9-17-2024

# Review of Quantum Mechanics Part 1

**Note:** Everyone is assumed to be familiar with grad level QM

Review of 2-level systems, Tensor Products of States, Operators, and Hilbert Spaces. Density Matrix formalism

## State vectors

("Rays" in Preskill)

Unique quantum state $\leftrightarrow$ unique state vector

$|\psi\rangle \in \mathcal{E}$ $\leftarrow$ State Space

## Scalar product

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

complex number

( $\mathcal{E}$ is a Hilbert Space )

## Linear Operators

$\forall\, |\psi\rangle \in \mathcal{E}: \quad A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$

## Projectors

$P_\psi = |\psi\rangle\langle\psi|$ $\leftarrow$ Projector on $|\psi\rangle$

$P_{\mathcal{E}_q} = \sum_{i=1}^{q} |\varphi_q^i\rangle\langle\varphi_q^i|$ $\leftarrow$ projector on subspace $\mathcal{E}_q$

Basis in $q$ dimensional $\mathcal{E}_q$

## Hermitian Operators

$A^+ = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^+$

Physical (measurable) quantities!

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## Linear Operators

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**Physical (measurable) quantities !**

## Eigenvalue Equation

$A|\psi\rangle = \lambda|\psi\rangle$

$A$ Hermitian

* Eigenvalues of $A$ are real-valued
* Eigenvectors $A|\psi\rangle = \lambda|\psi\rangle$, $A|\varphi\rangle = \mu|\varphi\rangle$ are orthogonal if $\lambda \neq \mu$
* Eigenvectors of $A$ form orthonormal basis in $\mathcal{E}$

## Commuting Observables

$[A,B] \equiv AB - BA = 0$ ⇨

$\exists$ orthonormal basis in $\mathcal{E}$ of common eigenvectors of $A, B$

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## C.S.C.O (Complete set of commuting observables)

Set $A, B, C \ldots$ such that basis $\exists$ in $\mathcal{E}$ of eigenvectors $|a_m, b_m, c_m \ldots\rangle$ uniquely labeled by the set of eigenvalues $a_m, b_m, c_m$

Example $H, L^2, L_z$ for the Hydrogen atom

## Unitary Operators

$U$ is unitary ⇒ $U^{-1} = U^{+} \leftrightarrow U^{+}U = UU^{+} = \mathbb{1}$

Scalar product invariant: $\langle\psi|\varphi\rangle = \langle\psi|U^{+}U|\varphi\rangle$

⇒ $U$ is a change of basis in $\mathcal{E}$

$U|v\rangle = \lambda|v\rangle$ ⇒ $\lambda = e^{i\theta}$

eigenvecs for $\lambda \neq \lambda'$ are orthogonal

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## Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in $\mathcal{E}$ if the expansion

$|\psi\rangle = \sum_i \langle u_i|\psi\rangle |u_i\rangle$ is unique and exists $\forall |\psi\rangle \in \mathcal{E}$

States $|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i|\psi\rangle \\ \vdots \end{bmatrix}$

Operators $A \Leftrightarrow \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$

# Review of Quantum Mechanics Part 1

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## Postulates of Quantum Mechanics

(1) At a fixed time $t$ the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{E}$.

(2) Every measurable physical quantity $\mathcal{A}$ is described by an operator $A$ acting in $\mathcal{E}$; this operator is an observable.

(3) The only possible result of a measurement of A physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable $A$.

(4) (Discrete non-degenerate spectrum) When the physical quantity $\mathcal{A}$ is measured on A system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue $a_n$ of the observable $A$ is:

$$\mathcal{P}(a_n) = |\langle a_n|\psi\rangle|^2 = \langle\psi|P_n|\psi\rangle$$

where $|a_n\rangle$ is the normalized eigenvector of $A$ associated with the eigenvalue $a_n$, and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

## Postulates of Quantum Mechanics

(5) If the measurement of the physical quantity $\mathcal{A}$ on the system in state $|\psi\rangle$ gives the result $a_n$, then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{after}\rangle = \frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle}$$

Degenerate case: use projector onto the Subspace associated with $a_n$.

(6) The time evolution of the state vector $|\psi(t)\rangle$ Is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the Bayesian Update Rule for "classical" probability distributions

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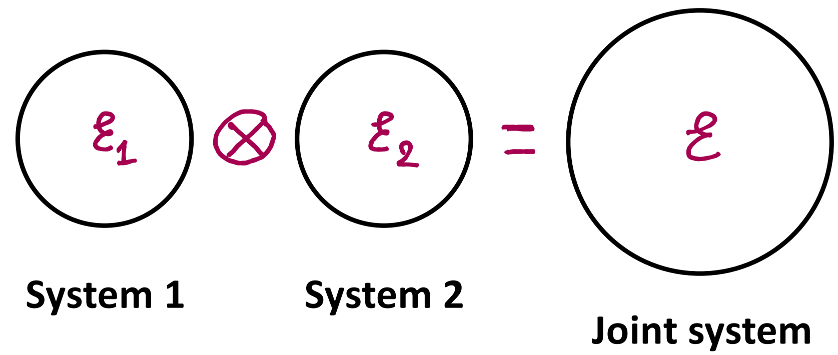
## Quantum Mechanics of systems that consist of multiple parts




**Def**: Let $\mathcal{E}_1$, $\mathcal{E}_2$ be vector spaces of dimension $N_1, N_2$

The vector space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is called the

Tensor Product of $\mathcal{E}_1$ and $\mathcal{E}_2$ iff

$\forall$ pairs $|\varphi(1)\rangle \in \mathcal{E}_1, |\chi(2)\rangle \in \mathcal{E}_2$, $\exists$ vector $\in \mathcal{E}$

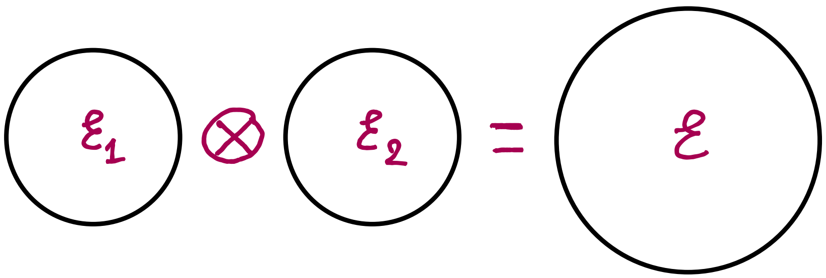
such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda|\varphi(1)\rangle \otimes \mu|\chi(2)\rangle = \lambda\mu\left[|\varphi(1)\rangle \otimes |\chi(2)\rangle\right]$$

# Review of Quantum Mechanics Part 1

## Quantum Mechanics of systems that consist of multiple parts




System 1 System 2 Joint system

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2. Distributive $|\varphi(1)\rangle \otimes \left[a|\chi_1(2)\rangle + b|\chi_2(2)\rangle\right]$

$$= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$$

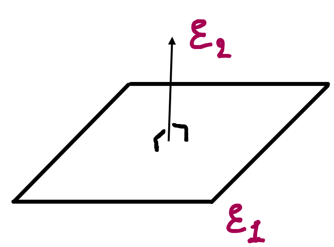
3. Bases $\{|u_i(1)\rangle\}$ in $\mathcal{E}_1$, $\{|v_\ell(2)\rangle\}$ in $\mathcal{E}_2$

$\Rightarrow$ $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$ is a basis in $\mathcal{E}$

Iff $N_1, N_2$ are finite, then $\mathrm{Dim}(\mathcal{E}) = N_1 \times N_2$

These properties $\Rightarrow$ The usual linear algebra works in $\mathcal{E}$

**Analogy**: Tensor product of 1D & 2D geometrical space




**Note**: $\mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

SP of vectors in $\mathcal{E}_1$ w/vectors in $\mathcal{E}_2$ not defined

# Review of Quantum Mechanics Part 1

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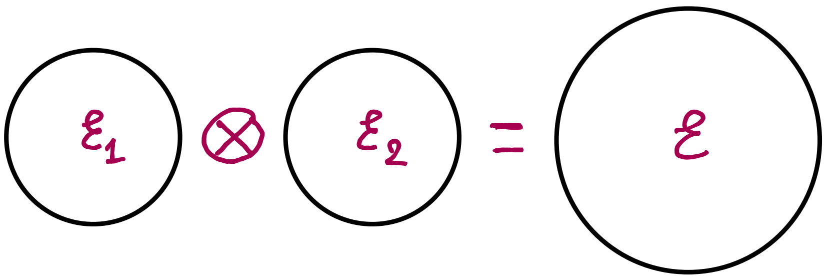


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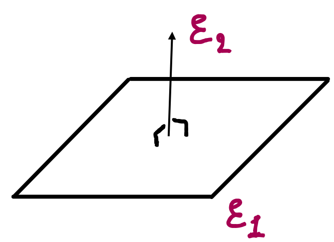


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**2. Distributive** $|\varphi(1)\rangle \otimes [a|\chi_1(2)\rangle + b|\chi_2(2)\rangle]$

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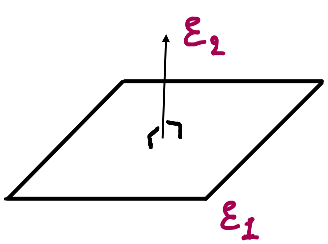


**Note:** $\mathcal{E}_1 \otimes \mathcal{E}_2 \neq$ 3D **geom. space**

SP **of vectors in** $\mathcal{E}_1$ **w/vectors in** $\mathcal{E}_2$ **not defined**

**Vectors in** $\mathcal{E}$ **Let** $|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$

$|\chi(2)\rangle = \sum b_\ell |v_\ell(2)\rangle$

**Then** $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,\ell} a_i b_\ell |u_i(1)\rangle \otimes |v_\ell(2)\rangle$

**Hugely important:**

**There are vectors in** $\mathcal{E}$ **that are not tensor products of vectors from** $\mathcal{E}_1, \mathcal{E}_2$

**General vector** $\in \mathcal{E}$ **can be written as**

$$|\psi\rangle = \sum_{i,\ell} c_{i\ell} |u_i(1)\rangle \otimes |v_\ell(2)\rangle$$

**How to see? There are** $N_1 \times N_2$ **prob. ampl's** $c_{i\ell}$

**These cannot all be written as** $a_i \times b_\ell$ **where the sets** $\{a_i\}, \{b_\ell\}$ **are valid probability amplitudes.**

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**Vectors in $\mathcal{E}$**

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**Example:** $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$

$|\chi(2)\rangle = b_1 |v_1(2)\rangle + b_2 |v_2(2)\rangle$

$\Rightarrow$ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_\ell(2)\rangle\}$

Product state $\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$ — 4 real-valued variables

General state $\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$ — 6 real-valued variables

$N$ qubits $\Rightarrow$
- product state $\rightarrow$ $2N$ real variables
- general state $\rightarrow$ $2^{N+1} - 2$ real var's

# Review of Quantum Mechanics Part 1

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$N$ qubits ⇨ product state → $2N$ real variables; general state → $2^{N+1}-2$ real var's

**Note**: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States

---

Back to the Linear Algebra engine of QM

**Scalar product**: $\left(\langle\varphi'(1)|\otimes\langle\chi'(2)|\right)\left(|\varphi(1)\rangle\otimes|\chi(2)\rangle\right)$
$$= \langle\varphi'(1)|\varphi(1)\rangle\langle\chi'(2)|\chi(2)\rangle$$

**Operators**: Let $A(1)$ act in $\mathcal{E}(1)$

The Extension $\tilde{A}(1)$ acting in $\mathcal{E}$ is defined by

$$\tilde{A}(1)\left[|\varphi(1)\rangle\otimes|\chi(2)\rangle\right] = \left(A(1)|\varphi(1)\rangle\right)\otimes|\chi(2)\rangle$$

Extension $\tilde{B}(2)$ of $B(2)$ into $\mathcal{E}$ is similar

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**Tensor Product of Operators**

$$[A(1)\otimes B(2)][|\varphi(1)\rangle\otimes|\chi(2)\rangle] = [A(1)|\varphi(1)\rangle]\otimes[B(2)|\chi(2)\rangle]$$

$$\Rightarrow A(1)\otimes B(2) = \tilde{A}(1)\tilde{B}(2)$$

special case: $\tilde{A}(1) = A(1)\otimes \mathbb{1}(2)$

$\tilde{B}(2) = \mathbb{1}(1)\otimes B(2)$

**Commutator**

$[\tilde{A}(1),\tilde{B}(2)] \equiv 0$ because $[A(1),\mathbb{1}(1)] = [B(2),\mathbb{1}(2)] = 0$

**Notation**: Obvious from context

$|\varphi(1)\rangle\otimes|\chi(2)\rangle \leftrightarrow |\varphi(1)\rangle|\chi(2)\rangle \leftrightarrow |\varphi(1)\chi(2)\rangle$

$A(1)\otimes B(2) \leftrightarrow A(1)B(2)$

$\tilde{A}(1) \leftrightarrow A(1)$

# Review of Quantum Mechanics Part 1

## Tensor Product of Operators

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## Eigenvalue problem in $\mathcal{E}$

Let $A(1)|\varphi_n^i(1)\rangle = a_n|\varphi_n^i(1)\rangle, \quad i = 1, \dots, g_n \Rightarrow$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle = a_n|\varphi_n^i(1)\chi(2)\rangle \quad \forall|\chi(2)\rangle \in \mathcal{E}_2$$

Can choose $|\chi(2)\rangle \in$ orthonormal basis in $\mathcal{E}_2$

$\Rightarrow g_i \times N_2$ - fold degeneracy of $a_n$ in $\mathcal{E}$

Furthermore
$$\left.\begin{array}{l} A(1)|\varphi_n^i(1)\rangle = a_n|\varphi_n^i(1)\rangle \\ B(2)|\chi_\ell^j(2)\rangle = b_\ell|\chi_\ell^j(2)\rangle \end{array}\right\} \Rightarrow$$

$$(A(1)+B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = (a_n+b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$A(1)B(2)|\varphi_n^i(1)\chi_\ell^j(2)\rangle = a_n b_\ell|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

$$f(A(1),B(2))|\varphi_n^i(1)\chi_\ell^j(2)\rangle = f(a_n,b_\ell)|\varphi_n^i(1)\chi_\ell^j(2)\rangle$$

Postulates of QM apply in $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$

$\Rightarrow$ **We are Done!**

# Review of Quantum Mechanics Part 2

9-19-2024

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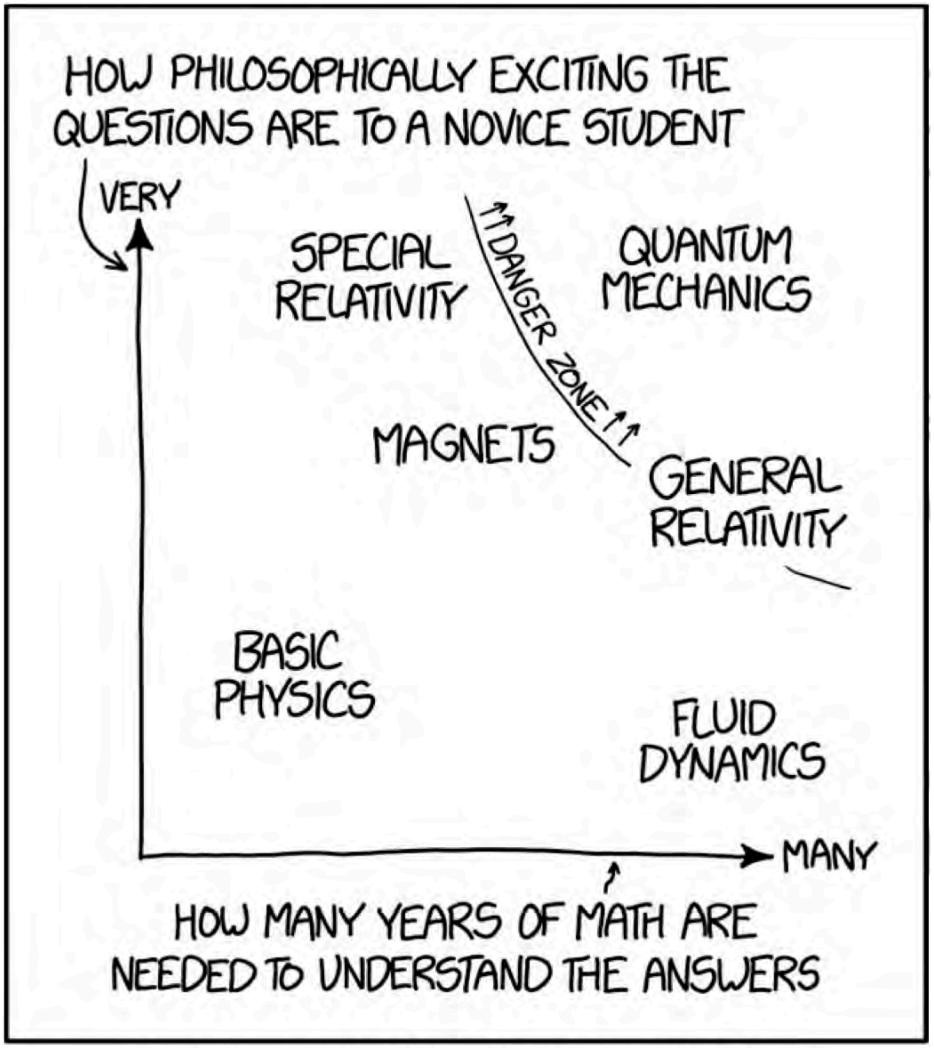
# Measurement on One Part of a System

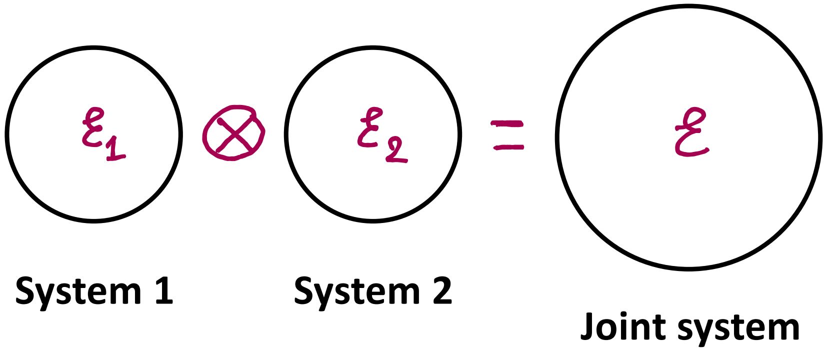



WHY SO MANY PEOPLE HAVE WEIRD IDEAS ABOUT QUANTUM MECHANICS

# Measurement on One Part of a System

9-24-2024

## Quantum Measurement on Bipartite Systems




**Consider the following:**

**Bipartite System**

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$

Observable on System 1

**Possible outcomes when measuring $\tilde{A}(1)$ ?**

$\{$Eigenvalues of $\tilde{A}(1)\} = \{$Eigenvalues of $A(1)\}$

$\tilde{g}_n = g_n \times N_2$ $\qquad$ $g_n$

- Same possible outcomes $a_n$ indep of $|\psi\rangle$
- Degeneracy in $\mathcal{E}$ increases by a factor $N_2$

**Projector:** $P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$

for eigenvalue $a_n$

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$

$$= \sum_{i=1}^{g_n} \sum_k |a_n^i(1) v_k(2)\rangle \langle a_n^i(1) v_k(2)|$$

Probability of outcome $a_n$, $|\psi\rangle$ general state $\in \mathcal{E}$

$$p(a_n) = \langle\psi|\tilde{P}_n(1)|\psi\rangle$$

$$= \sum_{i=1}^{g_n} \sum_k \langle\psi| a_n^i(1) v_k(2)\rangle \langle a_n^i(1) v_k(2)|\psi\rangle$$

Lower case p is a probability

Posterior state $\quad |\psi'\rangle = \frac{1}{\sqrt{p(a_n)}} \tilde{P}_n(1)|\psi\rangle$

# Measurement on One Part of a System

- Same possible outcomes $a_n$ indep of $|\psi\rangle$
- Degeneracy in $\mathcal{E}$ increases by a factor $N_2$

**Projector:** $P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$

for eigenvalue $a_n$

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$

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## Some Observations:

1. Basis $|v_k(2)\rangle$ arbitrary, no phys. significance
2. Product States Let $|\psi\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$

   If we measure $A(1)$ and observe $|a_n(1)\rangle$ then

   $$|\psi'\rangle \propto P_n(1)|\varphi(1)\rangle \otimes \mathbb{1}(2)|\chi(2)\rangle \propto |\varphi'(1)\rangle \otimes |\chi(2)\rangle$$

   still a product state

3. Entangled States

   Consider a pair of states where $n$ and $i$ labels the eigenvalues and degeneracies within the subspace $g_n$

   $$|\varphi(1)\rangle = \sum_n \sum_{i=1}^{g_n} a_{ni} |u_{ni}(1)\rangle, \quad |\chi(2)\rangle = \sum_k b_k |\chi_k(2)\rangle$$

   The corresponding product state is of the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k a_{ni} b_k |u_{ni}(1)\rangle |\chi_k(2)\rangle$$

   By comparison, the most general state in $\mathcal{E}$ has the form

   $$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k c_{nik} |u_{ni}(2)\rangle |\chi_k(2)\rangle$$

   If the $c_{nik}$ are all products of the type $a_{ni} b_k$ then $|\psi\rangle$ is a product state. Otherwise, $|\psi\rangle$ is entangled.

# Measurement on One Part of a System

- Same possible outcomes $a_n$ indep of $|\psi\rangle$
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Using the recipe to extend an operator into $\mathcal{E}$

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Probability of outcome $a_n$, $|\psi\rangle$ general state $\in \mathcal{E}$

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# Measurement on One Part of a System

9-24-2024

## Some Observations:

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   Consider a pair of states where *n* and *i* labels the eigenvalues and degeneracies within the subspace $g_n$

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## Some Observations: (Continued)

3. **Entangled States**

   If we measure $A(1)$ and observe the outcome $a_N$ then the posterior state is

   $$|\psi'\rangle \propto \left[P_N(1) \otimes \mathbb{1}(2)\right]|\psi\rangle \propto \sum_{i=1}^{g_N} \sum_k c_{Nik} \left[|u_{Ni}(1)\rangle \otimes |\chi_k(2)\rangle\right]$$

   Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

   $$|\psi'\rangle \propto |u_N(1)\rangle \otimes \sum_k |\chi_k(2)\rangle \propto \left[|u_N(1)\rangle \otimes |\chi(2)\rangle\right]$$

   Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out of the global state.

   The case $g_N > 1$ is more subtle. Once we measure $a_N$, we know system 1 resides in the degenerate subspace associated with the outcome $a_N$. Repeat measurements do not generate further information about which of the exact $|u_{Ni}(1)\rangle$ our system is in. Thus, the measurement removes some, but not all of the entanglement present in $|\psi\rangle$. To completely factorize the state we would need to measure a C.S.C.O. This will identify not only the degenerate subspace but also the specific state vector $|u_{Ni}(1)\rangle$.
   See Cohen-Tannoudji Chapter III, Complement $D_{III}$

# Measurement on One Part of a System

## Some Observations: (Continued)

### 3. Entangled States

If we measure $A(1)$ and observe the outcome $a_N$ then the posterior state is

$$|\psi'\rangle \propto \left[P_N(1)\otimes \mathbb{1}(2)\right]|\psi\rangle \propto \sum_{i=1}^{g_N}\sum_{k} c_{Ni;k}\left[|u_{Ni}(1)\rangle\otimes|\chi_k(2)\rangle\right]$$

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## Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle\varphi(1)|P_n(1)|\varphi(1)\rangle\langle\chi(2)|P_k(2)|\chi(2)\rangle$$

⇨ Outcomes $a_n$, $b_n$ are **Independent Random Var's**

Uncorrelated

## Physical Interpretation of Entangled States

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Global $|\psi\rangle$ cannot be written as $|\varphi(1)\rangle\otimes|\chi(2)\rangle$

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Conclusion: We cannot assign state vectors to the individual subsystems !

# Measurement on One Part of a System

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Conclusion: We cannot assign state vectors to the individual subsystems !

## Note:

Even though we cannot assign $|\varphi(1)\rangle$, $|\chi(2)\rangle$, it is still possible to have a local description of each subsystem on its own. It must be consistent with tensor product states, yet it must reduce the information that is locally available when the global $|\psi\rangle$ is entangled

## Density Matrix Formalism

**Definition**: A system for which we know only the probabilities $p_k$ of finding the system in state $|\psi_k\rangle$ is said to be in a statistical mixture of states. Shorthand: mixed state.

Shorthand for non-mixed state: pure state

# Density Matrix Description of 2-Level Atoms

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**Definition:** Density Operator for pure states

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

**Definition:** Density Matrix

$$|\psi(t)\rangle = \sum_n c_n(t)|u_n\rangle$$

$$\rho_{pn}(t) = \langle u_p|\rho(t)|u_n\rangle = c_p(t)\,c_n^*(t)$$

**Definition:** Density Operator for mixed states

$$\rho(t) = \sum_{\vec{k}} p_k\,\rho_k(t), \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $p_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

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**Definition**: Density Operator for mixed states

$$\rho(t) = \sum_{\vec{k}} \eta_k \rho_k(t), \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $\eta_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

Let $A$ be an observable w/eigenvalues $a_n$

Let $P_n$ be the projector on the eigen-subspace of $a_n$

For a pure state, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ , we have

(*) $Tr\,\rho(t) = \sum_n \rho_{nn}(t) = \sum_n |C_n|^2 = 1$

(*) $\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_p \langle\psi(t)|A|u_p\rangle\langle u_p|\psi(t)\rangle$

$$= \sum_p \langle u_p|\psi(t)\rangle\langle\psi(t)|A|u_p\rangle = \sum_p \langle u_p|\rho(t)A|u_p\rangle$$

$$= Tr[\rho(t)A] \quad (|u_p\rangle \text{ basis in } \mathcal{H})$$

(*) Let $P_n$ be the projector on eigensubspace of $a_n$

$$P(a_n) = \langle\psi(t)|P_n|\psi(t)\rangle = Tr[\rho(t)P_n]$$

(*) $\dot{\rho}(t) = |\dot{\psi}(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\dot{\psi}(t)|$

$$= \frac{1}{i\hbar}H|\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar}|\psi(t)\rangle\langle\psi(t)|H$$

$$= \frac{1}{i\hbar}[H,\rho]$$

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For a mixed state, $\rho(t) = \sum_{k} p_k \rho_k(t)$ , $\rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$

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$= Tr[\rho(t)A]$

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Density Operator formalism is general !

9-26-2024

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**Density Operator formalism is general !**

## Important properties of the Density Operator

(1) $\rho$ is Hermitian, $\rho^+ = \rho$ ⇨ $\rho$ is an observable

⇨ $\exists$ basis in which $\rho$ is diagonal

In this basis a pure state has one diagonal element $= 1$ , the rest $= 0$

(2) Test for purity.

Pure: $\rho^2 = \rho$ ⇨ $\mathrm{Tr}\,\rho^2 = 1$

Mixed: $\rho^2 \neq \rho$ ⇨ $\mathrm{Tr}\,\rho^2 < 1$

(3) Schrödinger evolution does not change the $p_k$

⇨ $\mathrm{Tr}\,\rho^2$ is conserved
pure states stay pure
mixed states stay mixed

Changing pure ⇨ mixed requires non-Hamiltonian evolution – see Cohen Tannoudji $D_{III}$ & $E_{III}$

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⇩

**Density Operator Formalism**

# Measurement on One Part of a System

9-26-2024

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**Density Operator Formalism**

## Connecting to Density Operator Formalism (OPTI 544)

**Density Operator:** $\boxed{\rho(t) = \sum_k p_k |\psi_k\rangle\langle\psi_k|}$

**Terminology:**

$|\psi\rangle$ known $\rightarrow$ pure state

$p_k, |\psi_k\rangle$ known $\rightarrow$ mixed state

**Properties**

(1) $\mathrm{Tr}\,\rho = 1$

(2) $\langle A\rangle = \mathrm{Tr}[\rho A]$

Degenerate subspace of *A*

(3) $P(a_n) = \mathrm{Tr}[\rho P_n]$, $P_n$: projector onto $\mathcal{E}_a$

(4) $\frac{d}{dt}\rho = \frac{1}{i\hbar}[H,\rho]$ Schrödinger Eq.

(5) $\rho$ pure $\rightarrow$ $\rho^2 = \rho$, $\mathrm{Tr}\,\rho^2 = 1$

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## Separate Description of Part of a System:

Let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

T.P. Basis $\{|u_i(1)\rangle\} \otimes \{|v_j(2)\rangle\}$

Density Operator $\rho$ in $\mathcal{E}$ ← Describes global system

Goal: To "reverse engineer" operators $\rho(1)$ in $\mathcal{E}_1$ and $\rho(2)$ in $\mathcal{E}_2$ such that they describe the systems independently

Our starting point is the global density operator

$$\boxed{\rho = \sum_{(ij)(k\ell)} \rho_{(ij)(k\ell)} |u_i v_j\rangle\langle u_k v_\ell|}$$

# Measurement on One Part of a System

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## Definition: Partial Trace

$$\rho(1) = \mathrm{Tr}_2\, \rho = \sum_q \langle v_q|\rho|v_q\rangle$$

$$= \sum_q \sum_{(ij)(k\ell)} \rho_{(ij)(k\ell)} \overbrace{\langle v_q||u_i v_j\rangle\langle u_k v_\ell||v_q\rangle}^{\delta_{jq}\delta_{\ell q}|u_i\rangle\langle u_k|}$$

$$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k|$$ ← **operator in** $\mathcal{E}_1$

## Properties of $\rho(1)$

**H.C.** c.c. numbers, swap kets & bras

**(1)** $\rho(1)^\dagger = \sum_{ik}\sum_q \rho^*_{(iq)(kq)} |u_i\rangle\langle u_k|$

$= \sum_{ik}\sum_q \rho_{(kq)(iq)} |u_k\rangle\langle u_i|$ ← **Relabel** $i \to k$, $k \to i$

$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| = \rho(1)$

**(2)** $\rho(1)$ **Hermitian** → **we can choose a basis** $\{|w_k(1)\rangle\}$
**So** $\rho(1)$ **is diagonal** → $\rho_{(iq)(kq)} \propto \delta_{ik}$

# Measurement on One Part of a System

9-26-2024

**Definition:** <u>Partial Trace</u>

$$\rho(1) = Tr_2\, \rho = \sum_q \langle v_q|\rho|v_q\rangle$$

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**(1)** $\rho(1)^{\dagger} = \sum_{ik}\sum_q \rho^*_{(iq)(kq)} |u_i\rangle\langle u_k|$

$$= \sum_{ik}\sum_q \rho_{(kq)(iq)} |u_k\rangle\langle u_i| \leftarrow \text{Relabel} \quad i \rightarrow k,\ k \rightarrow i$$

$$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| = \rho(1)$$

**(2)** $\rho(1)$ Hermitian → we can choose a basis $\{|w_k(1)\rangle\}$

So $\rho(1)$ is diagonal → $\rho_{(iq)(kq)} \propto \delta_{ik}$

**Thus** $$\rho(1) = \sum_k \sum_q \underbrace{\rho_{(kq)(kq)}}_{} |w_k\rangle\langle w_k|$$

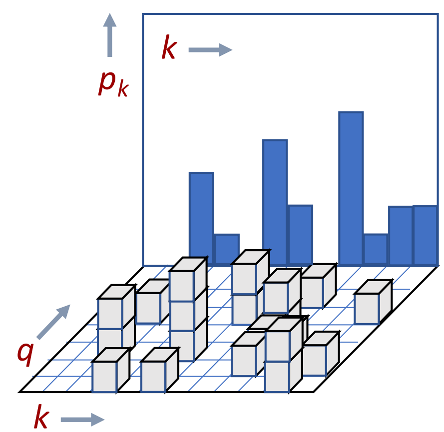
$$= \sum_k \pi_k |w_k\rangle\langle w_k|$$

**Note:**

**(1)** $\rho_{(kq)(kq)}$ = population of $|w_k v_q\rangle$, i.e. prob. of finding the global system in this state.

**(2)** $\pi_k = \sum_q \rho_{(kq)(kq)}$ is a <u>marginal probability</u>, i.e., the prob. of finding system 1 in $|w_k\rangle$, found by adding the probs $\rho_{(kq)(kq)}$ of finding the global system in the states $|w_k v_q\rangle$

**Visualization - Marginal Probability**



# Measurement on One Part of a System

**Definition:** <u>Partial Trace</u>

$$\boxed{\begin{aligned}\rho(1) &= Tr_2\,\rho = \sum_q \langle v_q|\rho|v_q\rangle \\ &= \sum_q \sum_{(ij)(kl)} \rho_{(ij)(kl)} \overbrace{\langle v_q||u_i v_j\rangle\langle u_k v_l||v_q\rangle}^{\delta_{jq}\delta_{lq}|u_i\rangle\langle u_k|} \\ &= \sum_{ik}\sum_q \rho_{(iq)(kq)}|u_i\rangle\langle u_k| \leftarrow \text{operator in } \mathcal{E}_1\end{aligned}}$$

**Properties of $\rho(1)$**

H.C. c.c. numbers, swap kets & bras

**(1)** $\rho(1)^\dagger = \sum_{ik}\sum_q \rho^*_{(iq)(kq)}|u_i\rangle\langle u_k|$

$= \sum_{ik}\sum_q \rho_{(kq)(iq)}|u_k\rangle\langle u_i| \leftarrow$ **Relabel** $i \rightarrow k$, $k \rightarrow i$

$= \sum_{ik}\sum_q \rho_{(iq)(kq)}|u_i\rangle\langle u_k| = \rho(1)$

**(2)** $\rho(1)$ **Hermitian** $\rightarrow$ **we can choose a basis** $\{|w_k(1)\rangle\}$

**So** $\rho(1)$ **is diagonal** $\rightarrow$ $\rho_{(iq)(kq)} \propto \delta_{ik}$

**Thus**

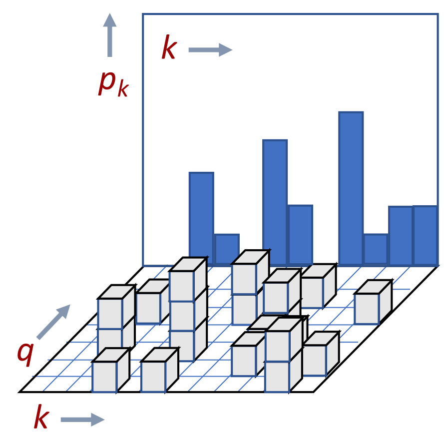
$$\boxed{\begin{aligned}\rho(1) &= \sum_k \sum_q \underbrace{\rho_{(kq)(kq)}}_{} |w_k\rangle\langle w_k| \\ &= \sum_k p_k |w_k\rangle\langle w_k|\end{aligned}}$$

**Note:**

**(1)** $\rho_{(kq)(kq)}$ **= population of** $|w_k v_q\rangle$**, i.e. prob. of finding the global system in this state.**

**(2)** $p_k = \sum_q \rho_{(kq)(kq)}$ **is a <u>marginal probability</u>, i.e., the prob. of finding system 1 in** $|w_k\rangle$**, found by adding the probs** $\rho_{(kq)(kq)}$ **of finding the global system in the states** $|w_k v_q\rangle$

**Visualization - Marginal Probability**



# Measurement on One Part of a System

**Thus**

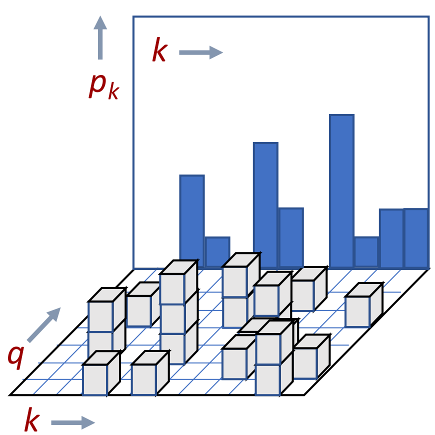
$$\boxed{\begin{aligned}\rho(1) &= \sum_k \sum_q \underbrace{\rho_{(kq)(kq)}}\, |u_k\rangle\langle u_k| \\ &= \sum_k p_k\, |u_k\rangle\langle u_k|\end{aligned}}$$

**Note:**

**(1)** $\rho_{(kq)(kq)}$ = population of $|u_k v_q\rangle$, i.e. prob. of finding the global system in this state.

**(2)** $p_k = \sum_q \rho_{(kq)(kq)}$ is a marginal probability, i.e., the prob. of finding system 1 in $|u_k\rangle$, found by adding the probs $\rho_{(kq)(kq)}$ of finding the global system in the states $|u_k v_q\rangle$

**Visualization - Marginal Probability**




**We define** $\rho(1) = Tr_2\, \rho$, $\rho(2) = Tr_1\, \rho$

**Partial Traces**

**Or**

**Reduced Density Operators**

**Note:** We already know these are Hermitian operators. Also,

$$\begin{aligned} Tr\,\rho &= \sum_n \sum_q \langle u_n v_q|\rho|u_n v_q\rangle \\ &= Tr_1(Tr_2\,\rho) = Tr_1(\rho(1)) \\ &= Tr_2(Tr_1\,\rho) = Tr_2(\rho(2)) \equiv 1 \end{aligned}$$

**Unit Trace Operators!**

**Expectation Values:**

$$\boxed{\begin{aligned}\langle \tilde{A}(1)\rangle &= Tr[\rho(1)\tilde{A}(1)] = \sum_{nq} \langle u_n v_q|\rho(1)\tilde{A}(1)|u_n v_q\rangle \\ &= \sum_{nq}\sum_{n'q'} \underbrace{\langle u_n v_q|\rho(1)|u_{n'} v_{q'}\rangle}_{\rho_{nn'}(1)} \underbrace{\langle u_{n'} v_{q'}|A(1)\otimes \mathbb{1}(2)|u_n v_q\rangle}_{\delta_{qq'}\langle u_{n'}|A(1)|u_n\rangle} \\ &= \sum_{nn'} \langle u_n|\rho(1)|u_{n'}\rangle\langle u_{n'}|A(1)|u_n\rangle \\ &= \sum_n \langle u_n|\rho(1)|u_n\rangle = \underline{\underline{Tr(\rho(1)A(1))}}\end{aligned}}$$

# Measurement on One Part of a System

## Additional Comments:

(1) Global state ≠ T. P. state

➡ Cannot assign states $|\varphi(1)\rangle, |\chi(2)\rangle$ to $S_1$, $S_2$

Can assign $\rho(1), \rho(2)$ ➡ Local description

(2) If $\rho$ is pure, $Tr\, \rho = 1$ , we still can have

$$Tr\, \rho(1)^2 \neq 1, \quad Tr\, \rho(2)^2 \neq 1$$

(2) If the Global state is a T. P., $|\psi\rangle = |\varphi(1)\rangle|\chi(2)\rangle$

then
$$\begin{cases} \sigma(1) = |\varphi(1)\rangle\langle\varphi(1)| \\ \tau(2) = |\chi(2)\rangle\langle\chi(2)| \\ \rho = \sigma(1) \otimes \tau(2) \end{cases}$$

(3) The Global state can itself be mixed. In that case a product state will have the following structure

$$\rho = \sigma(1) \otimes \tau(2) \Rightarrow \begin{cases} Tr_2 \left[\sigma(1) \otimes \tau(2)\right] = \sigma(1) \\ Tr_1 \left[\sigma(1) \otimes \tau(2)\right] = \tau(2) \end{cases}$$

# Measurement on One Part of a System

## Additional Comments:

(1) Global state ≠ T. P. state

Cannot assign states $|\varphi(1)\rangle, |\chi(2)\rangle$ to $S_1, S_2$

Can assign $\rho(1), \rho(2)$ Local description

(2) If $\rho$ is pure, $Tr\,\rho = 1$, we still can have

$$Tr\,\rho(1)^2 \neq 1, \quad Tr\,\rho(2)^2 \neq 1$$

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$$\begin{cases} \sigma(1) = |\varphi(1)\rangle\langle\varphi(1)| \\ \tau(2) = |\chi(2)\rangle\langle\chi(2)| \\ \rho = \sigma(1) \otimes \tau(2) \end{cases}$$

(3) The Global state can itself be mixed. In that case a product state will have the following structure

$$\rho = \sigma(1) \otimes \tau(2) \Rightarrow \begin{cases} Tr_2[\sigma(1) \otimes \tau(2)] = \sigma(1) \\ Tr_1[\sigma(1) \otimes \tau(2)] = \tau(2) \end{cases}$$

## Additional Comments:

(4) However, if $\rho(1) = Tr_2(\rho), \; \rho(2) = Tr_1(\rho)$

then in general $\rho' = \rho(1) \otimes \rho(2) \neq \rho$

(5) If the evolution of $\rho$ is Hamiltonian, $\dot{\rho} = \frac{1}{i\hbar}[H, \rho]$, we cannot in general find a $H(1)$ that allows an analogous equation for $\rho(1)$

## Note:

Hamiltonian evolution conserves the purity of $\rho$. However, if $\rho(1)$ is initially pure (unentangled $S_1, S_2$) the global evolution may entangle $S_1, S_2$ and cause $\rho(1)$ to become mixed.

$$\boxed{\text{Evolution of } \rho(1) \text{ is not Hamiltonian}}$$

# 2 Spins, EPR States (Preskill ch. 2.5)

## Basic Paradigm:

### Shared pair of spin-1/2 particles