Introduction and Overview (Preskills Notes)

Quantum Computing in the NISQ era and beyond

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Noisy Intermediate-Scale Quantum (NISQ) technology will be available in the near future. Quantum computers with 50-100 qubits may be able to perform tasks which surpass the capabilities of today's classical digital computers, but noise in quantum gates will limit the size of quantum circuits that can be executed reliably. NISQ devices will be useful tools for exploring many-body quantum physics, and may have other useful applications, but the 100-qubit quantum computer will not change the world right away — we should regard it as a significant step toward the more powerful quantum technologies of the future. Quantum technologists should continue to strive for more accurate quantum gates and, eventually, fully fault-tolerant quantum computing.

Review of Quantum Mechanics Part 1

Cohen-Tannoudji Ch. II & III, Preskill 2.1 & 2.3

Review of Quantum Mechanics Part 1

9-17-2024

Note: Everyone is assumed to be familiar with grad level QM

Review of 2-level systems, Tensor Products of States, Operators, and Hilbert Spaces. Density Matrix formalism

State vectors ("Rays" in Preskill)

Unique quantum state ⟷ unique state vector

$|\psi\rangle \in \mathcal{E}$ ← State Space

Scalar product $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

complex number

($\mathcal{E}$ is a Hilbert Space)

Linear Operators

$\forall\, |\psi\rangle \in \mathcal{E}: \; A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

Projectors $P_\psi = |\psi\rangle\langle\psi|$ ← Projector on $|\psi\rangle$

$P_{\mathcal{E}_q} = \sum_{i=1}^{q} |\varphi_q^i\rangle\langle\varphi_q^i|$ ← projector on subspace $\mathcal{E}_q$

Basis in q dimensional $\mathcal{E}_q$

Hermitian Operators $A^+ = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^+$

Physical (measurable) quantities !

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Basis in q **dimensional** $\mathcal{E}_q$

Hermitian Operators

$$A^+ = A$$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^+$

Physical (measurable) quantities!

Eigenvalue Equation

$$A|\psi\rangle = \lambda|\psi\rangle$$

A **Hermitian**

* **Eigenvalues of** A **are real-valued**
* **Eigenvectors** $A|\psi\rangle = \lambda|\psi\rangle$, $A|\varphi\rangle = \mu|\varphi\rangle$ **are orthogonal if** $\lambda \neq \mu$
* **Eigenvectors of** A **form orthonormal basis in** $\mathcal{E}$

Commuting Observables

$$[A,B] \equiv AB - BA = 0 \;\Rightarrow$$

$\exists$ **orthonormal basis in** $\mathcal{E}$ **of common eigenvectors of** A, B

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Eigenvalue Equation $A|\psi\rangle = \lambda|\psi\rangle$

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Commuting Observables

$[A,B] \equiv AB - BA = 0$ ⇨

$\exists$ orthonormal basis in $\mathcal{E}$ of common eigenvectors of A, B

C.S.C.O (Complete set of commuting observables)

Set $A, B, C\ldots$ such that basis $\exists$ in $\mathcal{E}$ of eigenvectors $|a_m, b_m, c_m\ldots\rangle$ uniquely labeled by the set of eigenvalues a_m, b_m, c_m

Example H, L^2, L_z for the Hydrogen atom

Unitary Operators

U is unitary ⇨ $U^{-1} = U^{\dagger} \Leftrightarrow U^{\dagger}U = UU^{\dagger} = \mathbb{1}$

Scalar product invariant: $\langle\psi|\varphi\rangle = \langle\psi|U^{\dagger}U|\varphi\rangle$

⇨ U is a change of basis in $\mathcal{E}$

$U|v\rangle = \lambda|v\rangle$ ⇨ $\lambda = e^{i\theta}$

eigenvecs for $\lambda \neq \lambda'$ are orthogonal

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Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in $\mathcal{E}$ if the expansion

$|\psi\rangle = \sum_i \langle u_i|\psi\rangle |u_i\rangle$ is unique and exists $\forall |\psi\rangle \in \mathcal{E}$

States $|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i|\psi\rangle \\ \vdots \end{bmatrix}$

Operators $A \Leftrightarrow \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$

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Postulates of Quantum Mechanics

(1) At a fixed time t the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{E}$.

(2) Every measurable physical quantity $\mathcal{A}$ is described by an operator A acting in $\mathcal{E}$; this operator is an observable.

(3) The only possible result of a measurement of A physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable A .

(4) (Discrete non-degenerate spectrum)
When the physical quantity $\mathcal{A}$ is measured on A system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:

$$\mathcal{P}(a_n) = |\langle a_n|\psi\rangle|^2 = \langle\psi|P_n|\psi\rangle$$

where $|a_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n , and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

Postulates of Quantum Mechanics

(5) If the measurement of the physical quantity $\mathcal{A}$ on the system in state $|\psi\rangle$ gives the result a_n, then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{after}\rangle = \frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle}$$

Degenerate case: use projector onto the Subspace associated with a_n .

(6) The time evolution of the state vector $|\psi(t)\rangle$ Is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for “classical” probability distributions

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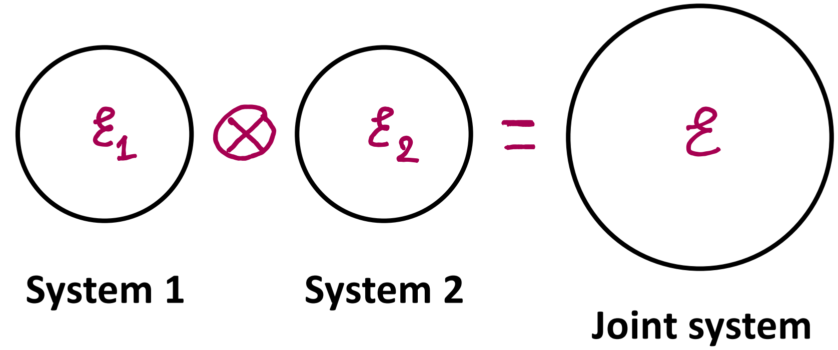
Quantum Mechanics of systems that consist of multiple parts



Def: Let $\mathcal{E}_1$, $\mathcal{E}_2$ be vector spaces of dimension N_1, N_2

The vector space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is called the

Tensor Product of $\mathcal{E}_1$ and $\mathcal{E}_2$ iff

$\forall$ pairs $|\varphi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2$, $\exists$ vector $\in \mathcal{E}$

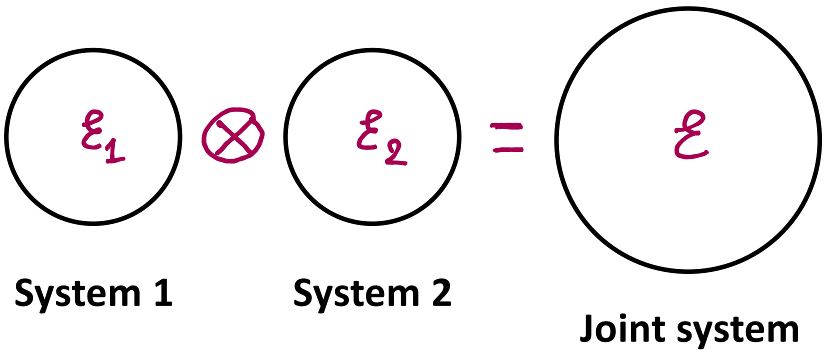
such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda|\varphi(1)\rangle \otimes \mu|\chi(2)\rangle = \lambda\mu\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right]$$

Review of Quantum Mechanics Part 1

Quantum Mechanics of systems that consist of multiple parts



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2. Distributive $|\varphi(1)\rangle \otimes \left[a|\chi_1(2)\rangle + b|\chi_2(2)\rangle\right]$

$$= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$$

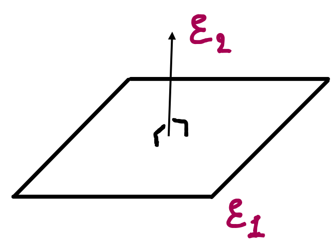
3. Bases $\{|u_i(1)\rangle\}$ in $\mathcal{E}_1$, $\{|v_\ell(2)\rangle\}$ in $\mathcal{E}_2$

$\Rightarrow$ $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$ is a basis in $\mathcal{E}$

Iff N_1, N_2 are finite, then $Dim(\mathcal{E}) = N_1 \times N_2$

These properties $\Rightarrow$ The usual linear algebra works in $\mathcal{E}$

Analogy: Tensor product of 1D & 2D geometrical space

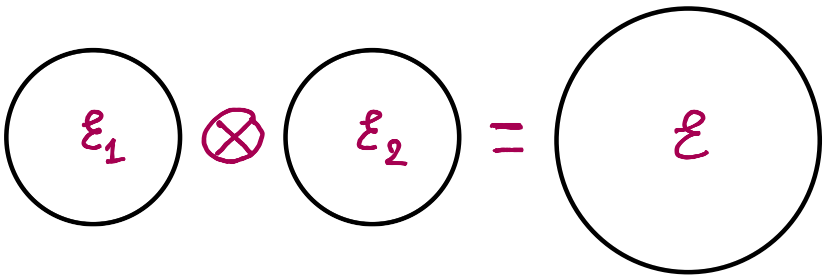


Note: $\mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$ not defined

Review of Quantum Mechanics Part 1

Quantum Mechanics of systems that consist of multiple parts



System 1 System 2 Joint system

Def: Let $\mathcal{E}_1, \mathcal{E}_2$ be vector spaces of dimension N_1, N_2

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Tensor Product of $\mathcal{E}_1$ and $\mathcal{E}_2$ iff

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1. The association is linear with respect to multiplication with complex numbers

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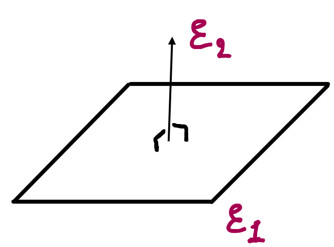
Analogy: Tensor product of 1D & 2D geometrical space



Note: $\mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$

not defined

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2. Distributive $|\varphi(1)\rangle \otimes [a|\chi_1(2)\rangle + b|\chi_2(2)\rangle]$
$= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$

3. Bases $\{|u_i(1)\rangle\}$ **in** $\mathcal{E}_1$, $\{|v_\ell(2)\rangle\}$ **in** $\mathcal{E}_2$

⇒ $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$ **is a basis in** $\mathcal{E}$

Iff N_1, N_2 **are finite, then** $Dim(\mathcal{E}) = N_1 \times N_2$

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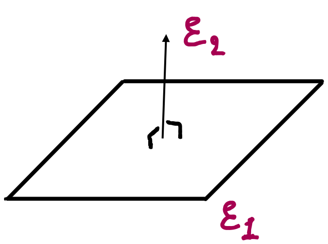
Analogy: Tensor product of $1D$ & $2D$ **geometrical space**



Note: $\mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ **geom. space**

SP **of vectors in** $\mathcal{E}_1$ **w/vectors in** $\mathcal{E}_2$ **not defined**

Vectors in $\mathcal{E}$

Let $|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$
$|\chi(2)\rangle = \sum b_\ell |v_\ell(2)\rangle$

Then $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,\ell} a_i b_\ell |u_i(1)\rangle \otimes |v_\ell(2)\rangle$

Hugely important:

There are vectors in $\mathcal{E}$ **that are not tensor products of vectors from** $\mathcal{E}_1, \mathcal{E}_2$

General vector $\in \mathcal{E}$ **can be written as**

$$|\psi\rangle = \sum_{i,\ell} c_{i\ell} |u_i(1)\rangle \otimes |v_\ell(2)\rangle$$

How to see ? There are $N_1 \times N_2$ **prob. ampl's** $c_{i\ell}$

These cannot all be written as $a_i \times b_\ell$ **where the sets** $\{a_i\}, \{b_\ell\}$ **are valid probability amplitudes.**

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Vectors in $\mathcal{E}$ Let $|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$

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How to see? There are $N_1 \times N_2$ prob. ampl's $c_{i\ell}$

These cannot all be written as $a_i \times b_\ell$ where the sets $\{a_i\}, \{b_\ell\}$ are valid probability amplitudes.

Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$

$|\chi(2)\rangle = b_1 |v_1(2)\rangle + b_2 |v_2(2)\rangle$

$\Rightarrow$ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$

Product state $\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$ General state $\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$

4 real-valued variables 6 real-valued variables

N qubits $\Rightarrow$ $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

Review of Quantum Mechanics Part 1

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⇨ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$

Product state $\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$ ↑ 4 real-valued variables

General state $\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$ ↑ 6 real-valued variables

N qubits ⇨ $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

Note: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

Back to the Linear Algebra engine of QM

Scalar product: $\left(\langle \varphi'(1)| \otimes \langle \chi'(2)|\right)\left(|\varphi(1)\rangle \otimes |\chi(2)\rangle\right)$

$= \langle \varphi'(1)|\varphi(1)\rangle \langle \chi'(2)|\chi(2)\rangle$

Operators: Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

$\tilde{A}(1)\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right] = \left(A(1)|\varphi(1)\rangle\right) \otimes |\chi(2)\rangle$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

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Tensor Product of Operators

$$[A(1)\otimes B(2)][|\varphi(1)\rangle\otimes|\chi(2)\rangle] = [A(1)|\varphi(1)\rangle]\otimes[B(2)|\chi(2)\rangle]$$

$$\Rightarrow A(1)\otimes B(2) = \tilde{A}(1)\tilde{B}(2)$$

special case: $\tilde{A}(1) = A(1)\otimes \mathbb{1}(2)$

$\tilde{B}(2) = \mathbb{1}(1)\otimes B(2)$

Commutator

$[\tilde{A}(1), \tilde{B}(2)] \equiv 0$ because $[A(1), \mathbb{1}(1)] = [B(2), \mathbb{1}(2)] = 0$

Notation: Obvious from context

$$|\varphi(1)\rangle\otimes|\chi(2)\rangle \leftrightarrow |\varphi(1)\rangle|\chi(2)\rangle \leftrightarrow |\varphi(1)\chi(2)\rangle$$

$$A(1)\otimes B(2) \leftrightarrow A(1)B(2)$$

$$\tilde{A}(1) \leftrightarrow A(1)$$

Review of Quantum Mechanics Part 1

Tensor Product of Operators

$$[A(1) \otimes B(2)][|\varphi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\varphi(1)\rangle] \otimes [B(2)|\chi(2)\rangle]$$

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$$\tilde{A}(1) \leftrightarrow A(1)$$

Eigenvalue problem in $\mathcal{E}$

Let $A(1)|\varphi_n^i(1)\rangle = a_n|\varphi_n^i(1)\rangle, \quad i = 1, \ldots, g_n \Rightarrow$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle = a_n|\varphi_n^i(1)\chi(2)\rangle \quad \forall |\chi(2)\rangle \in \mathcal{E}_2$$

Can choose $|\chi(2)\rangle \in$ orthonormal basis in $\mathcal{E}_2$

$\Rightarrow$ $g_i \times N_2$ - fold degeneracy of a_n in $\mathcal{E}$

Furthermore $\left.\begin{array}{l} A(1)|\varphi_n^i(1)\rangle = a_n|\varphi_n^i(1)\rangle \\ B(2)|\chi_\ell^j(2)\rangle = b_\ell|\chi_\ell^j(2)\rangle \end{array}\right\} \Rightarrow$

$$(A(1) + B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = (a_n + b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$A(1)B(2)|\varphi_n^i(1)\chi_\ell^j(2)\rangle = a_n b_\ell|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$f(A(1), B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = f(a_n, b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

Postulates of QM apply in $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$

$\Rightarrow$ **We are Done!**

Review of Quantum Mechanics Part 2

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(3) The only possible result of a measurement of A physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable A.

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Postulates of Quantum Mechanics

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where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for "classical" probability distributions

Bayes rule and the updating of probabilities

The Bayesian Update Rule

Consider two stochastic variables A and B. The joint, conditional, and univariate probabilities are related as follows:

$$\left.\begin{array}{l} P(A,B)=P(A|B)\,P(B) \\ P(A,B)=P(B|A)\,P(A) \end{array}\right\} \Rightarrow P(B|A)=\frac{P(A|B)\,P(B)}{P(A)}$$

Thus, with knowledge of $P(A)$ and $P(B)$ we can update our prior knowledge $P(B|A)$ when new information, $P(A|B)$, becomes available.

There are subtleties when working with a mix of probability densitity funticons (pfd's) and discrete data points. Let

α : continuous variable with pdf $p(\alpha)$

B : random discret data point

$p(B|\alpha)$: likelihood function

The Bayesian Update Rule generalizes like this:

$$p(\alpha|B)\,d\alpha = \frac{p(B|\alpha)\,p(\alpha)\,d\alpha}{P(B)}$$

where $P(B)=\int_{-\infty}^{\infty} p(B|\alpha)\,p(\alpha)\,d\alpha$ is a number

and therefore

$$\boxed{p(\alpha|B) \propto p(B|\alpha)\,p(\alpha)}$$

See https://math.mit.edu/~dav/05.dir/clss13-slidesall.pdf Page 17

Bayes rule and the updating of probabilities

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and therefore

$$\boxed{p(\alpha|B) \propto p(B|\alpha)\,p(\alpha)}$$

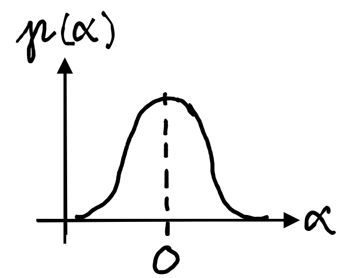
See https://math.mit.edu/~dav/05.dir/clss13-slidesall.pdf Page 17

Bayesian Update of Classical Information

Consider a classical particle located somewhere on the α– axis. The Bayesian interpretation holds that a probability distribution quantifies prior knowledge, in this example about the position of the particle.

Let $p(\alpha)$ be the probability density for finding the particle at position α. We assume this pdf is a Gaussian centered at $\alpha = 0$.

$$p(\alpha) = \frac{1}{\sqrt{2\pi\sigma_\alpha^2}}\, e^{-\alpha^2/2\sigma_\alpha^2}$$



Next, we measure the position of the particle without disturbing it. The measurement has finite resolution, i. e., there is a change of observing the particle at B even if the actual position is α. This resolution is quantified by the likelihood Function $p(B|\alpha)$

<u>Bayes rule</u> and the updating of probabilities

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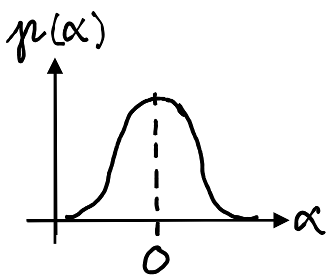

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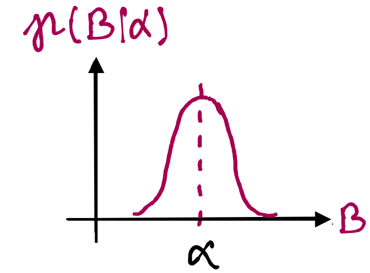
Bayesian Update of Classical Information, cont.

Let $p(B|\alpha)$ be a Gaussian,

$$P(B|\alpha) = \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-B^2/2\sigma_B^2}$$



Post-measurement, we can use Bayes Rule to update our knowledge of the position of the particle given that we observed B :

$$\boxed{\underbrace{p(\alpha|B)}_{G_{pm}} \propto \underbrace{p(B|\alpha)}_{G_1}\, \underbrace{p(\alpha)}_{G_2}}$$

The product of two Gaussians is a Gaussian, and therefore G_{pm} is also a Gaussian.

Furthermore, there are exact expressions for the means and σ's of the products, see, e. g.
http://www.lucamartino.altervista.org/2003-003.pdf

Bayes rule and the updating of probabilities

Bayesian Update of Classical Information, cont.

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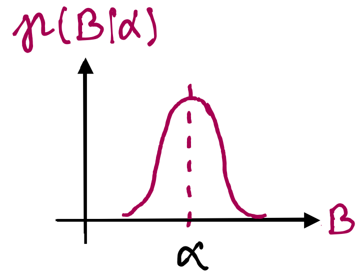

Post-measurement, we can use Bayes Rule to update our knowledge of the position of the particle given that we observed B :

$$\boxed{\underbrace{p(\alpha|B)}_{G_{pm}} \propto \underbrace{p(B|\alpha)}_{G_2}\,\underbrace{p(\alpha)}_{G_1}}$$

The product of two Gaussians is a Gaussian, and therefore G_{pm} is also a Gaussian.

Furthermore, there are exact expressions for the means and σ's of the products, see, e. g.
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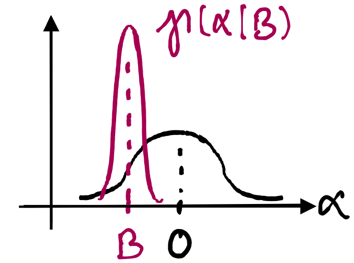
Physical Interpretation, Sharp Measurement

Now let $\sigma_{B|\alpha} \ll \sigma_\alpha$. In that case the pdf's will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-(\alpha-B)^2/\sigma_{B|\alpha}^2}$$



Here we learn a lot from the measurement, and this leads to a large update of our Prior. In this example there will be a large change In the mean and uncertainty that we assign post-measurement. The resulting pdf looks much more like the resolution function than the pdf for the original Gaussian $p(\alpha)$.

Bayes rule and the updating of probabilities

9-19-2024

Physical Interpretation, Sharp Measurement

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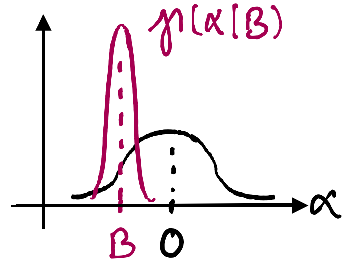

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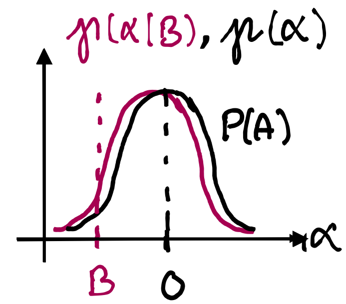
Physical Interpretation, Unsharp Measurement

Now let $\sigma_{B|\alpha} \approx \sigma_\alpha$. In that case the pdf's will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-\alpha^2/2\sigma_{B|\alpha}^2}$$

$$\approx p(\alpha)$$



Here we learn little from the measurement and this leads to at most a minor update of our Prior. In this example there will be at most a modest change in the mean and uncertainty that we assign post-measurement. The result looks like a slightly shifted and broadened version of the original $p(\alpha)$