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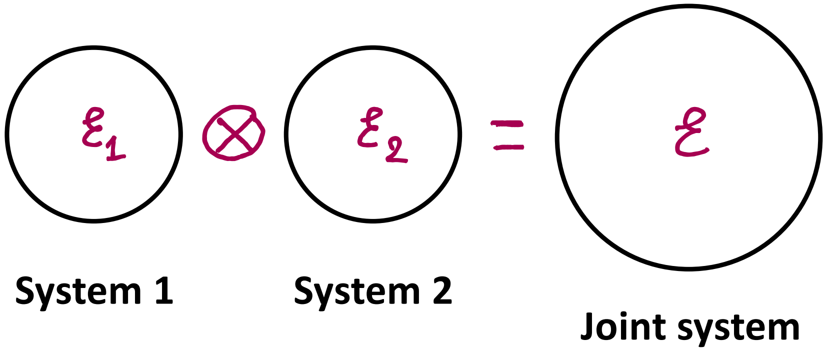
Measurement on One Part of a System

Cohen-Tannoudji Ch. III, Complement D_{III}

Measurement on One Part of a System

Quantum Measurement on Bipartite Systems



Consider the following:

Bipartite System

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$

Observable on System 1

Possible outcomes when measuring $\tilde{A}(1)$?

$$\{\text{Eigenvalues of } \tilde{A}(1)\} = \{\text{Eigenvalues of } A(1)\}$$

$\tilde{g}_n = g_n \times N_2$ $\qquad$ g_n

$\Rightarrow$ Same possible outcomes a_n indep of $|\psi\rangle$; Degeneracy in $\mathcal{E}$ increases by a factor N_2

Projector:

$$P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$$

for eigenvalue a_n

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$

$$= \sum_{i=1}^{g_n} \sum_k |a_n^i(1) v_k(2)\rangle\langle a_n^i(1) v_k(2)|$$

Probability of outcome a_n, $|\psi\rangle$ general state $\in \mathcal{E}$

$$p(a_n) = \langle\psi|\tilde{P}_n(1)|\psi\rangle$$

Lower case p is a probability

$$= \sum_{i=1}^{g_n} \sum_k \langle\psi|a_n^i(1) v_k(2)\rangle\langle a_n^i(1) v_k(2)|\psi\rangle$$

Posterior state $\quad |\psi'\rangle = \frac{1}{\sqrt{p(a_n)}} \tilde{P}_n(1)|\psi\rangle$

Measurement on One Part of a System

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Some Observations:

1. Basis $|v_k(2)\rangle$ arbitrary, no phys. significance
2. Product States Let $|\psi\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$

   If we measure $A(1)$ and observe $|a_n(1)\rangle$ then

   $$|\psi'\rangle \propto P_n(1)|\varphi(1)\rangle \otimes \mathbb{1}(2)|\chi(2)\rangle \propto |\varphi'(1)\rangle \otimes |\chi(2)\rangle$$

   still a product state

3. Entangled States

   Consider a pair of states where n and i labels the eigenvalues and degeneracies within the subspace g_n

   $$|\varphi(1)\rangle = \sum_n \sum_{i=1}^{g_n} a_{ni} |u_{ni}(1)\rangle, \quad |\chi(2)\rangle = \sum_k b_k |\chi_k(2)\rangle$$

   The corresponding product state is of the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k a_{ni} b_k |u_{ni}(1)\rangle|\chi_k(2)\rangle$$

   By comparison, the most general state in $\mathcal{E}$ has the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k c_{nik} |u_{ni}(2)\rangle|\chi_k(2)\rangle$$

   If the c_{nik} are all products of the type $a_{ni} b_k$ then $|\psi\rangle$ is a product state. Otherwise, $|\psi\rangle$ is entangled.

Measurement on One Part of a System

Some Observations:

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Some Observations: (Continued)

3. **Entangled States**

   If we measure $A(1)$ and observe the outcome a_N then the posterior state is

   $$|\psi'\rangle \propto [P_N(1) \otimes \mathbb{1}(2)]|\psi\rangle \propto \sum_{i=1}^{g_N} \sum_k c_{Nik} [|u_{Ni}(1)\rangle \otimes |\chi_k(2)\rangle]$$

   Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

   $$|\psi'\rangle \propto |u_N(1)\rangle \otimes \sum_k |\chi_k(2)\rangle \propto [|u_N(1)\rangle \otimes |\chi(2)\rangle]$$

   Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out in the global state.

   The case $g_N > 1$ is more subtle. Once we measure a_N, we know system 1 resides in the degenerate subspace associated with the outcome a_N. Repeat measurements do not generate further information about which of the exact $|u_{Ni}(1)\rangle$ our system is in. Thus, the measurement removes some, but not all of the entanglement present in $|\psi\rangle$. To completely factorize the state we would need to measure a C.S.C.O. This will identify not only the degenerate subspace but also the specific state vector $|u_{Ni}(1)\rangle$.

   See Cohen-Tannoudji Chapter III, Complement D_{III}

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Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle \varphi(1)|P_n(1)|\varphi(1)\rangle \langle \chi(2)|P_k(2)|\chi(2)\rangle$$

⇨ Outcomes a_n, b_n are **Independent Random Var's**

Uncorrelated

Physical Interpretation of Entangled States

From (3) above, measuring $A(1)$, $B(2)$

Global $|\psi\rangle$ cannot be written as $|\varphi(1)\rangle \otimes |\chi(2)\rangle$

$$P(a_n, b_k) = \langle \psi|P_n(1)P_k(2)|\psi\rangle$$

In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Measurement on One Part of a System

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Conclusion: We cannot assign state vectors to the individual subsystems !

Note:

Even though we cannot assign $|\varphi(1)\rangle$, $|\chi(2)\rangle$, it is still possible to have a local description of each subsystem on its own. It must be consistent with tensor product states, yet it must reduce the information that is locally available when the global $|\psi\rangle$ is entangled

Density Matrix Formalism