General Theory of Quantum Measurement (Preskill ch. 3)

10-10-2024

Von Neumann's Theory of Measurement

System $\xleftrightarrow{\text{interaction}}$ Meter (Pointer)

System Observable *M*

Pointer observable (position *X* of a free particle)

Hamiltonian for the coupled System and Meter

$$\boxed{H = H_0 + \frac{1}{2m}P^2 + \lambda M P}$$

system, free particle, interaction, momentum

System-Meter interaction correlates *M* and *x*

➡ Measure *x* → indirect measurement of *M*

Standard Quantum Limit (example)

Heisenberg: $\Delta x \Delta p = \frac{\hbar}{2}$ ➡ $\Delta x(t)^2 \sim \Delta x(0)^2 + \left(\frac{\hbar t}{2m\,\Delta x(0)}\right)^2$

Interaction time t ➡ $\Delta x(t) \gtrsim \Delta X_{SQL} \sim \sqrt{\frac{\hbar t}{m}}$

Heavy pointer, Strong interaction ➡ $\boxed{H = \lambda M P}$

Note: *P* is the generator of translations along *x*

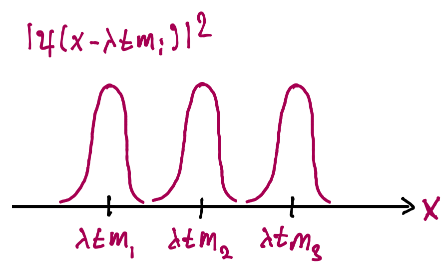
➡ Time evolution $U(t) = e^{-i\lambda t M P/\hbar}$

If	then
$M = \sum_a m_a \lvert a\rangle\langle a\rvert$	$U(t) = \sum_a \lvert a\rangle\langle a\rvert\, e^{-i\lambda t m_a P/\hbar}$

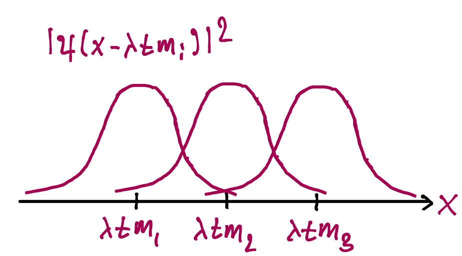
$$U(t)\sum_a \alpha_a |a\rangle \otimes |\psi(x)\rangle = \sum_a \alpha_a |a\rangle \otimes \underbrace{|\psi(x-\lambda t m_a)\rangle}$$

translation along *x* $\propto$ m_a

Projective



Non - Projective



General Theory of Quantum Measurement (Preskill ch. 3)

10-10-2024

Bob's OM has 3 outcomes m_a w/projectors $E_a \in \mathcal{H}$

If Alice only prepares states $\rho_A \in \mathcal{H}_A$ then

$$P(m_a) = \mathrm{Tr}[\rho_A E_a] = \mathrm{Tr}[E_A \rho_A E_A E_a]$$

$$= \mathrm{Tr}[\rho_A \underbrace{E_A E_a E_A}_{F_A}] \equiv \mathrm{Tr}[\rho_A F_A]$$

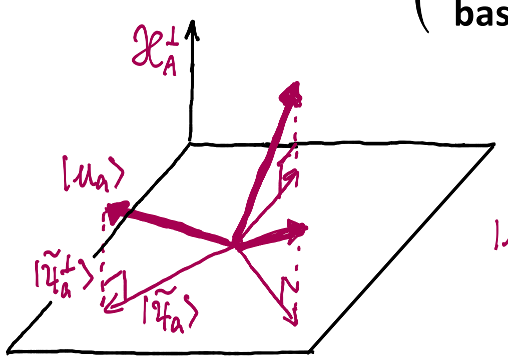
norm ≤1

$$= \langle u_a | \rho_A | u_a \rangle = \langle \tilde{\psi}_A | \rho_A | \tilde{\psi}_A \rangle$$

$$= \lambda_a \langle \psi_a | \rho_A | \psi_a \rangle$$

number ≤1 normalized

Geometric visualization: (like an over complete basis in 2D subspace)



$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle$

We can now define effective measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_A \rho_A]$$

Properties:

Each F_A is Hermitian & non-negative Evals $P(m_a) \geq 0$

Individual F_A are not projectors unless $\lambda_a = 1$

$\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A$ ← identity on $\mathcal{H}_A$

POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a 's are non-negative & $\sum_a F_a = \mathbb{1}$

General Theory of Quantum Measurement (Preskill ch. 3)

10-10-2024

Bob's OM has **3** outcomes m_a w/projectors $E_a \in \mathcal{H}$

If Alice only prepares states $\rho_A \in \mathcal{H}_A$ then

$$P(m_a) = Tr[\rho_A E_a] = Tr[E_A \rho_A E_A E_a]$$

$$= Tr[\rho_A \underbrace{E_A E_a E_A}_{F_A}] \equiv Tr[\rho_A F_A]$$

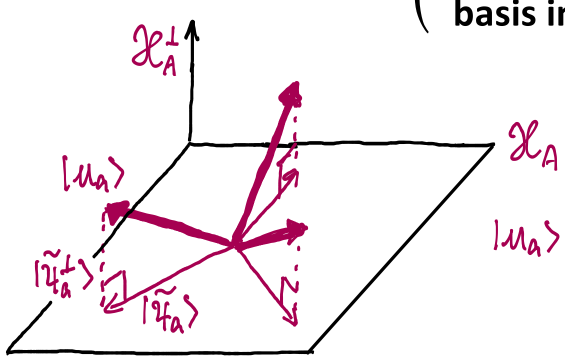
norm ≤1

$$= \langle u_a | \rho_A | u_a \rangle = \langle \tilde{\psi}_A | \rho_A | \tilde{\psi}_A \rangle$$

$$= \lambda_a \langle \psi_a | \rho_A | \psi_a \rangle$$

number ≤1 — normalized

Geometric visualization: (like an over complete basis in 2D subspace)



$$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle$$

We can now define **effective** measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = Tr[E_a \rho_A] = Tr[F_A \rho_A]$$

Properties:

Each F_A is Hermitian & non-negative Evals $P(m_a) \geq 0$

Individual F_A are not projectors unless $\lambda_a = 1$

$\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A$ ← identity on $\mathcal{H}_A$

POVM : **P**ositive **O**perator **V**alued **M**easure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a's are non-negative & $\sum_a F_a = \mathbb{1}$

General Theory of Quantum Measurement (Preskill ch. 3)

10-10-2024

We can now define effective measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow \quad P(m_a) = Tr[E_a \rho_A] = Tr[F_A \rho_A]$$

Properties:

Each F_A is Hermitian & non-negative Evals $P(m_a) \geq 0$

Individual F_A are not projectors unless $\lambda_a = 1$

$\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A \leftarrow$ identity on $\mathcal{H}_A$

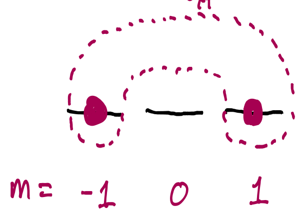
POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a 's are non-negative & $\sum_a F_a = \mathbb{1}$

Example : POVM on Qubit encoded in Qutrit

$^{87}Rb(F=1)$ atomic HF state

$\mathcal{H}_A$



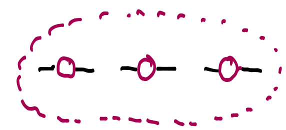
Unitary Transformation

$U: |\tilde{\psi}_1\rangle + |\tilde{\psi}_1^\perp\rangle \to |-1\rangle$

$U: |\tilde{\psi}_2\rangle + |\tilde{\psi}_2^\perp\rangle \to |0\rangle$

$U: |\tilde{\psi}_3\rangle + |\tilde{\psi}_3^\perp\rangle \to |1\rangle$

$\mathcal{H}$



Alice prepares in $\mathcal{H}_A$

Bob measures $E_m = |m\rangle\langle m| \in \mathcal{H}$

Choose the map U

$\Rightarrow$ any 1 qubit, 3 outcome POVM we want

Theorem: Any POVM can be realized by adding to $\mathcal{H}_A$ an orthogonal complement $\mathcal{H}_A^\perp$

If N F_a's are desired, where $N > Dim\, \mathcal{H}_A$

then we need $Dim(\mathcal{H}_A + \mathcal{H}_A^\perp) \geq N$

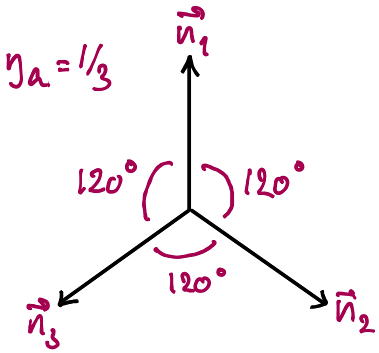
(Preskill 3.1.4)

General Theory of Quantum Measurement (Preskill ch. 3)

10-10-2024

Toy Example: One Qubit POVM, illustrates different capabilities of OM & non-OM POVM's

Pick 3 unit vectors s. t. $\sum_a y_a \vec{n}_a = 0, \ \sum_a y_a = 1$



Measurement operators

$$F_a = 2 y_a |\uparrow_{\vec{n}_a}\rangle\langle\uparrow_{\vec{n}_a}| \ \Rightarrow \ \sum_a F_a = \mathbb{1}$$

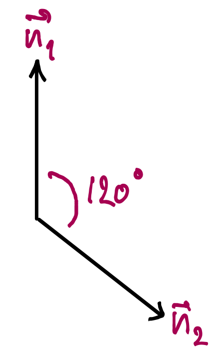
For the above & following, note that

$$|\uparrow_{\vec{n}_2}\rangle = \cos(60°)|\uparrow_{\vec{n}_1}\rangle + \sin(60°)|\downarrow_{\vec{n}_1}\rangle = \frac{1}{2}|\uparrow_{\vec{n}_1}\rangle + \frac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

$$|\uparrow_{\vec{n}_3}\rangle = \cos(60°)|\uparrow_{\vec{n}_1}\rangle + \sin(-60°)|\downarrow_{\vec{n}_1}\rangle = \frac{1}{2}|\uparrow_{\vec{n}_1}\rangle - \frac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability



How can Bob best tell the difference ?

OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Bob's guess

Alice sends
- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 3/4$ — Bob's guess: $|\uparrow_{\vec{n}_2}\rangle$

(Bob's guess for the first two outcomes: $|\uparrow_{\vec{n}_1}\rangle$)

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$