^\dagger = \rho_A'$
* ρ_A' has unit trace: $\mathrm{Tr}\,\rho_A' = \sum_\mu \mathrm{Tr}[\rho_A M_\mu^\dagger M_\mu]$
* ρ_A' is positive:

$${}_A\langle\psi|\rho_A'|\psi\rangle_A = \sum_\mu \left({}_A\langle\psi| M_\mu\right)\rho_A\left(M_\mu^\dagger|\psi\rangle_A\right) \geq 0$$

Used $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ and
Trace invariance under cyclic permutation

Theorem: Given some $\$$ with an operator-sum representation, we can choose $\mathcal{H}_B$ and find the corresponding unitary U_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Note: $\$$ maps density operators to density operators because

* ρ'_A is Hermitian: $\rho_A'^{\dagger} = \sum_\mu M_\mu \rho_A^\dagger M_\mu^\dagger = \rho'_A$

* ρ'_A has unit trace: $\mathrm{Tr}\,\rho'_A = \sum_\mu \mathrm{Tr}\left[\rho_A \underbrace{M_\mu^\dagger M_\mu}_{=1}\right]$

* ρ'_A has unit trace:

$${}_A\langle\psi|\rho'_A|\psi\rangle = \sum_\mu \left({}_A\langle\psi|M_\mu\right)\rho_A\left(M_\mu^\dagger|\psi\rangle\right) \geq 0$$

Used $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ and
Trace invariance under cyclic permutation

> **Theorem**: Given some $\$$ with an operator-sum representation, we can choose $\mathcal{H}_B$ and find the corresponding unitary U_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$

Note:

* Superoperators provide a formalism to describe decoherence, i. e., maps from pure to mixed states

* Unitary evolution is a special case with only one term in the operator-sum expansion

* Two or more terms ➡ initial pure states $\in \mathcal{H}_A$ become entangled w/states $\in \mathcal{H}_B$ due to U_{AB}
  ➡ mixed final state ρ'_A

* Superoperators can be concatenated to form new ones, $\$ = \$_1 \$_2$

> **Theorem**: If $(\$)^{-1}(\$) = \mathbb{1}$ then $\$$ must necessarily be unitary

Non-unitary evolution cannot be reversed
➡ "arrow of time"

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Note:

* Superoperators provide a formalism to describe decoherence, i. e., maps from pure to mixed states

* Unitary evolution is a special case with only one in the operator-sum expansion

* Two or more terms → initial pure states $\in \mathcal{H}_A$ become entangled w/states $\in \mathcal{H}_B$ due to U_{AB}
  → mixed final state ρ_A'

* Superoperators can be concatenated to form new ones, $\$ = \$_1 \$_2$

Theorem: If $(\$)^{-1}(\$) = \mathbb{1}$ then $\$$ must necessarily be unitary

Non-unitary evolution cannot be reversed
→ "arrow of time"

We summarize:

A mapping $\$: \rho \to \rho'$ where ρ, ρ' are density operators, is a mapping of operators to operators that satisfy

(0) $\$$ is Linear

(1) $\$$ preserves Hermiticity

(2) $\$$ is Trace preserving

(3) $\$$ is completely positive,

$\$_A \otimes \mathbb{1}_B$ positive in $\mathcal{H}_A \otimes \mathcal{H}_B$ for all $\mathbb{1}_B$

Krauss Representation Theorem

Any $\$$ satisfying (0) – (3) has an Operator-Sum Representation

See Preskill, Ch. 3.2 for more on Superoperator formalism

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Measurement as a Superoperator:

Not covered in Lectures

Von Neumann: Entangle System *A* with Meter *B*

$$U_{AB}: |\varphi\rangle_A |0\rangle_B \to \sum_\mu M_\mu |\varphi\rangle_A |\mu\rangle_B \quad (1)$$

Orthogonal measurement on *B* in "pointer basis" $|\mu\rangle_B$ yields outcome μ and tells us the meter is in $|\mu\rangle_B$.

This projects out a state $|\psi_\mu\rangle_{AA}\langle\psi_\mu| = \dfrac{M_\mu|\varphi\rangle_{AA}\langle\varphi|M_\mu^\dagger}{{}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A}$

with probability $P(\mu) = {}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A$

Generally ρ_A is mixed

Meas. on *B* projects out $\rho_A^\mu = \dfrac{M_\mu \rho_A M_\mu^\dagger}{\mathrm{Tr}[M_\mu \rho_A M_\mu^\dagger]}$

with probability $P(\mu) = \mathrm{Tr}_A[M_\mu^\dagger M_\mu \rho_A] = \mathrm{Tr}_A[F_\mu \rho_A]$

This is a POVM with elements

$$F_\mu = M_\mu^\dagger M_\mu, \quad \sum_\mu F_\mu = \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}_A \quad (2)$$

If no access to the measurement outcome then

$$\rho_A \to \rho_A' = \sum_\mu P(\mu)\rho_A^\mu = \sum_\mu M_\mu \rho_A M_\mu^\dagger = \$(\rho_A)$$

Superoperator

Most general measurement: POVM $\{F_\mu\}$ on ρ_A

In this case we have $\begin{cases} P(\mu) = \mathrm{Tr}_A[F_\mu \rho_A] \\ \rho_A' = \sum_\mu \sqrt{F_\mu}\, \rho_A \sqrt{F_\mu} \end{cases}$

Note that F_μ Hermitian $\to$ $\sqrt{F_\mu}$ Hermitian

$\sum_\mu F_\mu = \mathbb{1}_A$ Follows from the operator sum Normalization condition (2) above

Compare w/ (1) above to see this POVM has the unitary representation

$$U_{AB}: |\varphi\rangle_A |0\rangle_B \to \sum_\mu \sqrt{F_\mu}\, |\varphi\rangle_A |\mu\rangle_B$$

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

If no access to the measurement outcome then

$$\rho_A \to \rho_A' = \sum_\mu P(\mu)\,\rho_A^\mu = \sum_\mu M_\mu \rho_A M_\mu^\dagger = \$(\rho_A)$$

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Note that F_μ **Hermitian** → $\sqrt{F_\mu}$ **Hermitian**

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Follows from the operator sum Normalization condition (2) above

Compare w/ (1) above to see <u>this POVM</u> has the <u>unitary representation</u>

$$U_{AB}: |\varphi\rangle_A|0\rangle_B \rightarrow \sum_\mu \sqrt{F_\mu}|\varphi\rangle_A|\mu\rangle_B$$

Summary:

* The discussion so far highlights the relationship between measurement and decoherence. We can always view the latter as the environment Doing a measurement and extracting information that we cannot retrieve. The loss of information causes an initial pure state to evolve into a statistical mixture, which is the definition of decoherence

* Sometimes we can “guess” what kind of “measurements” the environment implements. This is useful in the modeling of decohering “Quantum Channels”

* The example that follows is based on the first of four examples of decohering quantum channels given in Preskills notes. These will be particularly Relevant for those of you working in the area of Quantum communication over quantum photonic Networks.

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Decohering Quantum Channels
– a simple example –

Communication Scenario:

imperfect quantum channel

Alice ⟶ $ ⟶ Bob

↑

what happens to a qubit
when transmitted ?

Alternatively: Transmission in time

imperfect evolution

Alice ⟶ $ ⟶ Bob

↑

evolution of qubit in
presence of environment?

These are generic input-output maps !

Example: Depolarizing Channel

Probability of error $= p$, 3 types, equal probability

(1) Bit flip $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$ ⇒ $|\psi\rangle \to \sigma_1 |\psi\rangle$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2) Phase flip $|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$ ⇒ $|\psi\rangle \to \sigma_3 |\psi\rangle$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(3) Both $|0\rangle \to i|1\rangle$, $|1\rangle \to -i|0\rangle$ ⇒ $|\psi\rangle \to \sigma_2 |\psi\rangle$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Unitary Representation: Channel is a unitary map on $\mathcal{H}_A \otimes \mathcal{H}_E$

One choice (not unique, can always find one)

$$\boxed{U_{AE} : |\psi\rangle_A |0\rangle_E \to \sqrt{1-p}\, |\psi\rangle_A |0\rangle_E + \sqrt{\frac{p}{3}} \left[\sigma_1^A |\psi\rangle_A |1\rangle_E + \sigma_2^A |\psi\rangle_A |2\rangle_E + \sigma_3^A |\psi_A\rangle |3\rangle_E \right]}$$

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Note:

The 4 orthogonal states in $\mathcal{H}_E$ keep records of what happened. If available through measurement in $\mathcal{H}_E$ the errors would in principle be reversible. We must have $\mathrm{Dim}\,\mathcal{H}_E \geq 4$ to allow 4 distinct evolutions

On Operator Form: we have

$$U_{AE} = \sqrt{1-p}\,\mathbb{1}_{AE} + \sqrt{\frac{p}{3}}\left(\sigma_1^A |1\rangle_E\, {}_E\langle 0| + \sigma_2^A |2\rangle_E\, {}_E\langle 0| + \sigma_3^A |3\rangle_E\, {}_E\langle 0|\right)$$

$$\Rightarrow \rho_A' = \mathrm{Tr}_E\left[U_{AE}\left(\rho_A |0\rangle_E\, {}_E\langle 0|\right) U_{AE}^\dagger\right]$$

$$= \sum_\mu \underbrace{{}_E\langle \mu | U_{AE} | 0\rangle_E}_{M_\mu} \rho_A\, {}_E\langle 0 | U_{AE}^\dagger | \mu\rangle_E = \sum_\mu M_\mu \rho_A M_\mu^\dagger$$

From this we find Kraus operators

$$\boxed{M_0 = \sqrt{1-p}\,\mathbb{1}_A, \quad M_1 = \sqrt{\frac{p}{3}}\,\sigma_1^A, \quad M_2 = \sqrt{\frac{p}{3}}\,\sigma_2^A, \quad M_3 = \sqrt{\frac{p}{3}}\,\sigma_3^A}$$

Check ($\sigma_i^2 = 1$) : $\sum_\mu M_\mu^\dagger M_\mu = \left(1 - p + 3\frac{p}{3}\right)\mathbb{1} = \mathbb{1}$

From earlier:

$$\rho_A' = \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0\rangle_B}\, \rho_A\, {}_B\langle 0 | U_{AB}^\dagger | \mu\rangle_B$$

operator M_μ acting on ρ_A

Evolution of the Qubit:

$$\boxed{\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1^A \rho_A \sigma_1^A + \sigma_2^A \rho_A \sigma_2^A + \sigma_3^A \rho_A \sigma_3^A\right)}$$

Bloch Sphere representation:

Bloch vector

Let: $\rho_A = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + P_3\sigma_3\right)$ (1)

Choose $\hat{z}_3$ along $\vec{P} = (0, 0, P_3)$

Sub in expression for ρ_A' above and use

$$\sigma_1\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_2 = -\sigma_3, \quad \sigma_3\sigma_3\sigma_3 = \sigma_3$$

Can show that (Math details)

$$\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1\rho_A\sigma_1 + \sigma_2\rho_A\sigma_2 + \sigma_3\rho_A\sigma_3\right)$$

$$= (1-p)\frac{1}{2}\left(\mathbb{1} + P_3\sigma_3\right) + \frac{p}{3}\left[\frac{1}{2}\left(\mathbb{1} - P_3\sigma_3\right) + \frac{1}{2}\left(\mathbb{1} - P_3\sigma_3\right) + \frac{1}{2}\left(\mathbb{1} + P_3\sigma_3\right)\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \left(1 - \frac{4p}{3}\right)P_3\sigma_3\right] = \frac{1}{2}\left(\mathbb{1} + P_3'\sigma_3\right) \Rightarrow P_3' = \left(1 - \frac{4p}{3}\right)P_3$$

By symmetry of (1) we have $\boxed{\vec{P}' = \left(1 - \frac{4}{3}p\right)\vec{P}}$

Uniform shrinking Bloch Sphere

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Evolution of the Qubit:

$$\boxed{\rho_A \to \rho_A' = (1-\eta)\rho_A + \frac{\eta}{3}\left(\sigma_1^A \rho_A \sigma_1^A + \sigma_2^A \rho_A \sigma_2^A + \sigma_3^A \rho_A \sigma_3^A\right)}$$

Bloch Sphere representation:

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Let: $\rho_A = \frac{1}{2}(\mathbb{1} + \vec{P}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbb{1} + P_3\sigma_3)$ **(1)**

Choose $\vec{z}_3$ along $\vec{P} = (0, 0, P_3)$

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$$\sigma_1\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_2 = -\sigma_3\,, \quad \sigma_3\sigma_3\sigma_3 = \sigma_3$$

Can show that (Math details)

$$\rho_A \to \rho_A' = (1-\eta)\rho_A + \frac{\eta}{3}\left(\sigma_1\rho_A\sigma_1 + \sigma_2\rho_A\sigma_2 + \sigma_3\rho_A\sigma_3\right)$$

$$= (1-\eta)\frac{1}{2}(\mathbb{1} + P_3\sigma_3) + \frac{\eta}{3}\left[\frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} + P_3\sigma_3)\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \left(1 - \frac{4\eta}{3}\right)P_3\sigma_3\right] = \frac{1}{2}(\mathbb{1} + P_3'\sigma_3) \Rightarrow P_3' = \left(1 - \frac{4\eta}{3}\right)P_3$$

By symmetry of **(1)** we have $\boxed{\vec{P}' = \left(1 - \frac{4}{3}\eta\right)\vec{P}}$

Uniform shrinking Bloch Sphere

Continuous limit:

$$\eta = \Gamma dt \Rightarrow P_3(t + dt) = \left(1 - \frac{4}{3}\Gamma dt\right)P(t)$$

$$\frac{dP_3}{dt} = -\frac{4}{3}\Gamma P_3 \Rightarrow P_3(t) = P_3(0)\, e^{-4/3\,\Gamma t}$$

Bloch Sphere shrinking at constant rate

$$\boxed{\vec{P}(t) = \vec{P}(0)\, e^{-4/3\,\Gamma t}}$$

This turns out to be identical to the Master Equation result

Other Examples:

- **Phase Damping (Bloch sphere shrinks along *x* , *y*)**
- **Amplitude Damping (Bloch sphere shrinks along *z*)**

Quantum Information Theory (Preskill ch. 5)

Main Topics of QIT:

(1) Transmission of classical info over quantum channels

(2) Information/disturbance tradeoff in QM

(3) Quantifying entanglement

(3) Transmission of quantum info over quantum channels

Our Program: (1) & (4) ~ 3 Lectures

Key Concept – Incompressible information content

Classical Measure:	Shannon Entropy
Quantum Measure:	von Neumann Entropy

Review of Classical Information Theory

(Shannon for Dummies, Preskill 5.1)

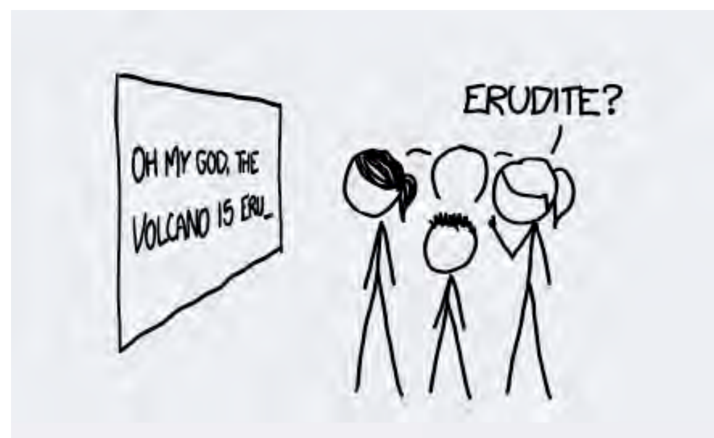
Shannon, 1948: Core findings of classical info theory

(1) How much data can be compressed (Redundancy)

(2) Reliable communication rate over noisy channel (Redundancy needed to protect against errors)

Shannon Entropy and Data Compression

(Shannon's noiseless coding theorem)



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Shannon Entropy and Data Compression

(Shannon's noiseless coding theorem)

Message = String of letters chosen from $\{a_1, a_2, \ldots, a_k\}$

A priori probability of occurrence: $p(a_x)$, $\sum p(a_x) = 1$

Basic Question: given message w/ $n \gg 1$ letters

Can we compress to length $< n$?

Quantum Information Theory (Preskill ch. 5)

Basic idea of Data Compression:

* Assign integer code letter to each typical string
* This block code has $2^{nH(p)}$ letters
* Each code letter specified by $n\,H(p)$ bits

$0 \le p \le 1 \rightarrow 0 \le H(p) \le 1$

$H(p) = 1$ only for $p = 1/2$

Block code compresses message for $p \neq 1/2$

Generalization:

k letters, prob. $p(x)$

Ensemble $X = \{x, p(x)\}$ of letters

n - letter string $\rightarrow$ x occurs $\sim n\,p(x)$ times

\# of typical strings $\sim \dfrac{n!}{\prod_x [n\,p(x)]!} \sim 2^{-nH(X)}$

$$\boxed{H(X) = -\sum_x p(x) \operatorname{Log} p(x)} \leftarrow$$

Shannon entropy !

We will see that $H(X)$ quantifies how much info is conveyed, on average, by a letter drawn from the ensemble (X) (alphabet)

Note: Boltzman Entropy

$$\boxed{S = -k \sum_i p_i \operatorname{Log} p_i}$$

Here the sum is over the microstates consistent with the given microstate. Assuming all microstates are equally likely, the System will be in the macrostate with the largest S.

Quantum Information Theory (Preskill ch. 5)

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