General Theory of Quantum Measurement (Preskill ch. 3)

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Von Neumann's Theory of Measurement

System $\longleftrightarrow$ interaction $\longleftrightarrow$ Meter (Pointer)

System Observable M — Pointer observable (position X of a free particle)

Hamiltonian for the coupled System and Meter

$$\boxed{H = \underbrace{H_0}_{\text{system}} + \underbrace{\frac{1}{2m}P^2}_{\text{free particle}} + \underbrace{\lambda M P}_{\text{interaction}}}$$

System-Meter interaction correlates M and x

➡ Measure x → indirect measurement of M

Standard Quantum Limit (example)

Heisenberg: $\Delta x \Delta p = \frac{\hbar}{2}$ ➡ $\Delta x(t)^2 \sim \Delta x(0)^2 + \left(\frac{\hbar t}{2m \Delta x(0)}\right)^2$

Interaction time t ➡ $\Delta x(t) \gtrsim \Delta x_{SQL} \sim \sqrt{\frac{\hbar t}{m}}$

Heavy pointer, Strong interaction ➡ $\boxed{H = \lambda M P}$

Note: P is the generator of translations along x

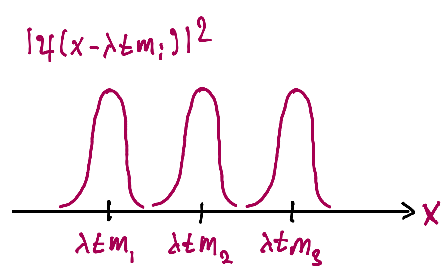
➡ Time evolution $U(t) = e^{-i\lambda t M P/\hbar}$

If	then
$M = \sum_a m_a \vert a\rangle\langle a\vert$	$U(t) = \sum_a \vert a\rangle\langle a\vert\, e^{-i\lambda t m_a P/\hbar}$

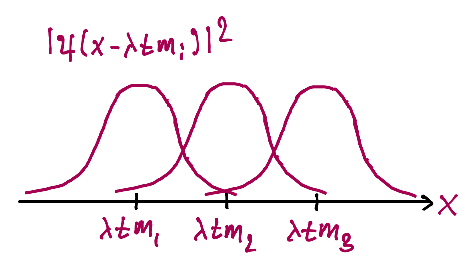
$$U(t)\sum_a \alpha_a |a\rangle \otimes |\psi(x)\rangle = \sum_a \alpha_a |a\rangle \otimes \underbrace{|\psi(x-\lambda t m_a)\rangle}_{\text{translation along } x \,\propto\, m_a}$$

translation along $x \propto m_a$

Projective



Non - Projective



General Theory of Quantum Measurement (Preskill ch. 3)

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$$U(t) \sum_a \alpha_a \lvert a\rangle \otimes \lvert \psi(x)\rangle = \sum_a \alpha_a \lvert a\rangle \otimes \underbrace{\lvert \psi(x - \lambda t m_a)\rangle}$$

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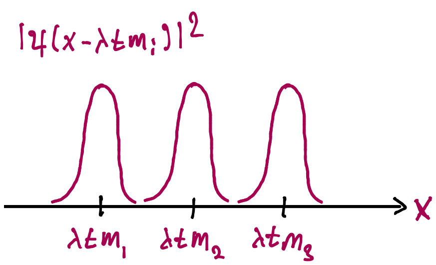

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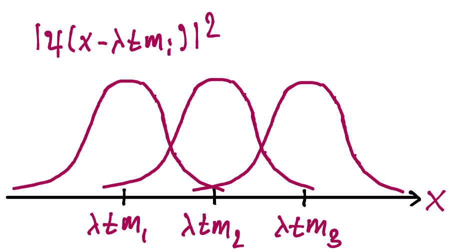
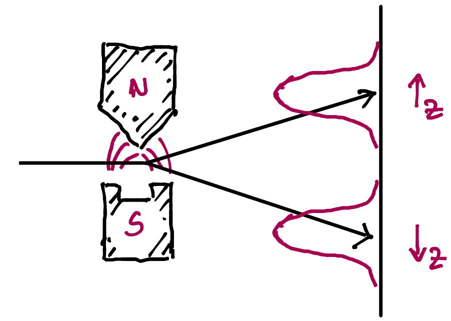
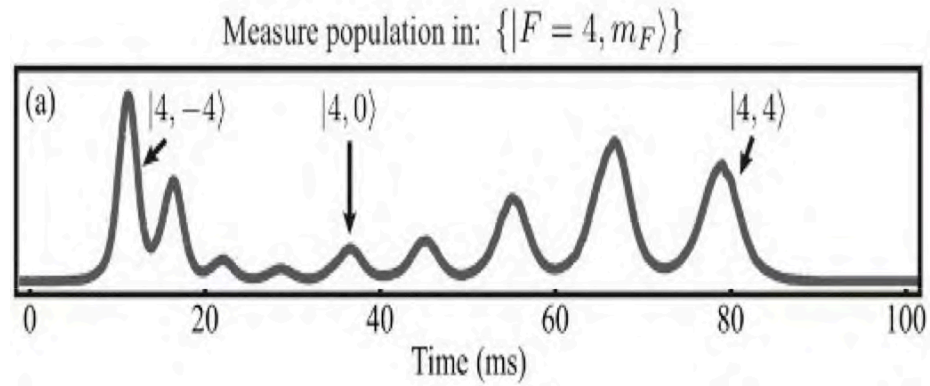


Familiar Paradigm:
Stern-Gerlach analysis

* System & Meter are 2 diff. deg.s of freedom



***S* >1/2 version of SGA :**



We have learned that

* We can engineer a Von Neumann measurement to access information about a desired observable *M*.
* We can implement more general types of measurements, including both projective and non-projective.
* We need to expand our theory of measurement beyond the type defined in the postulates of QM

General Theory of Quantum Measurement (Preskill ch. 3)

Orthogonal Measurement (OM)

Consider a set of measurements $\{E_a\}$ such that

$$E_a = E_a^\dagger \qquad E_a E_{a'} = \delta_{aa'} E_a \qquad \sum_a E_a = \mathbb{1}$$

orthogonal projectors complete set

We can associate such a set with any observable $M = \sum_a m_a E_a$

This allows us to restate the measurement postulates:

An Orthogonal Measurement of an observable M is described by a collection of operators $\{E_a\}$,

$$E_a = E_a^\dagger \qquad E_a E_{a'} = \delta_{aa'} E_a \qquad \sum_a E_a = \mathbb{1}$$

The outcome m_a occurs w/prob. $P(m_a) = \langle\psi| E_a |\psi\rangle$

⇨ the state collapses as $|\psi\rangle \to E_a|\psi\rangle / \sqrt{P(m_a)}$

Mixed state: $P(m_a) = \mathrm{Tr}[E_a \rho]$, $\rho \to E_a \rho E_a / P(m_a)$

m_a degenerate: E_a projects onto subspace

Can we generalize to a broader class? - Yes!

General Theory of Quantum Measurement (Preskill ch. 3)

Bob's OM has 3 outcomes m_a w/projectors $E_a \in \mathcal{H}$

If Alice only prepares states $\rho_A \in \mathcal{H}_A$ then

$$P(m_a) = \mathrm{Tr}[\rho_A E_a] = \mathrm{Tr}[E_A \rho_A E_A E_a]$$

$$= \mathrm{Tr}[\rho_A \underbrace{E_A E_a E_A}_{F_A}] \equiv \mathrm{Tr}[\rho_A F_A]$$

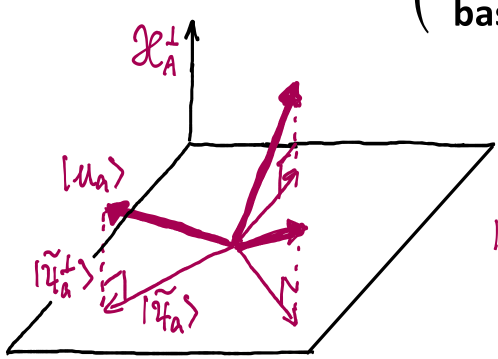
norm ≤1

$$= \langle u_a|\rho_A|u_a\rangle = \langle\tilde{\psi}_A|\rho_A|\tilde{\psi}_A\rangle$$

$$= \lambda_a \langle\psi_a|\rho_A|\psi_a\rangle$$

number ≤1 normalized

Geometric visualization: (like an over complete basis in 2D subspace)



$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle$

We can now define effective measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_A \rho_A]$$

Properties:

* Each F_A is Hermitian & non-negative $\Rightarrow P(m_a) \geq 0$
* Individual F_A are not projectors unless $\lambda_a = 1$
* $\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A$ ← identity on $\mathcal{H}_A$

POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a's are non-negative & $\sum_a F_a = \mathbb{1}$

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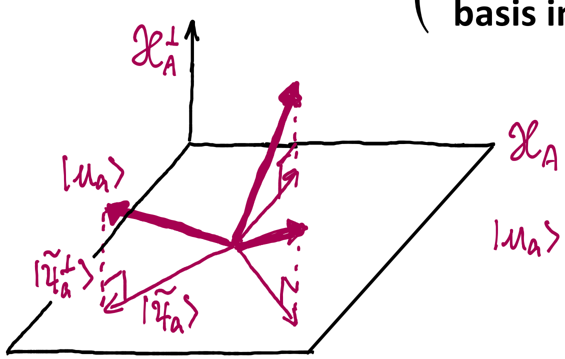

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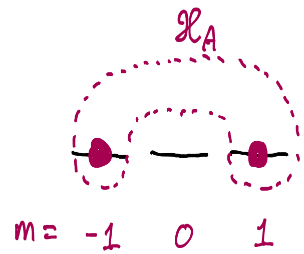
Example : POVM on Qubit encoded in Qutrit

$^{87}Rb(F=1)$ atomic HF state



Unitary Transformation

$U: |\tilde{\psi}_1\rangle + |\tilde{\psi}_1^\perp\rangle \rightarrow |-1\rangle$

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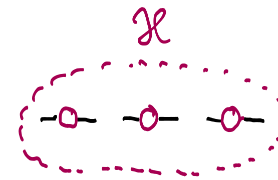
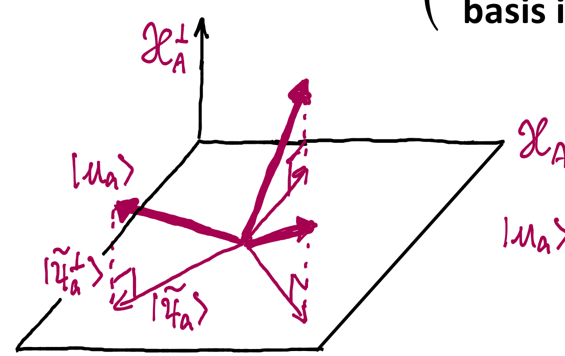


Alice prepares in $\mathcal{H}_A$

Bob measures $E_m = |m\rangle\langle m| \in \mathcal{H}$

Geometric visualization: (like an over complete basis in 2D subspace)



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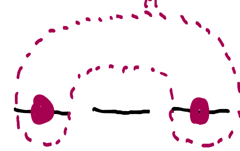

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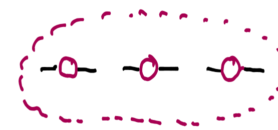

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Choose the map U

$\Rightarrow$ any 1 qubit, 3 outcome POVM we want

<u>Theorem</u>: Any POVM can be realized by adding to $\mathcal{H}_A$ an orthogonal complement $\mathcal{H}_A^\perp$

If N F_a's are desired, where $N > \mathcal{D}im\, \mathcal{H}_A$

then we need $\mathcal{D}im(\mathcal{H}_A + \mathcal{H}_A^\perp) \geq N$

(Preskill 3.1.4)

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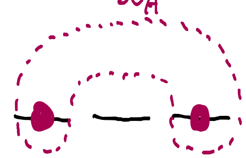

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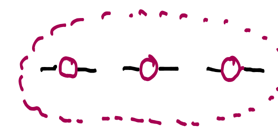

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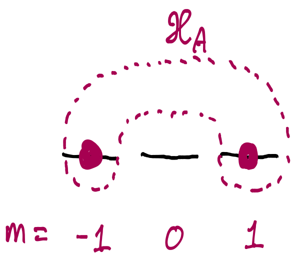

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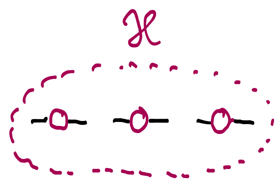

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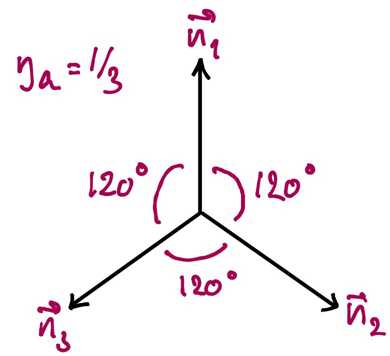
Toy Example: One Qubit POVM, illustrates different capabilities of OM & non-OM POVM's

Pick 3 unit vectors s. t. $\sum_a y_a \vec{n}_a = 0, \ \sum_a y_a = 1$



Measurement operators

$F_a = 2y_a |\uparrow_{\vec n_a}\rangle\langle\uparrow_{\vec n_a}|$ ⇨ $\sum_a F_a = \mathbb{1}$

For the above & following, note that

$|\uparrow_{\vec n_2}\rangle = \cos(60^\circ)|\uparrow_{\vec n_1}\rangle + \sin(60^\circ)|\downarrow_{\vec n_1}\rangle = \frac{1}{2}|\uparrow_{\vec n_1}\rangle + \frac{\sqrt3}{2}|\downarrow_{\vec n_1}\rangle$

$|\uparrow_{\vec n_3}\rangle = \cos(60^\circ)|\uparrow_{\vec n_1}\rangle + \sin(-60^\circ)|\downarrow_{\vec n_1}\rangle = \frac{1}{2}|\uparrow_{\vec n_1}\rangle - \frac{\sqrt3}{2}|\downarrow_{\vec n_1}\rangle$

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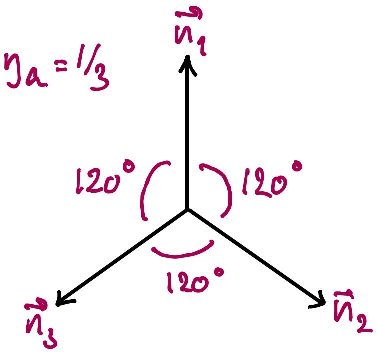
$y_a = 1/3$



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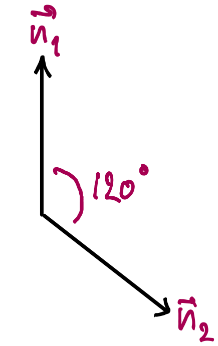
Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability



How can Bob best tell the difference ?

OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Bob's guess ?

Alice sends:
- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets:
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 3/4$ — Bob's guess: $|\uparrow_{\vec{n}_2}\rangle$

(Bob's guess for the first two outcomes: $|\uparrow_{\vec{n}_1}\rangle$)

Fidelity of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \frac{1}{2}\times 1 + \frac{1}{2}\left(3/4 \times 1 + 1/4 \times 1/4\right) = \frac{29}{32} \simeq \underline{0.9063}$$

(a) (b) (c) (d) (Quite good)

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$

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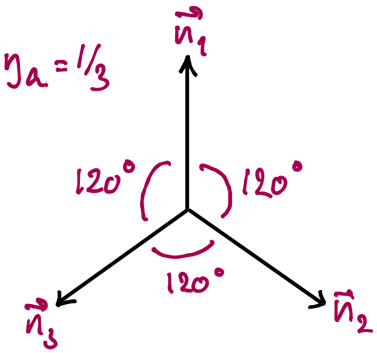

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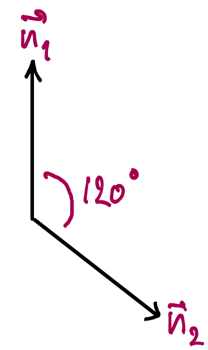

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- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$ — Bob's guess: $|\uparrow_{\vec{n}_1}\rangle$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 3/4$ — Bob's guess: $|\uparrow_{\vec{n}_2}\rangle$

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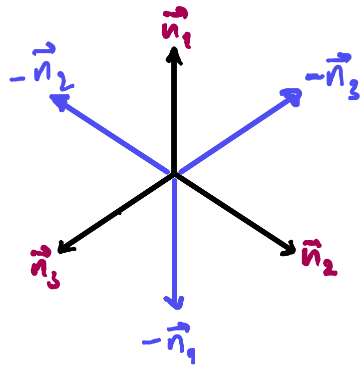
Instead

Bob does
the POVM

$$\boxed{F_a = \frac{2}{3}\,|\downarrow_{\vec n_a}\rangle\langle\downarrow_{\vec n_a}|}$$



Alice sends

- $|\uparrow_{\vec n_1}\rangle \rightarrow$ Bob gets
  - $|\downarrow_{\vec n_1}\rangle$ w/ $P = 0$
  - $|\downarrow_{\vec n_2}\rangle$ w/ $P = 1/2$
  - $|\downarrow_{\vec n_3}\rangle$ w/ $P = 1/2$
- $|\uparrow_{\vec n_2}\rangle \rightarrow$ Bob gets
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  - $|\downarrow_{\vec n_2}\rangle$ w/ $P = 0$
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Bob gets

- $|\downarrow_{\vec n_1}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec n_2}\rangle$
- $|\downarrow_{\vec n_2}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec n_1}\rangle$
- $|\downarrow_{\vec n_3}\rangle \rightarrow$ Bob is not sure

Fidelity of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \underbrace{\tfrac{1}{2}}_{P(know)} \times 1 + \underbrace{\tfrac{1}{2}}_{P(don't\ know)}\left(\tfrac{1}{2}\times 1 + \tfrac{1}{2}\times\tfrac{1}{4}\right) = \frac{13}{16} = \underline{0.8125}$$

(a) (b) (c)

(a) A sent $|\uparrow_{\vec n_1}\rangle$ or $|\uparrow_{\vec n_2}\rangle$, B knows which one w/ $P = 1/2$ $(\mathcal{F} = 1)$

(b) A sent $|\uparrow_{\vec n_1}\rangle$ or $|\uparrow_{\vec n_2}\rangle$, B DK, correct guess w/ $P = 1/2$ $(\mathcal{F} = 1)$

(c) A sent $|\uparrow_{\vec n_1}\rangle$ or $|\uparrow_{\vec n_2}\rangle$, B DK, wrong guess w/ $P = 1/2$ $(\mathcal{F} = 1/4)$

Note: If in (c) Bob guesses $|\downarrow_{\vec n_3}\rangle$ w/ $\mathcal{F} = 3/4$ he gets a slightly better fidelity of

$$\mathcal{F}_{POVM} = \frac{14}{16} = \underline{0.8750}$$

However: if Bob sticks with Heralded Success he will have a subensemble w/ $\mathcal{F}_{POVM} = 1$!

$$|\uparrow_{\vec n_2}\rangle = \cos(60^\circ)\,|\uparrow_{\vec n_1}\rangle + \sin(60^\circ)\,|\downarrow_{\vec n_1}\rangle = \tfrac{1}{2}|\uparrow_{\vec n_1}\rangle + \tfrac{\sqrt{3}}{2}|\downarrow_{\vec n_1}\rangle$$

$$|\uparrow_{\vec n_3}\rangle = \cos(60^\circ)\,|\uparrow_{\vec n_1}\rangle + \sin(-60^\circ)\,|\downarrow_{\vec n_1}\rangle = \tfrac{1}{2}|\uparrow_{\vec n_1}\rangle - \tfrac{\sqrt{3}}{2}|\downarrow_{\vec n_1}\rangle$$

General Theory of Quantum Measurement (Preskill ch. 3)

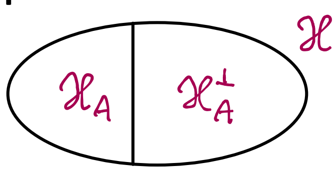
How to do it?

We can effectively do non-OM's in part of Hilbert space if we can add extra dimensions to $\mathcal{H}$:

$$\boxed{\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^\perp \quad \text{or} \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$$

<u>Direct Sum</u> Implementation

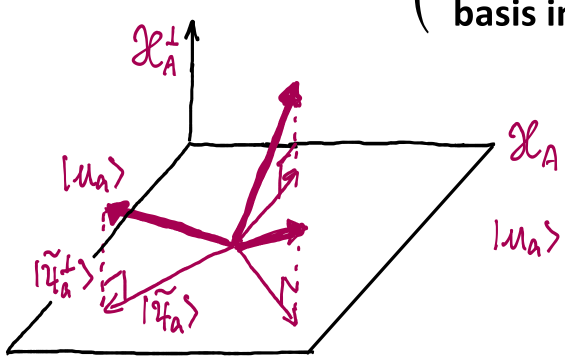
Consider $\mathcal{H}_A \oplus \mathcal{H}_A^\perp$



Alice prepares states $\rho_A \in \mathcal{H}_A$

Bob (and/or Alice) makes OM $\{E_a\}$ in $\mathcal{H}$, $E_a = |u_a\rangle\langle u_a|$

Geometric visualization: $\left(\text{like an over complete basis in 2D subspace}\right)$



$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle$

We can now define <u>effective</u> measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_A \rho_A]$$

<u>Properties</u>:

* Each F_A is Hermitian & non-negative $\Rightarrow P(m_a) \geq 0$
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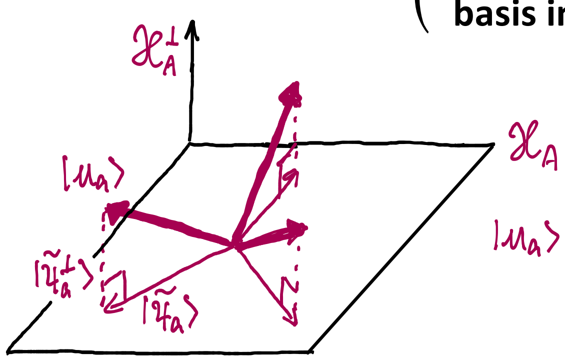

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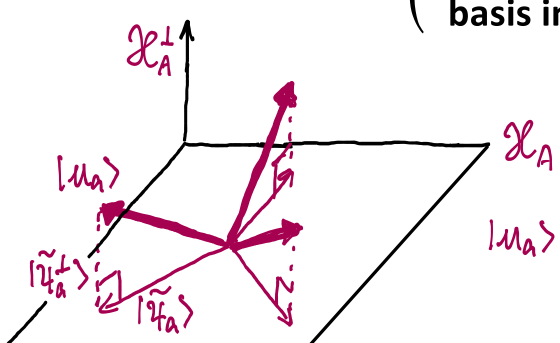

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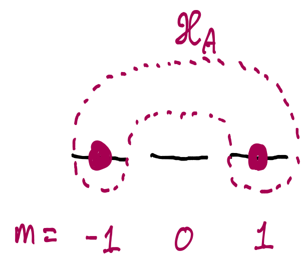

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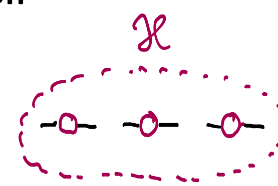

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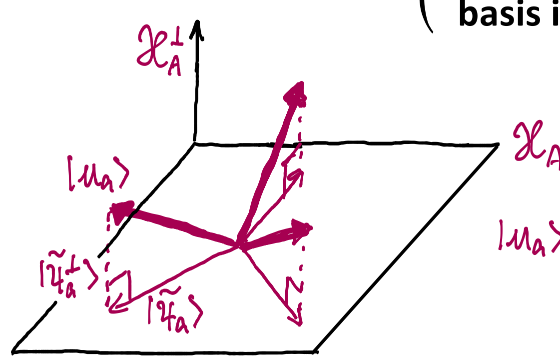

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General Theory of Quantum Measurement (Preskill ch. 3)

Bob's OM has 3 outcomes m_a w/projectors $E_a \in \mathcal{H}$

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norm ≤1

$$= \langle u_a | \rho_A | u_a \rangle = \langle \tilde{\psi}_A | \rho_A | \tilde{\psi}_A \rangle$$

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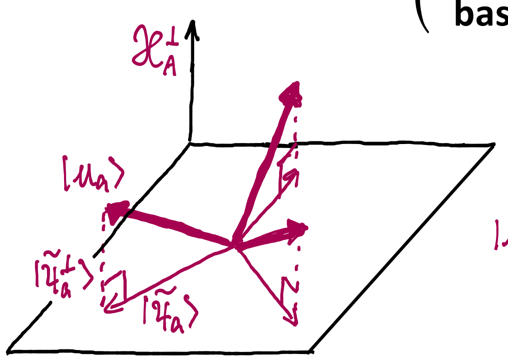

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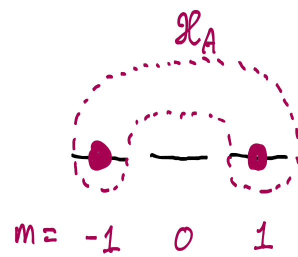

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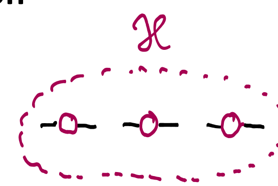

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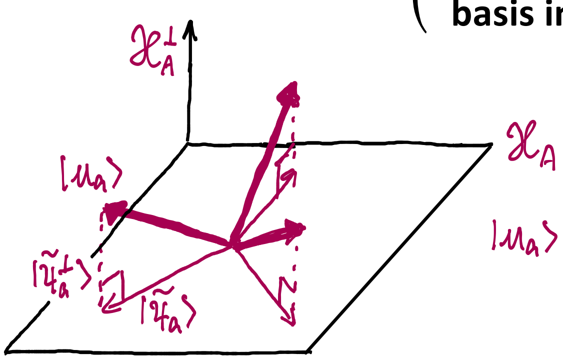

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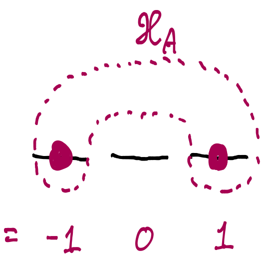

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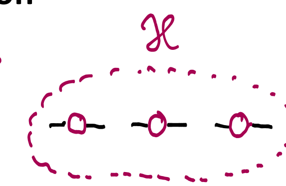

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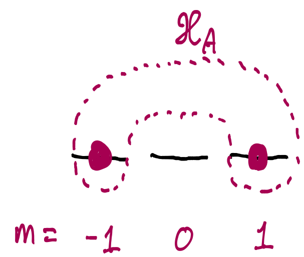

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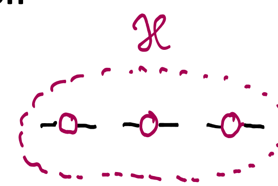

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